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Supporting Information for

A Model for Positive Corona Inception from Charged Ellipsoidal Thundercloud Hydrometeors

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Contents of this file

1. Text S1 to S3

2. Figures S1 to S4

Additional Supporting Information (Files uploaded separately)

1. effective_ ionization_ coefficient.m

2. onset_ field.m

3. ionization_ integral.m

4. onset_ voltage.m

5. Towsend_coeff_delta_Extended.xlsx

Introduction The supporting information included in this document pertains to the calculations used for solving the corona onset criterion for the onset surface field of the charged hydrometeor. While these full derivations are not necessary in the paper and can in principle be done by the reader, providing them supports reproduction of the study. Specifically, derivations for the distance ρ_{ab} to the point of photon absorption (Text S1), the electric field $E(\rho, r, \theta)$ of a conducting ellipsoid (Text S2), and the distance r_{max} between the tip of the ellipsoid and the position of the breakdown field E_k (Text S3) are presented. The calculations are supported by Figures S1 to S4. The MATLAB code used to implement the model is given as additional supporting information and uploaded separately. Comments are provided in the MATLAB scripts for clarity and an Excel file is also included to be able to run these scripts.

Text S1. Derivation of the distance ρ_{ab}

The distance ρ_{ab} to the point of photon absorption, which will now be referred to as the 'observation point', is in the direction of the electric field. This direction is given by the bisector of the two straight lines from the focal points of the spheroid to the observation point (Curtright et al., 2020). The bisector ρ_{ab} is depicted in Figure S1.

Since the origin of the spherical coordinate system in Figure S1 is placed at the tip of the ellipsoid on the positive z-axis, the lines from the focal points to the observation point given by $\rho^2 = x^2 + y^2 + z^2$ become in spherical coordinates:

$$\begin{aligned} \rho_1^2 &= (r\sin\theta)^2 + (a - d + r\cos\theta)^2 = r^2 + (a-d)^2 + 2(a-d)r\cos\theta, \\ \rho_2^2 &= (r\sin\theta)^2 + (a + d + r\cos\theta)^2 = r^2 + (a+d)^2 + 2(a+d)r\cos\theta, \end{aligned} \tag{1}$$

with coordinates as given in Figure S1 and the linear eccentricity $d^2 = a^2 - b^2$.

Using the law of cosines, the angle 2β between these two lines ρ_1 and ρ_2 is given by:

$$\cos(2\beta) = \frac{1}{2\rho_1\rho_2}\left(\rho_1^2 + \rho_2^2 - 4d^2\right). \tag{2}$$

Using the trigonometric identity $\cos(2\beta) = 2\cos^2\beta - 1$ the cosine of the angle between ρ_{ab} and ρ_1 is found:

$$\cos(\beta) = \sqrt{\frac{1}{2} + \frac{1}{4\rho_1\rho_2}(\rho_1^2 + \rho_2^2 - 4d^2)}. \tag{3}$$

The acute angle ϕ between ρ_1 and the z-axis follows from:

$$\sin\phi = \frac{r\sin\theta}{\rho_1}. \tag{4}$$

Since the acute angle between ρ_{ab} and the z-axis is given by $\phi - \beta$, it can be concluded that

$$\rho_{ab} = \frac{r\sin\theta}{\sin(\phi - \beta)}, \tag{5}$$

where β and ϕ are given by Equations 3 and 4, respectively. The final expression for ρ_{ab} can be formulated more compactly by applying the trigonometric identity $\sin(\phi - \beta) = \sin(\phi)\cos(\beta) - \cos(\phi)\sin(\beta)$. Figure S1 shows that $\cos\phi = (r\cos(\theta) + a - d)/\rho_1$. Moreover, the law of sines applied to the triangle with the observation point and the two focal points as vertices in Figure S1, in combination with the trigonometric identity $\sin(2\beta) = 2\sin(\beta)\cos(\beta)$, gives $\sin(\beta) = \sin(\phi)d/\cos(\beta)\rho_2$. Finally, the following expression for ρ_{ab} is obtained:

$$\rho_{ab} = \frac{\sqrt{2}{\rho_1}^2\sqrt{\frac{\left(4\sqrt{a^2-b^2}(a+r\cos(\theta))+{\rho_1}^2\right)(2ar\cos(\theta)+b^2+\rho_1\rho_2+r^2)}{\rho_1\rho_2}}}{{\rho_1}^2 + \rho_1\rho_2}, \tag{6}$$

with ρ_1 and ρ_2 the straight lines from the two focal points of the ellipsoid to the observation point (see also the supplementary materials) given by

$$\rho_1 = \sqrt{r^2 + (a - \sqrt{a^2 - b^2})^2 + 2(a - \sqrt{a^2 - b^2}) r \cos\theta}, \tag{7}$$

$$\rho_2 = \sqrt{r^2 + (a + \sqrt{a^2 - b^2})^2 + 2(a + \sqrt{a^2 - b^2}) r \cos\theta}, \tag{8}$$

This expression reduces to $\rho_{ab} = \sqrt{(r\sin\theta)^2 + (\rho_0 + r\cos\theta)^2}$ for a sphere of radius $a = b = \rho_0$.

