Well-posedness and exponential stability for a nonlinear wave equation with acoustic boundary conditions

George J. Bautista * Juan Límaco † Leyter Potenciano-Machado ‡

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Abstract

In this paper, we prove the well-posedness of a nonlinear wave equation coupled with boundary conditions of Dirichlet and acoustic type imposed on disjoints open boundary subsets. The proposed nonlinear equation models small vertical vibrations of an elastic medium with weak internal damping and a general nonlinear term. We also prove the exponential decay of the energy associated with the problem. Our results extend the ones obtained by Frota-Goldstein [18] and Limaco-Clark-Frota-Medeiros [26] to allow weak internal dampings and removing the dimensional restriction $1 \leq n \leq 4$. The method we use is based on a finite-dimensional approach by combining the Faedo-Galerkin method with suitable energy estimates and multiplier techniques.

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1 Introduction

Let Ω be an open, bounded and connected set of $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 1$. Assume that its boundary $\Gamma = \partial\Omega$ is a compact and regular $(n-1)$-manifold. Additionally, suppose that Γ is divided into two disjoint sets in the following sense:

$$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing,$$

where Γ_0 and Γ_1 are connected subsets of Γ both with positive measure in $\mathbb{R}^{n-1}$. For $T > 0$, we denote by $Q = \Omega \times (0,T)$, the time-cylinder of $\mathbb{R}^{n+1}$ with lateral boundary $\Sigma = \Gamma \times (0,T)$. According to the above decomposition of Γ, we have

$$\Sigma = \Sigma_0 \cup \Sigma_1, \quad \Sigma_0 = \Gamma_0 \times (0,T), \quad \Sigma_1 = \Gamma_1 \times (0,T).$$

*Facultad de Ingeniería , Escuela Profesional de Ingeniería Civil, Universidad Tecnológica de los Andes, Sede Abancay, Av. Perú 700, Apurímac, Peru. E-mail: geojbs25@gmail.com
†Institute of Mathematics, Federal University of Fluminense, UFF, Rio de Janeiro, RJ, Brasil. E-mail: jlimaco@id.uff.br
‡Department of Mathematics and Statistics, University of Jyäskylä, Jyäskylä, Finland. E-mail: leyter.m.potenciano@gmail.com

This paper aims to prove the existence and uniqueness of global regular weak solutions (see Definition 3.1) to the following nonlinear initial-boundary value problem

$$
(1) \quad \begin{cases}
u''(x,t) - M\left(\int_\Omega |u(x,t)|^2_{\mathbb{R}} dx\right) \Delta u(x,t) + \Phi(u(x,t)) + \beta u'(x,t) = 0, & \text{in } Q \\
u(x,t) = 0, & \text{on } \Sigma_0, \\
f(x)\upsilon''(x,t) + g(x)\upsilon'(x,t) + h(x)\upsilon(x,t) = -\rho u'(x,t), & \text{on } \Sigma_1 \\
\partial_\nu u(x,t) = \upsilon'(x,t), & \text{on } \Sigma_1, \\
u(x,0) = u_0(x),\ u'(x,0) = u_1(x), & \text{in } \Omega \\
\upsilon(x,0) = \upsilon_0(x);\ \upsilon'(x,0) = \partial_\nu u_0(x), & \text{on } \Gamma.
\end{cases}
$$

We also prove the exponential decay of the energy associated with (1), see (13). Here Δ and ∂_ν represent the usual Laplace and $\nu \cdot \nabla$ operators in $\mathbb{R}^n$, respectively. Moreover, ν is the outward unit normal vector at Γ. The real constants β and ρ are positive. The functions M, f, g, h are nonnegative functions with M, f and h strictly positive; and u_0, u_1 and υ_0 are initial conditions.

Let us explain the motivation to consider the nonlinear model (1) and declare previous references. The rigorous mathematical foundation of acoustic wave propagation goes back to the seminal works of Beale, and Rosencrans [4, 5, 6]. They introduced and studied the well-posedness and spectral properties of the linear wave equation

$$u'' - c^2 \Delta u = 0, \quad \text{in } Q$$

subject to full acoustic boundary condition to model acoustic wave behavior of a fluid (gas) undergoing small irrotational perturbations from rest in a bounded medium. Here c stands for the sound speed of the medium. This corresponds to the case when $M = c^2$, $\Phi \equiv 0$ and $\beta = 0$ in $(1)_1$. In this physical framework, it is natural to assume that the medium's boundary acts like a spring or a resistive harmonic oscillator in response to the excess pressure in the fluid. Such a boundary is called locally reacting. See for instance [35, pp. 259-264]. Thus, the normal boundary displacement $\upsilon(x,t)$ must satisfy the following spring equation, see $(1)_3$:

$$(2) \qquad f(x)\upsilon''(x,t) + g(x)\upsilon'(x,t) + h(x)\upsilon(x,t) = -\text{excess pressure} = -\rho u'(x,t), \quad (x,t) \in \Sigma,$$

where for $x \in \partial\Omega$, $f(x)$ represent the fluid's mass, $g(x)$ and $h(x)$ act as weighted dissipative and linear boundary displacement factors, respectively. Here ρ is the fluid's density and u' the velocity potential. The above equation must be complemented with an impenetrable condition. Namely, the boundary Σ should act as an impenetrable barrier: the medium's interior fluid does not go out Q and is not disturbed by some other exterior fluid. Mathematically, it reads as the following Neumann condition, see $(1)_4$:

$$(3) \qquad \partial_\nu u(x,t) = \upsilon'(x,t), \quad (x,t) \in \Sigma.$$

We refer the reader to Beale and Rosencrans' work [4, 5, 6] and the reference therein for a comprehensive physical explanation of the linear wave equation with full acoustic boundary. Among other results, Beale and Rosencrans [4, 5, 6] proved existence, uniqueness and spectral properties to $(1)_1$ with $M = c^2$, $\Phi = 0$ and $\beta = 0$; and subject to (2)-(3), by using a suitable energy-norm and Semigroup Theory.

For completeness, we also mention results strongly connected with Control Theory. Motivated by vibration controllability matters, the authors in [21] considered localized internal damping with partial acoustic boundary conditions to study

$$u''(x,t) - \Delta u(x,t) + \omega(x)u'(x,t) = 0, \quad \text{in } Q.$$

The nonlinear coefficient $\omega \in L^\infty(\Omega)$ is a cutoff function with zero values close to $\partial\Omega$. They obtained uniform decay rates for the damped wave equations with nonlinear acoustic boundary conditions using the multipliers method. See also [38] for polynomial decay of the energies with smooth initial data. Note that this corresponds to the case $M \equiv 1$, $\Phi \equiv 0$ and $\beta = \omega(x)$ in $(1)_1$. In [22], the authors studied a wave equation with semilinear porous acoustic boundary conditions. They showed several decay rates of the energies depending on the damping's location, namely, interior or boundary damping. This corresponds to the case when $M \equiv 1$, $\Phi(u)$ and $\beta u'$ are replaced, respectively, by $\Phi(u')$ and $\beta(x)u$ in $(1)_1$. Their method relies on showing that the energy associated with the system satisfies certain ODE. Explicit decay rates are obtained in cases where the involved ODE is explicitly solved. Additionally, the solution blows up roughly when the interior source dominates the interior damping term and if the boundary source dominates the boundary damping. For related results, including numerical implementation and Laplace-Beltrami operator instead of Δ, we refer the reader to [1]-[2], [9]-[10], and the references therein.

On the other hand, it is well-known that the evolution and behavior of acoustic waves are highly nonlinear in real applications. In counterpart, mixed acoustic boundary conditions are easy to imagine. One can think of a music auditorium designed so that a portion of the boundary (for example, the ceiling) absorbs noise and another portion absorbs or reflects acoustic waves (for example, the lateral walls and floor). The former can be modelled by $(1)_1$ with acoustic boundary condition like $(1)_3$-$(1)_4$, and the latter by assuming zero boundary Dirichlet data like $(1)_2$. This class of problems was first considered by Frota and Goldstein [18] to study the problem

$$u'' - M\left(\int_\Omega |u(x,t)|_{\mathbb{R}}^2 dx\right) + \beta|u'|^\alpha u' = 0, \quad \text{in } Q, \tag{4}$$

where $\alpha > 1$ and $\beta > 0$. This class of models is known in the literature as of Carrier type. To avoid repeatedly mentioning mixed boundary conditions, we kindly refer readers to see details of such conditions in the cited article. Unless otherwise indicated, the below-listed results are also considered with these boundary conditions. At this point, it is enough to know that they are similar to the ones in (1). The authors in [18] proved the existence and uniqueness of global solutions to (4) by employing the Faedo-Galerkin method. It is worth mentioning that Frota and Goldstein's work [18] was motivated to study nonlinear wave equations of Carrier and Kirchhoff type [8, 11]. Many authors have extended Frota, and Goldstein's seminal work [18]. Remarkable examples of nonlinearities with weak and strong dissipative mechanisms were considered, including damping, delay, and memory. In this direction, we quote the results in [1]-[3], [15]-[26], [23]-[25], [28]-[29], [31]-[33], [36]-[37], [39], [45]-[46]; and the references therein. The main idea in proving the works mentioned above is getting a-priori estimates in appropriate finite-dimensional Hilbert spaces to perform a Faedo-Galerkin method combined with multipliers methods (energy estimates) and Semigroup Theory. We also mention the stability results in [19], where the authors studied the energy decay associated with nonlinear wave equations of Carrier type. The novelty in their results is that the acoustic boundary conditions are imposed with non-locally reacting properties. See also [42]-[45] for more details.