Text S2. Derivation of the electric field $E(\rho, r, \theta)$

The electric field of a conducting ellipsoid has been derived analytically by Köhn and Ebert (2015) for the prolate spheroid case and by Curtright et al. (2020) for arbitrary dimensions. The derivation of the electric field strength E yields

$$E(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left(\prod_{k=1}^{3} \frac{1}{\sqrt{a_k^2 + \Theta}} \right) \Bigg/ \sqrt{\left(\sum_{m=1}^{3} \frac{x_m^2}{(a_m^2 + \Theta)^2} \right)}, \tag{9}$$

where Q is the charge on the ellipsoid surface, ϵ_0 is the vacuum permittivity, and $a_1 = a_x = b$, $a_2 = a_y = b$ and $a_3 = a_z = a$ are the semi-axes of the ellipsoid. Moreover, the Θ-equipotentials follow from

$$\sum_{k=1}^{3} \frac{x_k^2}{a_k^2 + \Theta} = 1, \text{ for } \Theta > 0. \tag{10}$$

Solving equation (10) for Θ gives

$$\Theta = \frac{1}{2}\left(\sqrt{\left(-a^2-b^2+x^2+y^2+z^2\right)^2+4\left(-a^2b^2+a^2x^2+a^2y^2+b^2z^2\right)}-a^2-b^2+x^2+y^2+z^2\right). \tag{11}$$

Substituting this in equation (9) and converting to spherical coordinates (ρ', θ', ϕ') with the origin at the center of the ellipsoid yields an expression for the electric field in terms of ρ', the radial coordinate from the center of the ellipsoid, and θ', the azimuthal angle for the origin at the center of the ellipsoid. Using the trigonometric relations

$$\theta' = \tan^{-1}\left(\frac{r\sin(\theta)}{a + r\cos(\theta)}\right) \tag{12}$$

and

$$\cos\left(2\tan^{-1} u\right) = (1-u^2)/(1+u^2), \tag{13}$$

with $u = \frac{r\sin(\theta)}{a+r\cos(\theta)}$, the obtained electric field expression can be rewritten in the desired θ coordinate. Note that the conversion from θ' to θ can be done in multiple (equivalent) ways using the tangent, sine or cosine. Now, the radial coordinate ρ' needs to be converted to the ρ coordinate along the electric field direction. This is done using the expression for ρ_{ab}, which is the distance to point of photoionization along the ρ coordinate and is given by equation (6). By simple trigonometry the following relation can be derived

$$\rho' = \sqrt{2a\Delta - \Delta^2 + \rho^2 + 2\Delta r\cos(\theta)}, \tag{14}$$

with $\Delta = \Delta(r,\theta) = a - \sqrt{\rho_{ab}^2 + \frac{1}{2}r^2\cos(2\theta) - \frac{r^2}{2}} + r\cos(\theta)$ the distance between the center of the ellipsoid and the intercept of ρ_{ab} with the major axis and with ρ_{ab} given by equation (6). Writing out equation (14) gives

$$\rho' = \sqrt{\frac{2ar\cos(\theta)\left(3a^2 - 3b^2 + \rho^2\right) + 2a^2\rho^2 + a^2\rho_1\rho_2 - b^2\rho^2 - b^2\rho_1\rho_2 + \rho^2 r^2 + \rho^2\rho_1\rho_2 + p}{2a^2 + 2ar\cos(\theta) - b^2 + r^2 + \rho_1\rho_2}}, \tag{15}$$

with $p = 2a^4 - a^2b^2 + 2a^2r^2\cos(2\theta) + a^2r^2 - b^4 - 2b^2r^2\cos(2\theta) - b^2r^2$. Thus, equation (14) allows us to convert ρ' to ρ. However, as r is also a function of ρ' through $\rho' = \sqrt{a^2 + r^2 + 2ar\cos(\theta)}$ (or equivalently $r = \sqrt{\frac{1}{2}a^2\cos(2\theta) - \frac{a^2}{2} + \rho'^2} - a\cos(\theta)$), this derivation is not complete. Substituting the expression for r in terms of ρ' in equation (14) gives an equation that is not analytically solvable. Ideally, an electric field would be obtained that is only a function of ρ and θ. As an analytical expression is required for the model, the expression for the electric field in terms of ρ, r and θ is now accepted, given by

$$E(\rho, r, \theta) = \frac{2b^2 E_0 \rho'}{\sqrt{a^2 - b^2 + q + \rho'^2}\left(-a^2 + b^2 + q + \rho'^2\right)\sqrt{\frac{q}{\frac{(b^2-a^2)(a^2+2ar\cos(\theta)+r^2\cos(2\theta))}{a^2+2ar\cos(\theta)+r^2} + q + \rho'^2}}}, \tag{16}$$

with the shorthand $q = \sqrt{\frac{2\rho'^2(b^2-a^2)(a^2+2ar\cos(\theta)+r^2\cos(2\theta))}{a^2+2ar\cos(\theta)+r^2} + \left(a^2 - b^2\right)^2 + \rho'^4}$ and where ρ' is converted to ρ using equation (15). As eventually the $\gamma = 1$ equation is solved numerically, the dependence of r is not a problem as long as the resulting α_{eff} (which depends on the electric field) behaves correctly.