Limaco et al. [26] considered acoustic wave equations with quadratic nonlinearities when the dimension is restricted to $2 \le n < 4$. They studied the case $M(\lambda) = a + b\lambda$ with $a > 0$, $b > 0$, and $\Phi(u) = u^2$. More precisely, Limaco et al. [26] considered equation (1) with $(1)_1$ replaced by

$$u''(x,t) - M\left(a + b\int_\Omega |u(x,t)|^2_{\mathbb{R}} dx\right)\Delta u(x,t) + u^2(x,t) + \beta u'(x,t) = 0, \quad \text{in } Q. \tag{5}$$

In this case, the existence and uniqueness of solutions were performed by imposing the initial data in bounded sets. It particularly holds when the initial data is small enough; see [26, Theorem 1.1, condition (1.8)]. We point out that the damping considered by Limaco et al. [26] ($\beta u'$) is weaker than the one considered by Frota and Goldstein [18] ($\beta|u'|^\alpha u'$), see (4). It induces several no mild consequences in the analysis. Thus, the method used in [18] can not be straightforwardly implemented. Roughly, the weaker damping introduces integral terms of the form $\int_\Omega u^p\,dx$ ($p \in \mathbb{N}$ and odd) into the energy estimates, which can not be easily absorbed due to sign issues, see (10) for details. This technical difficulty was overcome by choosing initial data in suitable bounded sets. This idea goes back to Tartar [40]. We will closely follow this clever idea in our proof. This is the main reason to consider initial data small enough or lying in a bounded set: see condition (12) in Theorem 2.1 below. See the discussion in Remark 2.1. We also mention the very recent well-posedness, and stability results for Kirchhoff type equation with acoustic boundary conditions [41] for

$$u''(x,t) - M\left(\int_\Omega |\nabla u(x,t)|^2_{\mathbb{R}} dx\right)\Delta u(x,t) + \beta u'(x,t) = 0.$$

Motivated by previous results, in this paper, we prove the well-posedness of the nonlinear wave equation with acoustic boundary conditions (1). See Theorem 2.1. Exponential decay of the energy is also obtained. See Theorem 2.2. In the following sense, our results extend the ones declared in [18] and [26]. We allow dampings weaker than $\beta|u'|^\alpha u'$, namely, $\beta u'$; and nonlinearities of the form $\Phi(u) = u^p$ with $p > 1$ so that the case $p = 2$ is included. Additionally, $M(\lambda) = a + b\lambda$ is extended to $M \in C^1([0,\infty), \mathbb{R})$ under certain growth condition, see (9). Finally, the restriction on the dimension ($2 \le n \le 4$) is removed.

The remaining part of this paper is organized as follows. In Section 2, we introduce some important notation, and we state our main results declaring the precise conditions on the involved parameters, functions, and initial data. The well-posedness problem for the nonlinear system (1) will be discussed in Section 3. We perform the Galerkin method to derive useful energy estimates. In Section 4, we prove the exponential decay of the solutions for system (1). Finally, some additional comments and open problems are presented in Section 5.

2 Notation and statement of main results

We first introduce a few notations. Let us consider a sub-space of $H^1(\Omega)$ larger than $H^1_0(\Omega)$, which is represented by V and defined by

$$V = \left\{v \in H^1(\Omega); \gamma_0(u) = 0 \text{ a. e. on } \Gamma_0\right\}.$$

Here $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ stands for the continuous trace operator. It is well known [18] that the Poincaré inequality is also true in V. This motivates to consider the inner product and norm in V as

$$((u,v)) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx \quad \text{ and } \|u\|_V^2 = \int_\Omega |\nabla u(x)|_{\mathbb{R}}^2\, dx.$$

Henceforth, the symbols $(\cdot,\cdot), (\cdot,\cdot)_\Gamma, ((\cdot,\cdot)), |\cdot|^2, |\cdot|_\Gamma^2$ and $\|\cdot\|^2$ denote the inner products and the norms in the Hilbert spaces $L^2(\Omega), L^2(\Gamma)$ and V, respectively.

Due to the continuity of the trace mapping γ_0, and the continuous embedding of V into L^j, $j = 2,3,4$, we deduce that there exist positive real constants C_j, $j = 0,1,2,3$, satisfying

$$\begin{aligned} |\gamma_0(\varphi)|_\Gamma \le C_0\|\varphi\|; \quad |\varphi| \le C_1\|\varphi\|; \quad |\varphi|_{L^3(\Omega)} \le C_2\|\varphi\| \\ \text{and } |\varphi|_{L^4(\Omega)} \le C_3||\varphi||, \text{ for all } \varphi \in V. \end{aligned} \tag{6}$$

Let $p \in \mathbb{R}$ be chosen so that

$$p > 1, \quad \text{ if } n = 1,2; \qquad 1 < p \le \frac{n}{n-2}, \quad \text{ if } n \ge 3.$$

Under these conditions, we deduce that there exist positive constants C_p, C_{2p} and $C_{n,p-1}$, so that

$$\left| \begin{array}{l} \|\varphi\|_{L^{p+1}(\Omega)} \le C_p\|\varphi\|, \\ \|\varphi\|_{L^{2p}(\Omega)} \le C_{2p}\|\varphi\|, \\ \|\varphi\|_{L^{n(p-1)}(\Omega)} \le C_{n,p-1}\|\varphi\|, \\ \text{for all } \varphi \in V. \end{array} \right. \tag{7}$$

Let move to consider the conditions imposed over the functions M, f, g, h; and the initial conditions u_0, u_1 and v_0. To ensure the strictly positive conditions, we assume $f,g,h \in C^0(\Gamma)$ being real-valued functions and the existence of positive constants f_1, g_1 and h_1 such that

$$0 < f_1 \le f(x),\ 0 < \frac{C_0^2\beta\rho}{2} \le g_1 \le g(x) \text{ and } 0 < h_1 \le h(x) \text{ for all } x \in \Gamma. \tag{8}$$

Above $C_0 > 0$ is defined in (6) and β and ρ as in (1). Additionally, we assume $M \in C^1([0,\infty);\mathbb{R})$ is strictly positive and satisfies the following growth condition

$$0 < r_0 \leqslant M(\lambda), \quad \frac{|M'(\lambda)\lambda^{1/2}|}{M(\lambda)} \leqslant r_1, \quad \text{ for all } \lambda \geqslant 0, \tag{9}$$

where r_0 and r_1 are positive suitable constants. Note that the functional studied in [26], namely $M(\lambda) = a + b\lambda$ —see also (5)— trivially holds condition (9). In the same fashion, assume that there exist positive constants b_0, b_1, and b_2 such that $\Phi \in C^1(\mathbb{R})$ is a real function and

$$\left| \begin{array}{l} |\Phi(s)| \le b_0|s|^p, \\ \\ |\Phi'(s)| \le b_1|s|^{p-1}, \\ \\ |\overline{\Phi}(s)| \le b_2|s|^{p+1}, \quad \overline{\Phi}(s) := \displaystyle\int_0^s \Phi(\sigma)d\sigma \end{array} \right. \tag{10}$$

For simplicity, and after normalization, we should consider $b_0 = b_1 = b_2 = 1$.

The existence of regular weak solution for the mixed problem (1) is established by assuming $u_0 \in V$, $u_1 \in H^2(\Omega)$ and $\upsilon_0 \in L^2(\Gamma)$. The following positive constant will repeatedly appear in our computations

$$\Lambda_0 = \frac{3\rho}{2}|u_1|^2 + \frac{3\rho}{8}|u_0|^2 + M\left(|u_0|^2\right)\left[\rho\|u_0\|^2 + \left(\left|f^{1/2}\frac{\partial u_0}{\partial\eta]}\right|_\Gamma^2 + \left|h^{1/2}\upsilon_0\right|_\Gamma^2\right)\right] + \frac{2}{p+1}\rho C_p^{p+1}\|u_0\|^{p+1}. \tag{11}$$

Now, we are in a position to state the first main result of this paper.

THEOREM 2.1. *Assume that hypotheses* (6)-(10) *are satisfied. Then, for each* $(u_0, u_1, \upsilon_0) \in V \cap H^2(\Omega) \times V \times L^2(\Gamma)$ *such that*

$$\frac{4C_1}{\sqrt{r_0}}\left|M'\left(|u_0|^2\right)\right|\Lambda_0 + \frac{2C_1^2\left(M'\left(|u_0|^2\right)\right)^2}{\rho\beta r_0^2}\Lambda_0^2 + \frac{\rho\beta}{2}\left(\frac{2}{\rho r_0}\right)^{\frac{p-1}{2}} C_p^{p+1}\Lambda_0^{\frac{p-1}{2}} < \frac{\rho\beta r_0}{8}, \tag{12}$$

there exists a unique regular global weak solution (u, υ) *of* (1).

REMARK 2.1. *We emphasize that the initial conditions* u_0, u_1, *and* υ_0 *can not be arbitrarily large. They must belong to an a-priori bounded set in Sobolev spaces. Indeed, suppose that, for instance,* u_0 *is arbitrarily large so that we can construct a sequence* $(u_{0,n})_{n\in\mathbb{Z}}$ *with*

$$\lim_{n\to\infty}|u_{0,n}| = +\infty.$$

For each $n \in \mathbb{Z}$, *let us denote by* $\Lambda_{0,n}$ *the positive number in* (11) *with* u_0 *replaced by* $u_{0,n}$. *Since* $\frac{3\rho}{8}|u_{0,n}|^2 \le \Lambda_{0,n}$, *we deduce*

$$\lim_{n\to\infty}\Lambda_{0,n} = +\infty.$$

Consequently, inequality (12) *is not verified when* Λ_0 *is replaced by* $\Lambda_{0,n}$, *and when* $|n|$ *is large enough. In this sense, Theorem 2.1 holds, roughly speaking, when either initial conditions are small or when they lie in Sobolev-bounded sets. The question concerning the optimality of condition* (12) *is reserved for upcoming work.*

Assume the hypotheses from Theorem 2.1. Let us consider the energy associated to system (1) as

$$E[u,\upsilon](t) = |u(t)|^2 + \left|u'(t)\right|^2 + M\left(|u(t)|^2\right)\|u(t)\|^2 + M\left(|u(t)|^2\right)\left|g^{1/2}\upsilon_m'(t)\right|_\Gamma^2, \tag{13}$$

where (u, v) is a solution to (1). Set

$$h_2 = \max_{x\in\Gamma}|h(x)|_{\mathbb{R}}, \quad \text{and} \quad \alpha_0 > \frac{16h_2C_0^2}{3\rho\beta}. \tag{14}$$

We restrict the energy decay analysis to couples (u, v) satisfying

$$u(x,t) = v(x,t), \quad \text{on } \Sigma_1,$$

and

$$f = g = \alpha_0 h, \quad \text{on } \Gamma. \tag{15}$$

Let us define

$$\mathcal{Y}_0 = \frac{4C_1}{\sqrt{r_0}} \left| M'\left(|u_0|^2\right) \right| \Lambda_0 + \frac{2C_1^2 \left(M'\left(|u_0|^2\right) \right)^2}{\rho\beta r_0^2} \Lambda_0^2 + \frac{\rho\beta}{2} \left(\frac{2}{\rho r_0} \right)^{\frac{p-1}{2}} C_p^{p+1} \Lambda_0^{\frac{p-1}{2}}.$$

Next, we state our second main result: the exponential decay of the energy.

THEOREM 2.2. *Suppose that* (14)-(15) *holds. Then, for each* $(u_0, u_1, \upsilon_0) \in V \cap H^2(\Omega) \times V \times L^2(\Gamma)$ *such that*

$$\mathcal{Y}_0 < \frac{\rho\beta r_0}{16}, \tag{16}$$

there exists a unique regular global weak solution (u, υ) *of* (1)*, satisfying*

$$|E[u, \upsilon](t)| \leq C e^{-\overline{C}t}, \quad t > 0$$

for some positive constants C *and* $\overline{C}$*, depending on a priori assumptions but independent of time. In particular, the energy goes to zero when times goes to infinity.*

3 Existence and uniqueness of solutions

The concept of solution for the mixed problem (1) is established in the definition below.

DEFINITION 3.1. *A regular global weak solution for the nonlinear initial-boundary value problem (1.1) is a pair of real-valued functions* (u, υ)*, with* $u : \Omega \times [0, \infty) \to \mathbb{R}$ *and* $\upsilon : \Gamma \times [0, \infty) \to \mathbb{R}$*, such that for each fixed* $T > 0$ *it satisfies:*

$$\left| \begin{array}{l} u \in L^\infty(0,T;V), u' \in L^\infty(0,T;V) \cap L^2\left(0,T;L^2(\Omega)\right), u'' \in L^\infty\left(0,T;L^2(\Omega)\right), \\ \upsilon, \upsilon', \upsilon'' \in L^\infty\left(0,T;L^2(\Gamma)\right), \\ \displaystyle\int_Q \left[u''\xi + M\left(|u|^2\right) \nabla u \nabla \xi + \Phi(u)\xi + \beta u'\xi \right] dxdt = \int_{\Sigma_1} M\left(|u|^2\right) \upsilon' \gamma_0(\xi) dxdt, \\ \displaystyle\int_\Sigma \left[\rho\gamma_0\left(u'\right)\psi + f\upsilon''\psi + g\upsilon'\psi + h\upsilon\psi \right] dxdt = 0, \\ \textit{for all } \xi \in L^2(0,T;V) \textit{ and } \psi \in L^2\left(0,T;L^2(\Gamma)\right). \end{array} \right.$$

We can now pass to prove the main result of this section.

3.1 Proof of Theorem 2.1

3.1.1 Existence of solutions

The proof of the solutions' existence will be done using the Faedo-Galerkin method combined with a modification of the Tartar method (see [40]). Let $(w_i)_{i\in\mathbb{N}}$ and $(z_i)_{i\in\mathbb{N}}$ basis of $V \cap H^2(\Omega)$

and $L^2(\Gamma)$, respectively. For each $m \in \mathbb{N}$, we consider the ansatz

$$u_m(x,t) = \sum_{i=1}^{m} g_{i,m}(t) w_i(x) \quad \text{and} \quad \upsilon_m(x,t) = \sum_{j=1}^{m} h_{j,m}(t) z_j(x)$$

aimed to be the solutions to the approximate family problem

$$(17)\qquad \left| \begin{array}{l} (u''_m(t), w) + M\left(|u_m(t)|^2\right)[(\nabla u_m(t), \nabla w) - (\upsilon'_m(t), \gamma_0(w))_\Gamma] + (\Phi(u_m(t)), w) \\ +\beta\,(u'_m(t), w) = 0, \\ (\rho\gamma_0\,(u'_m(t)) + f\upsilon''_m(t) + g\upsilon'_m(t) + h\upsilon_m(t), z)_\Gamma = 0, \\ u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad V \cap H^2(\Omega), \quad u'_m(0) = u_{1m} \to u_1 \quad \text{in} \quad V, \\ \upsilon_m(0) = \upsilon_{0m} \to \upsilon_0 \quad \text{in} \quad L^2(\Gamma), \quad \upsilon'_m(0) = \partial_\nu u_{0m} \to \partial_\nu u_0 \quad \text{in} \quad L^2(\Gamma), \\ \text{ for all } w \in V_m = \operatorname{Span}\{w_1, \dots, w_m\} \text{ and } z \in Z_m = \operatorname{Span}\{z_1, \dots, z_m\}. \end{array} \right.$$

The system (17) has local solution $\{u_m, \upsilon_m\}$ in the interval $[0, t_m]$ and its extension to the whole semi-line $[0, \infty)$ is a consequence of the a priori estimates established below.

Estimate I - First, setting $w = 2u'_m$ and $z = 2\upsilon'_m$ in $(17)_1$ and $(17)_2$, respectively, we obtain

$$(18)\qquad \frac{d}{dt}\left|u'_m(t)\right|^2 + M\left(|u_m(t)|^2\right)\left[\frac{d}{dt}\|u_m(t)\|^2 - 2\left(\upsilon'_m(t), \gamma_0\left(u'_m(t)\right)\right)_\Gamma\right] +$$

$$+ 2\frac{d}{dt}\int_\Omega \overline{\Phi}(u_m(t))dx + 2\beta\left|u'_m(t)\right|^2 = 0,$$

where $\overline{\Phi}(s) = \displaystyle\int_0^s \Phi(\sigma)d\sigma$, and

$$(19)\qquad \rho\left(\gamma_0\left(u'_m(t)\right), \upsilon'(t)\right)_\Gamma + \frac{d}{dt}\left|f^{1/2}\upsilon'_m(t)\right|_\Gamma^2 + \frac{d}{dt}\left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2 + 2\left|g^{1/2}\upsilon'_m(t)\right|_\Gamma^2 = 0.$$

A straightforward computation shows

$$\frac{d}{dt}\left[M\left(|u_m(t)|^2\right)\left(\|u_m(t)\|^2 + \left|f^{1/2}\upsilon'_m(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right)\right]$$

$$= M\left(|u_m(t)|^2\right)\frac{d}{dt}\left(\|u_m(t)\|^2 + \left|f^{1/2}\upsilon'_m(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right) +$$

$$+ 2M'\left(|u_m(t)|^2\right)\left(u'_m(t), u_m(t)\right)\left(\|u_m(t)\|^2 + \left|f^{1/2}\upsilon'_m(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right).$$

Multiplying (18) by ρ, (19) by $M\left(|u_m(t)|^2\right)$ and adding the resulting expressions, yields

$$\frac{d}{dt}\left[\rho\left|u_m'(t)\right|^2+\rho M\left(|u_m(t)|^2\right)\|u_m(t)\|^2+2\rho\int_\Omega\Phi(u_m(t))dx+M\left(|u_m(t)|^2\right)\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2\right.$$
$$\left.+M\left(|u_m(t)|^2\right)\left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right]+2\rho\beta\left|u_m'(t)\right|^2+2M\left(|u_m(t)|^2\right)\left|g^{1/2}\upsilon_m^r(t)\right|_\Gamma^2$$
$$\leq I_{1,m}(t)+I_{2,m}(t), \tag{20}$$

where

$$I_{1,m}(t)=2\rho M'\left(|u_m(t)|^2\right)\left|u_m'(t)\right||u_m(t)|\,\|u_m(t)\|^2,$$

and

$$I_{2,m}(t)=2M'\left(|u_m(t)|^2\right)\left|u_m'(t)\|u_m(t)\right|\left[\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2+\left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right].$$

On the other hand, setting $w=u_m$ in $(17)_1$ allows us to deduce

$$\frac{d}{dt}\left(u_m'(t),u_m(t)\right)-\left|u_m'(t)\right|^2+M\left(|u_m(t)|^2\right)\|u_m(t)\|^2+\frac{\beta}{2}\frac{d}{dt}|u_m(t)|^2\leq I_{3,m}(t), \tag{21}$$

where

$$I_{3,m}(t)=C_0M\left(|u_m(t)|^2\right)\left|\upsilon'(t)\right|_\Gamma\|u_m(t)\|-\int_\Omega\Phi(u_m(t))u_m(t)dx.$$

The above $C_0>0$ is defined in (6). Multiplying (21) by $(\rho\beta)/2$ and adding the resulting expression with (20), we get

$$\begin{aligned}&\frac{d}{dt}\left[\rho\left|u_m'(t)\right|^2+\frac{\rho\beta}{2}\left(u_m'(t),u_m(t)\right)+\frac{\rho\beta^2}{4}|u_m(t)|^2+\rho M\left(|u_m(t)|^2\right)\|u_m(t)\|^2+\right.\\&\left.2\rho\int_\Omega\Phi(u_m(t))dx+M\left(|u_m(t)|^2\right)\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2+M\left(|u_m(t)|^2\right)\left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right]+\\&\frac{\rho\beta}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2+\frac{3\rho\beta}{2}\left|u_m'(t)\right|^2+2M\left(|u_m(t)|^2\right)\left|g^{1/2}\upsilon_m'(t)\right|_\Gamma^2\leq\\&\leq I_{1,m}(t)+I_{2,m}(t)+\frac{\rho\beta}{2}I_{3,m}(t).\end{aligned} \tag{22}$$

Next, we upper bound the terms on the right-hand-side of (22). In fact, if C_0, C_1 are the positive real constants defined by (6) and (7) (with $p=1$), then

$$I_{1,m}(t)\leq 2\rho C_1M'\left(|u_m(t)|^2\right)\left|u_m'(t)\right|\|u_m(t)\|^3,$$

$$I_{2,m}(t)\leq\frac{\rho\beta}{2}\left|u_m'(t)\right|^2+\frac{2C_1^2}{\rho\beta}\left(M'\left(|u_m(t)|^2\right)\right)^2\|u_m(t)\|^2\left[\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2+\left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right]^2,$$

and

$$\frac{\rho\beta}{2} I_{3,m}(t) \leq \frac{\rho\beta}{2}\left[\frac{1}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2 + \frac{C_0^2}{2}M\left(|u_m(t)|^2\right)\left|\upsilon'(t)\right|^2 + C_p^{p+1}\|u_m(t)\|^{p+1}\right]$$

$$\leq \frac{\rho\beta}{2}\left[\frac{1}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2 + \frac{1}{2}M\left(|u_m(t)|^2\right)\left|g^{1/2}\upsilon'(t)\right|^2 + C_p^{p+1}\|u_m(t)\|^{p+1}\right].$$

Inserting the last inequalities in (22) and using M's growth condition(9), we obtain

$$\frac{d}{dt}\left[\Lambda_m(t) + \frac{\rho}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2 + 2\rho\int_\Omega \Phi(u_m(t))dx\right] + \rho\beta\left|u_m'(t)\right|^2 \tag{23}$$

$$+\frac{3}{2}M\left(|u_m(t)|^2\right)\left|g^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \|u_m(t)\|^2\left[\frac{\rho\beta r_0}{4} - K_m(t)\right] \leq 0,$$

where

$$\Lambda_m(t) = \rho\left|u_m'(t)\right|^2 + \frac{\rho\beta}{2}\left(u_m'(t), u_m(t)\right) + \frac{\rho\beta^2}{4}|u_m(t)|^2 + \frac{\rho}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2 \tag{24}$$

$$+ M\left(|u_m(t)|^2\right)\left(\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right)$$

and

$$K_m(t) = \frac{2C_1^2}{\rho\beta}\left(M'\left(|u_m(t)|^2\right)\right)^2\left(\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right)^2 + \frac{\rho\beta C_p^{p+1}}{2}\|u_m(t)\|^{p-1} \tag{25}$$

$$+ 2C_1\rho\left|M'\left(|u_m(t)|^2\right)\right|\left|u_m'(t)\right|\|u_m(t)\|.$$

Lemma 3.1. *Let $P:\mathbb{R}\to\mathbb{R}$ be defined by $P(s) = \frac{\rho r_0}{2}s^2 - 2\rho C_p^{p+1}s^{p+1}$.
Then,*

$$W_m(t) = \frac{\rho}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2 + 2\rho\int_\Omega \overline{\Phi}(u_m(t))dx \geq P\left(\|u_m(t)\|\right).$$

Proof. From (10), we have

$$\left|2\rho\int_\Omega \overline{\Phi}(u_m(t))dx\right|_\mathbb{R} \leq 2\rho\int_\Omega |u_m(t)|^{p+1}dx \leq 2\rho C_p^{p+1}\|u_m(t)\|^{p+1},$$

where C_p is defined in (7). Thus,

$$2\rho\int_\Omega \overline{\Phi}(u_m(t))dx \geq -2\rho C_p^{p+1}\|u_m(t)\|^{p+1}.$$

From (9), we get

$$\frac{\rho}{2}M\left(|u_m(t)|^2\right)\|u_m(t)\|^2 \geq \frac{\rho r_0}{2}\|u_m(t)\|^2.$$

Therefore, from the two previous inequalities the lemma is proved. □

Remark 3.1. *We have that* $s_1 = 0$ *and* $s_2 = (r_0/2C_p^{p+1})^{1/p-1}$ *are critical points of* P. *Morever,* $P'(s) > 0$ *for all* s *in the open interval* (s_1, s_2).

Lemma 3.2. *For all* $m \in \mathbb{N}$, *the function* Λ_m *defined in* (24) *satisfies*

$$\text{(26)}\quad \Lambda_m(t) \geq \frac{\rho}{2}\left|u_m'(t)\right|^2 + \frac{\rho\beta^2}{8}\left|u_m(t)\right|^2 + \frac{\rho r_0}{2}\left\|u_m(t)\right\|^2 + r_0\left(\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right)$$

and

$$\begin{aligned}\Lambda_m(t) \leq {} & \frac{3\rho}{2}\left|u_m'(t)\right|^2 + \frac{3\rho}{8}\left|u_m(t)\right|^2 + \frac{\rho}{2}M\left(\left|u_m(t)\right|^2\right)\left\|u_m(t)\right\|^2 \\ & + M\left(\left|u_m(t)\right|^2\right)\left(\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2\right).\end{aligned} \tag{27}$$

Proof. From Cauchy-Schwartz inequality, we obtain

$$\left|\frac{\rho\beta}{2}\left(u_m'(t), u_m(t)\right)\right|_{\mathbb{R}} \leq \frac{\rho}{2}\left|u_m'(t)\right|^2 + \frac{\rho\beta^2}{8}\left|u_m(t)\right|^2.$$

Thus,

$$\frac{\rho\beta}{2}\left(u_m'(t), u_m(t)\right) \geq -\frac{\rho}{2}\left|u_m'(t)\right|^2 - \frac{\rho\beta^2}{8}\left|u_m(t)\right|^2. \tag{28}$$

From (24) and (28), we easily get (26). Moreover, (27) is deduced from (24). □

Lemma 3.3. *For all* $m \in \mathbb{N}$, *the function* K_m *defined in* (25) *satisfies the inequality*

$$\begin{aligned}K_m(t) \leq {} & \frac{2C_1^2}{\rho\beta r_0^2}\left(M'\left(\left|u_m(t)\right|^2\right)\right)^2\left[\Lambda_m(t)\right]^2 + \frac{4C_1}{\sqrt{r_0}}\left|M'\left(\left|u_m(t)\right|^2\right)\right|\Lambda_m(t) \\ & + \frac{\rho\beta}{2}\left(\frac{2}{\rho r_0}\right)^{(p-1)/2} C_p^{p+1}\left[\Lambda_m(t)\right]^{(p-1)/2}.\end{aligned} \tag{29}$$

Proof. From Lemma 3.2, we obtain

$$\left|u_m'(t)\right| \leq \sqrt{\frac{2}{\rho}}\sqrt{\Lambda_m(t)},$$

$$\left\|u_m(t)\right\| \leq \sqrt{\frac{2}{\rho r_0}}\sqrt{\Lambda_m(t)}$$

and

$$\left|f^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m(t)\right|_\Gamma^2 \leq \frac{1}{r_0}\Lambda_m(t).$$

Inserting these three inequalities into (25) we obtain (29). This proves the lemma. □

Remark 3.2. *In particular, taking* $t = 0$ *in* (27), *we have*

$$\Lambda_m(0) + W_m(0) \leq \Lambda_m(0) + \frac{\rho}{2}M\left(\left|u_{0m}\right|^2\right)\left\|u_{0m}\right\|^2 + 2\rho C_p^{p+1}\left\|u_{0m}\right\|^{p+1} \leq \Lambda_0,$$

where C_p *and* Λ_0 *are positive constants defined in* (7) *and* (11), *respectively.*

REMARK 3.3. *If $K_m(t) < (\rho\beta r_0)/8$, $t \geq 0$ then we have*

$$\frac{\rho\beta C_p^{p+1}}{2}\left\|u_m(t)\right\|^{p-1} \leq \frac{\rho\beta r_0}{8}, \quad t \geq 0$$

and this implies $\|u_m(t)\| \leq (r_0/4C_p^{p+1})^{1/p-1}$, $t \geq 0$ Thus, from Lemma 3.1 and Remark 3.1 we get

$$W_m(t) \geq P\left(\|u_m(t)\|\right) \geq 0, \quad t \geq 0.$$

Now, we prove that the hypothesis assumed in the Remark 3.3 is true for all $t \geq 0$.

LEMMA 3.4. *For all $m \in \mathbb{N}$, the function K_m defined in (25) satisfies the inequality*

$$K_m(t) < \frac{\rho\beta r_0}{8}, \quad \textit{for all } t \geq 0.$$

Proof. We use a contradiction argument. There exist $m \in \mathbb{N}$ and $t_0 > 0$ such that

$$K_m(t_0) \geq (\rho\beta r_0)/8.$$

By applying Lemma 3.3 at $t = 0$, combined with condition (12) we deduce

$$K_m(0) < (\rho\beta r_0)/8.$$

From continuity of K_m, we can see that there exists $t^* > 0$ such that

$$K_m(t) < \frac{\rho\beta r_0}{8} \quad \text{for all } 0 < t < t^*, \text{ and } K_m\left(t^*\right) = \frac{\rho\beta r_0}{8}. \tag{30}$$

Integrating (23) from 0 to t^*, using (30) and Remark 3.2, we get

$$\Lambda_m\left(t^*\right) + W_m\left(t^*\right) \leq \Lambda_m(0) + W_m(0) \leq \Lambda_0. \tag{31}$$

On the other hand, as $K_m\left(t^*\right) = (\rho\beta r_0)/8$, from Remark 3.3, we obtain that

$$W_m(t) \geq P\left(\|u_m(t)\|\right) \geq 0. \tag{32}$$

Thus, from (31)-(32), we deduce $\Lambda_m\left(t^*\right) \leq \Lambda_0$. Therefore, combining Lemma 3.3 with (12), gives

$$K_m(t^*) \leq \frac{2C_1^2}{\rho\beta r_0^2}\left(M'\left(|u_0(t)|^2\right)\right)^2[\Lambda_0]^2 + \frac{4C_1}{\sqrt{r_0}}\left|M'\left(|u_0|^2\right)\right|\Lambda_0$$

$$+\frac{\rho\beta}{2}\left(\frac{2}{\rho r_0}\right)^{(p-1)/2} C_p^{p+1}\left[\Lambda_0\right]^{(p-1)/2} < \frac{\rho\beta r_0}{8}.$$

This inequality contradicts (30). So the lemma is proved. $\square$

Thus, combining inequalitites from Lemma 3.2 and Lemma 3.4, and using Remark 3.3, we have

$$\begin{aligned}&\frac{\rho}{2}\left|u_m'(t)\right|^2 + \frac{\rho\beta^2}{8}\left|u_m(t)\right|^2 + \frac{\rho r_0}{2}\left\|u_m(t)\right\|^2 + r_0\left(f_1\left|v_m'(t)\right|_\Gamma^2 + h_1\left|v_m(t)\right|_\Gamma^2\right) + \\ &\rho\beta\int_0^t\left|u_m'(s)\right|^2 ds + \frac{3r_0 g_1}{2}\int_0^t\left|v_m'(s)\right|_\Gamma^2 ds \leq \Lambda_0.\end{aligned} \tag{33}$$

Remark 3.4. *Using Poincaré inequality with* (33), *we have*

$$|u_m(t)|^2 \in [0, C], \quad m \in \mathbb{N}, \quad t \geq 0.$$

for some $C > 0$. *Using the fact that* $M \in C^1([0,\infty))$ *with* (9), *we immediately assert that*

$$r_0 \leq M\left(|u_m(t)|^2\right) \leq r_3, \quad m \in \mathbb{N}, \quad t \geq 0$$

for some r_0 *and* r_3.

Estimate II - For all $m \in \mathbb{N}$, there exists $C > 0$, independent of m, such that

$$\left|u_m''(0)\right| \leq C, \quad \text{and} \quad \left|\upsilon_m''(0)\right| \leq C. \tag{34}$$

Taking $t = 0$ and considerig $w = u_m''(0)$ and $z = \upsilon_m''(0)$ in (17), we have

$$\begin{aligned} \left|u_m''(0)\right|^2 &\leq \left[M\left(|u_{0m}|^2\right)|\upsilon u_{0m}| + C_p^p \|u_{0m}\|^p + \beta |u_{1m}|\right]\left|u_m''(0)\right| \\ &\leq \frac{1}{2}\left[M\left(|u_0|^2\right)|\Delta u_0| + C_p^p \|u_0\|^p + \beta |u_1|\right]^2 + \frac{1}{2}\left|u_m''(0)\right|^2, \end{aligned} \tag{35}$$

and

$$\begin{aligned} f_1\left|\upsilon_m''(0)\right|^2 &\leq \left[g_2 C_4 \|u_{0m}\|_{H^2(\Omega)} + h_2|\upsilon_{0m}| + \rho C_0\|u_{1m}\|\right]\left|\upsilon_m''(0)\right| \\ &\leq \frac{1}{2f_1}\left[g_2 C_4 \|u_0\|_{H^2(\Omega)} + h_2|\upsilon_0| + \rho C_0\|u_1\|\right]^2 + \frac{f_1}{2}\left|\upsilon_m''(0)\right|^2, \end{aligned} \tag{36}$$

where $g_2 = \max_{x\in\Gamma}|g(x)|_{\mathbb{R}}$ and $h_2 = \max_{x\in\Gamma}|h(x)|_{\mathbb{R}}$. Therefore, from (35) and (36), we get (34).

Estimate III - Dividing $(1)_1$ by $M(|u_m(t)|)$ and differentiating the resulting expression with respect to the time variable t, we get

$$\begin{aligned} &\frac{(u_m'''(t), w)}{M\left(|u_m(t)|^2\right)} - 2I_{4,m}(t)\left(u_m''(t), w\right) + \left(\nabla u_m'(t), \nabla w\right) - \\ &(\upsilon_m''(t), \gamma_0(w))_\Gamma + \frac{(u_m'(t)\Phi'(u_m(t)), w)}{M\left(|u_m(t)|^2\right)} - 2I_{4,m}(t)\left(\Phi(u_m(t)), w\right) + \\ &\frac{\beta\left(u_m''(t), w\right)}{M\left(|u_m(t)|^2\right)} - 2\beta I_{4,m}(t)\left(u_m'(t), w\right) = 0, \end{aligned} \tag{37}$$

with

$$I_{4,m}(t) = \frac{M'\left(|u_m(t)|^2\right)\left(u_m'(t), u_m(t)\right)}{\left[M\left(|u_m(t)|^2\right)\right]^2}.$$

A straightforward computation shows

$$\frac{(u_m'''(t), u_m''(t))}{M\left(|u_m(t)|^2\right)} = \frac{1}{2}\frac{d}{dt}\left[\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}\right] + I_{4,m}(t)\left|u_m''(t)\right|^2,$$

and considering $w = \rho u_m''(t)$ in (37), we obtain

$$\frac{\rho}{2}\frac{d}{dt}\left[\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}\right] + \frac{\rho}{2}\frac{d}{dt}\left\|u_m'(t)\right\|^2 + \frac{\rho\beta\,|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)} - \rho\left(\upsilon_m''(t), \gamma_0(u_m''(t))\right)_\Gamma =$$
$$\rho I_{4,m}(t)\left|u_m''(t)\right|^2 - 2\rho I_{4,m}(t)\left(\Phi(u_m(t)), u_m''(t)\right) + 2\rho\beta I_{4,m}(t)\left(u_m'(t), u_m''(t)\right)$$
$$- \frac{\rho\left(u_m'(t)\Phi'(u_m(t)), u_m''(t)\right)}{M\left(|u_m(t)|^2\right)}. \tag{38}$$

Now, differentiating $(1)_2$ with respect to t and considering $z = \upsilon_m''(t)$, we have

$$\rho\left(\upsilon_m''(t), \gamma_0(u_m''(t))\right)_\Gamma + \frac{1}{2}\frac{d}{dt}\left|f^{1/2}\upsilon_m''(t)\right|_\Gamma^2 + \frac{1}{2}\frac{d}{dt}\left|h^{1/2}\upsilon_m'(t)\right|_\Gamma^2 + \left|g^{1/2}\upsilon_m''(t)\right|_\Gamma^2 = 0. \tag{39}$$

Adding (38) and (39), result in

$$\frac{1}{2}\frac{d}{dt}\left[\frac{\rho\,|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)} + \rho\left\|u_m'(t)\right\|^2 + \left|f^{1/2}\upsilon_m''(t)\right|_\Gamma^2 + \left|h^{1/2}\upsilon_m'(t)\right|_\Gamma^2\right] +$$
$$+ \frac{\rho\beta\,|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)} + \left|g^{1/2}\upsilon_m''(t)\right|_\Gamma^2 = \rho I_{4,m}(t)\left|u_m''(t)\right|^2 + 2\rho I_{4,m}(t)\left(\Phi(u_m(t)), u_m''(t)\right)$$
$$+ 2\rho\beta I_{4,m}(t)\left(u_m'(t), u_m''(t)\right) - \frac{\rho\left(u_m'(t)\Phi'(u_m(t)), u_m''(t)\right)}{M\left(|u_m(t)|^2\right)}. \tag{40}$$

From (9)-(10) and (33), we have

$$I_{4,m}(t)\left|u_m''(t)\right|^2 \le r_1\left|u_m'(t)\right|\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)} \le r_1\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}, \tag{41}$$

$$I_{4,m}(t)\left(\Phi(u_m(t)), u_m''(t)\right) \le r_1\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\frac{\|u_m(t)\|_{L^{2p}(\Omega)}^p\,|u_m''(t)|}{M\left(|u_m(t)|^2\right)}$$
$$\le r_1\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\frac{C_{2p}^{2p}\left\|u_m(t)\right\|^{2p} + |u_m''(t)|^2}{2M\left(|u_m(t)|^2\right)}$$
$$\le C_{2p}^{2p}\frac{r_1}{2r_0}\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\left(\frac{2}{\rho r_0}\Lambda_0\right)^p + \frac{r_1}{2}\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)},$$

and

$$I_{4,m}(t)\left(u_m'(t),u_m''(t)\right)\leq r_1\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\frac{|u_m'(t)|^2+|u_m''(t)|^2}{2M\left(|u_m(t)|^2\right)}$$

$$\leq C_1^2\frac{r_1}{2r_0}\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\left\|u_m'(t)\right\|^2+\frac{r_1}{2}\sqrt{\frac{2}{\rho}}\sqrt{\Lambda_0}\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}.$$

Furthermore, for $q_n = 2n/n-2$, we get $\frac{1}{n}+\frac{1}{q_n}+\frac{1}{2}=1$. Therefore, from (10), we have that

$$(42)\quad \frac{(u_m'(t)\Phi'(u_m(t)),u_m''(t))}{M\left(|u_m(t)|^2\right)}\leq\frac{\|u_m(t)\|_{L^{n(p-1)}(\Omega)}^{p-1}\|u_m'(t)\|_{L^{q_n}(\Omega)}|u_m''(t)|}{M\left(|u_m(t)|^2\right)}$$

$$\leq\frac{C_{n,p-1}^{p-1}\|u_m(t)\|^{p-1}C_{q_n}\|u_m'(t)\|\,|u_m''(t)|}{M\left(|u_m(t)|^2\right)}$$

$$\leq C_{n,p-1}^{p-1}\frac{C_{q_n}}{2}\left(\frac{2}{\rho r_0}\Lambda_0\right)^{(p-1)/2}\left[\frac{\|u_m'(t)\|^2}{r_0}+\frac{|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}\right].$$

Inserting (41)-(42) into (40), we obtain a couple of positive constants C_5 and C_6, such that

$$(43)\quad \frac{1}{2}\frac{d}{dt}\left[\frac{\rho\,|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}+\rho\left\|u_m'(t)\right\|^2+\left|f^{1/2}\upsilon_m''(t)\right|_\Gamma^2+\left|h^{1/2}\upsilon_m'(t)\right|_\Gamma^2\right]+$$

$$+\frac{\rho\beta\,|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}+\left|g^{1/2}\upsilon_m''(t)\right|_\Gamma^2\leq C_5+\frac{C_6}{2}\left[\frac{\rho\,|u_m''(t)|^2}{M\left(|u_m(t)|^2\right)}+\rho\left\|u_m'(t)\right\|^2\right].$$

Now integrating (43) from 0 to t we get

$$\frac{1}{2}\left[\frac{\rho\left|u_m''(t)\right|^2}{M\left(\left|u_m(t)\right|^2\right)}+\rho\left\|u_m'(t)\right\|^2+\left|f^{1/2}\upsilon_m''(t)\right|_\Gamma^2+\left|h^{1/2}\upsilon_m'(t)\right|_\Gamma^2\right] \tag{44}$$

$$+\int_0^t\left[\frac{\rho\beta\left|u_m''(s)\right|^2}{M\left(\left|u_m(s)\right|^2\right)}+\left|g^{1/2}\upsilon_m''(s)\right|_\Gamma^2\right]ds$$

$$\leq\frac{\rho}{2}\left[\frac{\left|u_m''(0)\right|^2}{M\left(\left|u_0\right|^2\right)}+\left\|u_{1m}\right\|^2\right]+\frac{1}{2}\left[f_2\left|\upsilon_m''(0)\right|_\Gamma^2+h_2\left|\frac{\partial u_{0m}}{\partial\eta}\right|_\Gamma^2\right]$$

$$+C_7\left(1+\int_0^t\left[\frac{\left|u_m''(s)\right|^2}{M\left(\left|u_m(s)\right|^2\right)}+\left\|u_m'(s)\right\|^2\right]ds\right),$$

where C_7 is a positive constant depending only on T.

On the other hand, from (9), $(17)_4$ and the estimate (34), we deduce that there exists $C_8>0$ (independent of m) such that

$$\frac{\rho}{2}\left[\frac{\left|u_m''(0)\right|^2}{M\left(\left|u_0\right|^2\right)}+\left\|u_{1m}\right\|^2\right]+\frac{1}{2}\left[f_2\left|\upsilon_m''(0)\right|_\Gamma^2+h_2\left|\frac{\partial u_{0m}}{\partial\eta}\right|_\Gamma^2\right]\leq C_8. \tag{45}$$

Therefore, inserting (45) into (44) and using the Gronwall inequality, yields

$$\frac{1}{2}\left[\frac{\rho\left|u_m''(t)\right|^2}{M\left(\left|u_m(t)\right|^2\right)}+\rho\left\|u_m'(t)\right\|^2+\left|f^{1/2}\upsilon_m''(t)\right|_\Gamma^2+\left|h^{1/2}\upsilon_m'(t)\right|_\Gamma^2\right] \tag{46}$$

$$+\int_0^t\left[\frac{\rho\beta\left|u_m''(s)\right|^2}{M\left(\left|u_m(s)\right|^2\right)}+\left|g^{1/2}\upsilon_m''(s)\right|_\Gamma^2\right]ds\leq C_9,$$

where $C_9=C_9\left(u_0,u_1,\upsilon_0,T\right)$ with $T>0$ fixed.

Thus, combining Remark 3.4, (8) and (46), we deduce the following estimate

$$\left|u_m''(t)\right|^2+\left\|u_m'(t)\right\|^2+f_1\left|\upsilon_m''(t)\right|_\Gamma^2+h_1\left|\upsilon_m'(t)\right|_\Gamma^2+\int_0^t\left[\left|u_m''(s)\right|^2+g_1\left|\upsilon_m''(s)\right|_\Gamma^2\right]ds\leq C_{10} \tag{47}$$

for some constant $C_{10}>0$.

Passage to Limit. Now we can prove the main result, namely, Theorem 2.1. Indeed, let $T > 0$ be fixed and arbitrary. As an immediate consequence of estimates (33) and (47), we get

$$\left| \begin{array}{lll} (u_m) & \text{bounded in} & L^\infty(0,T;V) \\ (u'_m) & \text{bounded in} & L^\infty(0,T;V) \cap L^2\left(0,T;L^2(\Omega)\right) \\ (u''_m) & \text{bounded in} & L^2\left(0,T;L^2(\Omega)\right) \\ (\upsilon_m), (\upsilon'_m), (\upsilon''_m) & \text{bounded in} & L^\infty\left(0,T;L^2(\Gamma)\right). \end{array} \right.$$

These facts allow us to deduce that there exist subsequences $(u_\mu)_{\mu\in\mathbb{N}}$ and $(\upsilon_\mu)_{\mu\in\mathbb{N}}$ of the sequences of approximations $(u_m)_{m\in\mathbb{N}}$ and $(\upsilon_m)_{m\in N}$ respectively, such that

$$(48) \qquad \left| \begin{array}{lll} u_\mu \overset{*}{\rightharpoonup} u & \text{in} & L^\infty(0,T;V); \\ u'_\mu \overset{*}{\rightharpoonup} u' & \text{in} & L^\infty(0,T;V); \\ u'_\mu \rightharpoonup u' & \text{in} & L^2\left(0,T;L^2(\Omega)\right); \\ u''_\mu \rightharpoonup u'' & \text{in} & L^2\left(0,T;L^2(\Omega)\right); \\ \upsilon_\mu \overset{*}{\rightharpoonup} \upsilon & \text{in} & L^2\left(0,T;L^2(\Gamma)\right); \\ \upsilon'_\mu \overset{*}{\rightharpoonup} \upsilon' & \text{in} & L^2\left(0,T;L^2(\Gamma)\right); \\ \upsilon''_\mu \overset{*}{\rightharpoonup} \upsilon'' & \text{in} & L^2\left(0,T;L^2(\Gamma)\right). \end{array} \right.$$

Since $V \overset{c}{\hookrightarrow} L^2(\Omega)$, the estimates $(48)_1$-$(48)_2$ and Aubin-Lions' Theorem provides us

$$u_\mu \to u \quad \text{a.e} \quad \text{in } Q = \Omega \times (0,T),$$

and, as $\Phi \in C^1(\mathbb{R})$, we obtains

$$(49) \qquad \Phi(u_\mu) \to \Phi(u) \quad \text{a.e} \quad \text{in } Q = \Omega \times (0,T).$$

From (7), (10) and (33), we get for some positive $C > 0$

$$(50) \qquad \int_\Omega |\Phi(u_\mu)|^2 dx \le \int_\Omega |u_\mu|^{2p} dx \le C_{2p}^{2p} \|u_\mu\|^{2p} \le C$$

Then, from (49)-(50) and using Lions' Lema yield

$$\Phi(u_\mu) \overset{*}{\rightharpoonup} \Phi(u) \quad \text{in } L^\infty(0,T;L^2(\Omega)).$$

The proof of the convergence for the other nonlinear term of (1) is quite similar to the ideas developed in [26]. Therefore it will be omitted. Combining all the above facts, we finally conclude that (u,υ) is a solution to (1).

3.1.2 Uniqueness of solutions

To prove the uniqueness of solutions, we rewrite (1) as

$$\left| \begin{array}{l} \displaystyle\int_\Omega \left[\frac{u''}{M\left(|u|^2\right)}\xi + \nabla u \nabla \xi + \frac{\Phi(u)}{M\left(|u|^2\right)}\xi + \beta\frac{u'}{M\left(|u|^2\right)}\xi\right] dx = \int_\Gamma \upsilon' \gamma_0(\xi) dx, \\ \displaystyle\int_\Gamma \left[\rho\gamma_0\left(u'\right)\psi + f\upsilon''\psi + g\upsilon'\psi + h\upsilon\psi\right] dx = 0, \end{array} \right.$$

for all $\xi \in L^2(0,T;V)$ and for all $\psi \in L^2\left(0,T;L^2(\Gamma)\right)$.

Let (u_1,v_1) and (u_2,v_2) be two regular global weak solutions to (1). Then, for all $T>0$ fixed, we have that $w=u_1-u_2$, $\theta=v_1-v_2$, $\xi=\rho w'$ and $\psi=\theta'$ satisfy

$$
(51)\qquad \left|\begin{array}{l}
\rho\displaystyle\int_\Omega \left(\frac{u_1''}{M\left(|u_1|^2\right)}-\frac{u_2''}{M\left(|u_2|^2\right)}\right)w'dx+\frac{\rho}{2}\frac{d}{dt}|\nabla w|^2dx-\rho\int_\Gamma \gamma_0\left(w'\right)\theta' dx+\\[2ex]
\rho\displaystyle\int_\Omega \left(\frac{\Phi(u_1)}{M\left(|u_1|^2\right)}-\frac{\Phi(u_2)}{M\left(|u_2|^2\right)}\right)w'dx+\beta\rho\int_\Omega \left(\frac{u_1'}{M\left(|u_1|^2\right)}-\frac{u_2'}{M\left(|u_2|^2\right)}\right)w'dx=0,\\[2ex]
\rho\displaystyle\int_\Gamma \gamma_0\left(w'\right)\theta' dx+\frac{1}{2}\frac{d}{dt}\left|f^{1/2}\theta'\right|_\Gamma^2+\frac{1}{2}\frac{d}{dt}\left|h^{1/2}\theta\right|_\Gamma^2+\left|g^{1/2}\theta'\right|_\Gamma^2=0,\\[2ex]
w(x,0)=0;\quad w'(x,0)=0;\quad \theta(x,0)=0\quad \text{and}\quad \theta'(x,0)=0.
\end{array}\right.
$$

Adding $(51)_1$ with $(51)_2$, integrating over $(0,t)$ the resulting identity, and using the third equation of (51), we obtain

$$
(52)\qquad \begin{aligned}
&\frac{\rho}{2}\|w\|^2+\frac{1}{2}\left|f^{1/2}\theta'\right|_\Gamma^2+\frac{1}{2}\left|h^{1/2}\theta\right|_\Gamma^2+\int_0^t\left|g^{1/2}\theta'\right|_\Gamma^2dt=\\
&-\rho\int_0^t\int_\Omega\left[\frac{w''}{M\left(|u_1|^2\right)}w'+u_2''\left(\frac{1}{M\left(|u_1|^2\right)}-\frac{1}{M\left(|u_2|^2\right)}\right)w'\right]dxdt\\
&-\rho\int_0^t\int_\Omega\left[\frac{\Phi(u_1)-\Phi(u_2)}{M\left(|u_1|^2\right)}w'+\Phi(u_2)\left(\frac{1}{M\left(|u_1|^2\right)}-\frac{1}{M\left(|u_2|^2\right)}\right)w'\right]dxdt\\
&-\beta\rho\int_0^t\int_\Omega\left[\frac{[w']^2}{M\left(|u_1|^2\right)}+u_2'\left(\frac{1}{M\left(|u_1|^2\right)}-\frac{1}{M\left(|u_2|^2\right)}\right)w'\right]dxdt.
\end{aligned}
$$

Now, we upper bound each term on the right-hand side of (52). In the upcoming computations, we denote a generic constant that can change line by line by $C>0$. It only depends on a priori assumptions. Let us start with the first term

$$
(53)\qquad \begin{aligned}
&-\rho\int_0^t\int_\Omega\frac{w''}{M\left(|u_1|^2\right)}w'dxdt\\
&=-\frac{\rho}{2}\int_0^t\int_\Omega\frac{1}{M\left(|u_1|^2\right)}\frac{d}{dt}\left|w'\right|_{\mathbb{R}}^2dxdt\\
&=-\frac{\rho}{2}\int_0^t\frac{d}{dt}\left(\frac{|w'|^2}{M\left(|u_1|^2\right)}\right)dt+\frac{\rho}{2}\int_0^t\left|w'\right|^2M'\left(|u_1|^2\right)\left(\frac{(u_1,u_1')}{\left[M\left(|u_1|^2\right)\right]^2}\right)dt\\
&=-\frac{\rho}{2}\frac{|w'|^2}{M\left(|u_1|^2\right)}+\frac{\rho}{2}\int_0^t\left|w'\right|^2M'\left(|u_1|^2\right)\left(\frac{(u_1,u_1')}{\left[M\left(|u_1|^2\right)\right]^2}\right)dt\\
&\leq-\frac{\rho}{2}\frac{|w'|^2}{M\left(|u_1|^2\right)}+\frac{r_1}{r_0}\int_0^T\left|u_1'\right|\left|w'\right|^2dt\leq-\frac{\rho}{2}\frac{|w'|^2}{M\left(|u_1|^2\right)}+C\int_0^T\left|w'\right|^2dt.
\end{aligned}
$$

On the other hand, using condition (9), we easily deduces

$$\left|M\left(|u_2|^2\right)-M\left(|u_1|^2\right)\right|=\left|\int_{|u_1|^2}^{|u_2|^2} M'(s)ds\right|\leq \left\|M'\right\|_{L^\infty}\left(|u_1|+|u_2|\right)|w|.$$

For the second term of (52), we have the following estimate:

$$\begin{aligned}
&-\rho\int_0^t\int_\Omega u_2''\left(\frac{1}{M\left(|u_1|^2\right)}-\frac{1}{M\left(|u_2|^2\right)}\right)w'dxdt\\
&\leq C\int_0^T\left[\left(|u_2|+|u_1|\right)|w|\int_\Omega \left|u_2''w'\right|_{\mathbb{R}}dx\right]dt\leq C\int_0^T\|w\||w'|dt.
\end{aligned}$$

For the third term of (52), we shall use the following fact. For all nonnegative constants α,β with $\alpha<\beta$, there exists $z\in(\alpha,\beta)$ such that

$$\Phi(\alpha)-\Phi(\beta)=\Phi'(z)(\alpha-\beta).$$

Hence, using conditions declared in (10), we get

$$\begin{aligned}
\left|-\rho\int_Q\frac{\Phi(u_1)-\Phi(u_2)}{M\left(|u_1|^2\right)}w'dxdt\right|_{\mathbb{R}}&=\left|\rho\int_Q\frac{\Phi'(z)w}{M\left(|u_1|^2\right)}w'dxdt\right|_{\mathbb{R}}\\
&\leq\frac{\rho}{r_0}\int_0^T\int_\Omega\left(|u_1|_{\mathbb{R}}^{p-1}+|u_2|_{\mathbb{R}}^{p-1}\right)|w|_{\mathbb{R}}|w'|_{\mathbb{R}}dxdt\\
&\leq\frac{C\rho}{r_0}\int_0^T\left(\|u_1\|_{L^{n(p-1)}(\Omega)}^{p-1}+\|u_2\|_{L^{n(p-1)}(\Omega)}^{p-1}\right)\|w\||w'|dt\\
&\leq C\int_0^T\|w\||w'|dt.
\end{aligned}$$

The remainder terms of (52) can be estimated similarly. Thus, we have

$$\begin{aligned}
&-\rho\int_0^t\int_\Omega\Phi(u_2)\left(\frac{1}{M\left(|u_1|^2\right)}-\frac{1}{M\left(|u_2|^2\right)}\right)w'dxdt\\
&\leq C\int_0^T\left[\left(|u_2|+|u_1|\right)|w|\int_\Omega|u_2|_{\mathbb{R}}^p\left|w'\right|_{\mathbb{R}}dx\right]dt\leq C\int_0^T\|w\||w'|dt,
\end{aligned}$$

and

$$\begin{aligned}
&-\beta\rho\int_0^t\int_\Omega\left[\frac{[w']^2}{M\left(|u_1|^2\right)}+u_2'\left(\frac{1}{M\left(|u_1|^2\right)}-\frac{1}{M\left(|u_2|^2\right)}\right)w'\right]dxdt\\
&\leq C\int_0^T|w'|^2dt+C\int_0^T\left[\left(|u_2|+|u_1|\right)|w|\int_\Omega\left|u_2'\right|_{\mathbb{R}}\left|w'\right|_{\mathbb{R}}dx\right]dt\\
&\leq C\int_0^T|w'|^2dt+C\int_0^T\|w\||w'|dt.
\end{aligned}\tag{54}$$

Plugging (53)-(54) into (52), we get

$$|w'|^2+\|w\|^2+\frac{1}{2}\left|f^{1/2}\theta'\right|_\Gamma^2+\frac{1}{2}\left|h^{1/2}\theta\right|_\Gamma^2+\int_0^t\left|g^{1/2}\theta'\right|_\Gamma^2dt\leq C\int_0^T\left(\left|w'\right|^2+\|w\|^2\right)dt$$

and hence

$$|w'|^2 + \|w\|^2 \le C \int_0^T \left(\left| w' \right|^2 + \|w\|^2 \right) dt.$$

Finally, Gronwall's inequality allow us to deduce that $w = 0$ and $\theta = 0$. This concludes the proof.

4 Decay of solutions

In this section, we demonstrate the exponential decay of the energy E, defined in (13), associated with the system (1). Consequently, the energy goes to zero when time goes to infinity. The strategy we use is mainly based on the multiplier method.

4.1 Proof of Theorem 2.2

Assume the hypothesis from Theorem 2.2. Since (16) holds, that is, $\mathcal{Y}_0 < \frac{\rho\beta r_0}{8}$; Theorem 2.1 ensures the existence of a unique regular global weak solution (u, υ) of (1).

Now, similar calculations done in (18)-(22) shows that

$$\begin{aligned} &\frac{d}{dt}\left[\overline{\Lambda}(t) + W(t)\right] + \frac{\rho\beta}{2} M\left(|u(t)|^2\right)\|u(t)\|^2 + \frac{3\rho\beta}{2}\left|u'(t)\right|^2 \\ &+2M\left(|u(t)|^2\right)\left|g^{1/2}\upsilon'(t)\right|_\Gamma^2 + 2\rho\beta\int_\Omega \Phi(u(t))u(t)dx \le I_1(t) + I_2(t) + I_3(t) + \frac{\rho\beta}{2} I_4(t), \end{aligned} \tag{55}$$

where

$$\begin{aligned} \overline{\Lambda}(t) =& \rho\left|u'(t)\right|^2 + \frac{\rho\beta}{2}\left(u'(t), u(t)\right) + \frac{\rho\beta^2}{4}|u(t)|^2 + \frac{\rho}{2} M\left(|u(t)|^2\right)\|u(t)\|^2 \\ &+ M\left(|u(t)|^2\right)\left|f^{1/2}\upsilon'(t)\right|_\Gamma^2, \end{aligned}$$

$$W(t) = \frac{\rho}{2} M\left(|u(t)|^2\right)\|u(t)\|^2 + 2\rho\int_\Omega \Phi(u(t))u(t)dx, \tag{56}$$

$$I_1(t) = 2\rho M'\left(|u(t)|^2\right)\left|u'(t)\right||u(t)|\,\|u(t)\|^2 \le 2\rho C_1 M'\left(|u(t)|^2\right)\left|u'(t)\right|\|u(t)\|^3,$$

$$\begin{aligned} I_2(t) =& 2M'\left(|u(t)|^2\right)\left|u'(t)\right||u(t)|\left|f^{1/2}\upsilon'(t)\right|_\Gamma^2 \\ \le& \frac{\rho\beta}{2}\left|u'(t)\right|^2 + \frac{2C_1^2}{\rho\beta}\left(M'\left(|u(t)|^2\right)\right)^2\|u(t)\|^2\left|f^{1/2}\upsilon'(t)\right|_\Gamma^4, \end{aligned}$$

$$\begin{aligned} I_3(t) =& 2M\left(|u(t)|^2\right)\left(h\upsilon(t), \upsilon'(t)\right) \le M\left(|u(t)|^2\right)\left(\varepsilon\left|h^{1/2}\upsilon(t)\right|_\Gamma^2 + \frac{1}{\varepsilon}\left|h^{1/2}\upsilon'(t)\right|_\Gamma^2\right) \\ \le& C_0^2\varepsilon h_2 M\left(|u(t)|^2\right)\|u(t)\|^2 + \frac{1}{\alpha_0\varepsilon} M\left(|u(t)|^2\right)\left|g^{1/2}\upsilon'(t)\right|_\Gamma^2, \quad \text{for all } \varepsilon > 0, \end{aligned}$$

and

$$\frac{\rho\beta}{2}I_4(t)=\frac{\rho\beta}{2}\left[C_0M\left(|u(t)|^2\right)\left|\upsilon'(t)\right|_\Gamma\|u(t)\|+\int_\Omega\Phi(u(t))u(t)dx\right] \tag{57}$$

$$\leq\frac{\rho\beta}{4}M\left(|u(t)|^2\right)\|u(t)\|^2+\frac{1}{2}M\left(|u(t)|^2\right)\left|g^{1/2}\upsilon'(t)\right|^2+\frac{\rho\beta}{2}C_p^{p+1}\|u(t)\|^{p+1},$$

with C_0, C_1 are positive real constants defined by (6) and $C_p>0$ is defined by (7).

From (55)-(57), we have that

$$\begin{aligned}&\frac{d}{dt}\left[\overline{\Lambda}(t)+W(t)\right]+\left(\frac{\rho\beta}{4}-C_0^2\varepsilon h_2\right)M\left(|u(t)|^2\right)\|u(t)\|^2+\rho\beta\left|u'(t)\right|^2\\&+\left(\frac{3}{2}-\frac{1}{\alpha_0\varepsilon}\right)M\left(|u(t)|^2\right)\left|g^{1/2}\upsilon'(t)\right|_\Gamma^2+2\rho\beta\int_\Omega\Phi(u(t))u(t)dx\leq\|u(t)\|^2\overline{K}(t),\end{aligned} \tag{58}$$

where

$$\overline{K}(t)=2\rho C_1M'\left(|u(t)|^2\right)\left|u'(t)\right|\|u(t)\|+\frac{2C_1^2}{\rho\beta}\left(M'\left(|u(t)|^2\right)\right)^2\left|f^{1/2}\upsilon'(t)\right|_\Gamma^4$$

$$+\frac{\rho\beta}{2}C_p^{p+1}\|u(t)\|^{p-1}.$$

From (15), we have that $\frac{\rho\beta}{8h_2C_0^2}>\frac{2}{3\alpha_0}$. Thus, taking $\varepsilon\in\left(\frac{2}{3\alpha_0},\frac{\rho\beta}{8h_2C_0^2}\right)$, results

$$\frac{\rho\beta}{4}-C_0^2\varepsilon h_2>\frac{\rho\beta}{8}>0\quad\text{and}\quad l_0=\frac{3}{2}-\frac{1}{\alpha_0\varepsilon}>0. \tag{59}$$

Therefore, from (58) and (59), we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\overline{\Lambda}(t)+W(t)\right]+\left(\frac{\rho\beta}{8}M\left(|u(t)|^2\right)-\overline{K}(t)\right)\|u(t)\|^2+\rho\beta\left|u'(t)\right|^2\\&+l_0M\left(|u(t)|^2\right)\left|g^{1/2}\upsilon'(t)\right|_\Gamma^2+2\rho\beta\int_\Omega\Phi(u(t))u(t)dx\leq 0.\end{aligned} \tag{60}$$

We proceed as in the proof of Lemma 3.4 to obtain

$$\overline{K}(t)<\frac{\rho\beta r_0}{16},\quad\text{for all } t\geq 0,$$

and, from Remark 3.3, we have that

$$W(t)\geq P\left(\|u(t)\|\right)\geq 0$$

where P is defined in Lemma 3.1.

By combining (60) and last above estimates, we deduce that

$$\frac{d}{dt}\left[\overline{\Lambda}(t)+W(t)\right]+\overline{C}_1E\left[u,\upsilon\right](t)+\overline{C}_2W(t)\leq 0. \tag{61}$$

The above constants are defined by

$$\overline{C}_1 = \min\left\{\rho\beta, \frac{\rho\beta r_0}{32C_0^2}, \frac{\rho\beta}{32}, l_0\right\} > 0,$$

and

$$\overline{C}_2 = \min\left\{\frac{\rho\beta}{32}, \beta(p+1)\right\} > 0.$$

On the other hand, we can proceed as in the proof of Lemma 3.2, to obtain

$$\overline{C}_3 E\left[u,v\right](t) \le \overline{\Lambda}(t) \le \overline{C}_4 E\left[u,v\right](t), \tag{62}$$

for some positive constants $\overline{C}_3$ and $\overline{C}_4$.

Hence, from (61) and (62), there exist $\overline{C}_5 = \min\left\{\frac{\overline{C}_1}{\overline{C}_4}, \overline{C}_2\right\} > 0$ such that

$$\frac{d}{dt}\left[\overline{\Lambda}(t) + W(t)\right] + \overline{C}_5\left[\overline{\Lambda}(t) + W(t)\right] \le 0,$$

and, from estimate above, it follows that

$$\overline{\Lambda}(t) + W(t) \le (\overline{\Lambda}(0) + W(0))e^{-\overline{C}_5 t}.$$

Moreover, from (56) and (13), the following inequality holds true

$$\begin{aligned} W(t) &\le \frac{\rho}{2}M\left(|u(t)|^2\right)\|u(t)\|^2 + \frac{2\rho C_p^{p+1}}{(p+1)r_0^{p+1}}\left[M\left(|u(t)|^2\right)\|u(t)\|^2\right]^{(p+1)/2}, \\ &\le \overline{C}_6\left[E\left[u,v\right](t) + \left(E\left[u,v\right](t)\right)^{(p+1)/2}\right], \end{aligned} \tag{63}$$

for some constant $\overline{C}_6 > 0$.
Since $W(t) \ge 0$, from (62)-(63), we have that

$$E\left[u,v\right](t) \le \overline{C}_7\left[E\left[u,v\right](0) + \left(E\left[u,v\right](0)\right)^{(p+1)/2}\right]e^{-\overline{C}_5 t},$$

for any $t \ge 0$ and some positive constants $\overline{C}_5, \overline{C}_7$.
Consequently,

$$\lim_{t\to+\infty} E\left[u,v\right](t) = 0$$

and the proof of Theorem 2.2 is complete.

5 Comments and open problems

It is possible to apply the ideas developed in this paper to study the well-posedness of the problem

$$u''(x,t) - M\left(t, \int_\Omega |u(x,t)|_{\mathbb{R}}^2 dx\right)\Delta u(x,t) + \Phi(x, u(x,t)) + \beta(x)u'(x,t) = 0,$$

with acoustic conditions on a part of the boundary, where $M_t(t,s) \leq 0$, $0 < \beta_0 < \beta(x) < \beta_1$ and

$$\left| \begin{array}{l} |\Phi(x,s)| \leq b_0|s|^p, \\ |\Phi_s(x,s)| \leq b_1|s|^{p-1}, \\ |\overline{\Phi}(x,s)| \leq b_2|s|^{p+1}, \quad \overline{\Phi}(x,s) = \displaystyle\int_0^s \Phi(x,\sigma)d\sigma \end{array} \right.$$

for some positive constants b_0, b_1, b_2.

There are some interesting issues growing out of this work that are worthy of further study:

- It remains open the study of the equation

  $$u''(x,t) - \text{ div } \left(M\left(x, \int_\Omega |u(x,t)|^2_{\mathbb{R}} dx\right) \nabla u(x,t) \right) + \Phi(u(x,t)) + \beta u'(x,t) = 0,$$

  with acoustic conditions on a nonempty part of the boundary.

  The problem without the nonlinear term in the operator

  $$p(x)u''(x,t) - \text{ div } (k(x)\nabla u(x,t)) + f_1(u(x,t)) + a(x)g_1(u'(x,t)) = 0,$$

  with acoustic conditions on a part of the boundary, was already studied by Cavalcanti et al. in [12].

- On the other hand, another interesting open problem to study are the properties for the equation

  $$u''(x,t) - A(t)u(x,t) + \Phi(u(x,t)) + \beta u'(x,t) = 0,$$

  with acoustic conditions on a part of the boundary, where

  $$A(t)u = \sum_{i,j=1}^{N} B_{ij}(u(.,t),t)\frac{\partial^2 u}{\partial x_i \partial x_j},$$

  $$\left| \begin{array}{l} B_{ij} = B_{ji} : L^1(\Omega) \times [0,T] \mapsto \mathbb{R}, \quad -\infty < \beta_0 \leq B_{ij} \leq \beta_1 < +\infty, \\ \\ B_{ij} \text{ is globally Lipschitz-continuous in } L^2(\Omega) \times [0,T], \text{ for all } i,j, \\ \\ \displaystyle\sum_{i,j=1}^{N} B_{ij}(z,t)\xi_i\xi_j \geq \alpha_0|\xi|^2, \forall \xi \in \mathbb{R}^N, (z,t) \text{ a.e. in } L^1(\Omega) \times [0,T] \\ \text{and some positive constant } \alpha_0. \end{array} \right.$$

  In particular, if

  $$B_{ij}(z,t) = \begin{cases} \displaystyle\int_\Omega |z(x,t)|^2_{\mathbb{R}} dx, & \text{for } i = j = 1, \\ \\ 0, & \text{for } i \neq 1, j \neq 1, \end{cases}$$

  we obtain the system (1), which was studied in this work.

  The way how the estimations (33), (34) and (47) were performed in this work, it is not possible to do so for the operator $A(t)u$ defined above. For a comprehensive study of parabolic equations with this operator, we refer the readers to [13]-[14] and the references therein.

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