Text S3. Derivation of the distance r_{max}

The distance r_{max} between the tip of the ellipsoid and the position of the breakdown field E_k can be found from the relation $\rho_{ab}(r_{max}, \theta.\phi) = \rho_c$. Here, ρ_{ab} is the bisector in the direction of the electric field, and ρ_c is the distance (in terms of the ρ coordinate) to the position of the breakdown field $E(\rho_c) = E_k$. Thus, to solve for r_{max}, ρ_c needs to be found first, where ρ_c is defined by the equation $E(\rho_c) = E_k$ (the electric field is derived in the supplementary materials). However, due to the complicated nature of the electric field of a conducting ellipsoid, this equation cannot be solved explicitly for ρ_c. Nevertheless, it turns out that there is a fairly good approximation for r_{max}.

The surface of a conducting ellipsoid is an equipotential. One can mistakenly think that the electric field is constant on the surface. From symmetry it then follows that the points where the electric field has a certain constant value, such as E_k, will lie on an ellipsoid. Though this is based on a false assumption, it might prove useful to approximate the surface $E = E_k$ as forming an ellipsoid. Simulating the electric field in COMSOL, specifically for the semi-axes $a = 5\,\mathrm{cm}$ and $b = 2\,\mathrm{cm}$, the field at the tip $E_0 = 95.2$ kV/cm and the breakdown field $E_k = 32$ kV/cm, results in the plot of Figure S2. This figure shows that near the tip of the ellipsoid in the (x, z)-plane, where x is the horizontal coordinate and z the vertical coordinate, the line of $E = E_k$ approximately follows the shape of an ellipse around the conducting ellipse. The validity of the approximation can be tested, by importing the data points where $E = E_k$ into MATLAB, and fitting these using the equation of an ellipse. This ellipse has unknown semi-axes a' and b', which are the fitting parameters, and is centered at the center of the conducting ellipsoid (here at $(0, 0)$) because of symmetry. The fitting equation is thus $z = a'\sqrt{1 - (x/b')^2}$. The data points imported from COMSOL and the fit through these points is depicted in Figure S3, where $(x, z) = (0, 5)$ is the position of the tip of the ellipsoid. It follows that the data points indeed approximately lie on an ellipse near the tip of the conducting ellipsoid, as the fit agrees very well with the data points.

It can thus be concluded that the surface where $E = E_k$ can be approximated as an ellipsoid. Because of symmetry, only an ellipse in the (x, z)-plane or (y, z)-plane needs to be considered. The next step is to find an analytical expression for r_{max} going from the origin at $r = 0$ at the tip of the conducting ellipsoid to the ellipse where $E = E_k$.

While r_{max} cannot be found from the electric field for arbitrary θ, it can be found for $\theta = 0$ and $\theta = \pi/2$. First, the electric field is reformulated in terms of only r and θ. Then, filling in $\theta = 0$ and solving $E(r, 0) = \frac{b^2 E_0}{2ar+b^2+r^2} = E_k$ for r gives:

$$r_{max}(\theta = 0) = \sqrt{a^2 + \frac{b^2(E_0 - E_k)}{E_k}} - a. \tag{17}$$

Obtaining $r_{max}(\theta = \pi/2)$ proves more difficult. Filling in $\theta = \pi/2$ in the electric field expression yields:

$$E(r, \theta = \pi/2) = \frac{2b^2 E_0 \sqrt{\frac{(a^2+r^2)\left((b^2-a^2)\left(\frac{2a^2}{a^2+r^2}-1\right)+\sqrt{4a^2r^2+b^4-2b^2r^2+r^4}+a^2+r^2\right)}{\sqrt{4a^2r^2+b^4-2b^2r^2+r^4}+2a^2-b^2+r^2}}}{\sqrt[4]{4a^2r^2 + b^4 - 2b^2r^2 + r^4}\left(\sqrt{4a^2r^2 + b^4 - 2b^2r^2 + r^4} + b^2 + r^2\right)}. \tag{18}$$

Setting the above expression equal to E_k and solving for r leads to a case known as 'Casus irreducibilis'. For an irreducible degree 3 polynomial with three real roots, it has been proven that complex numbers need to be introduced to express the solution in roots of any degree, even though the solution is real (Wantzel, 1843). Solving $E(r, \theta = \pi/2) = E_k$ for r, for example using software like Mathematica, leads to 6 solutions containing imaginary parts. Setting b close to a, it is found that one of these solutions approaches the solution of a sphere, accompanied by a very small imaginary part. For example, for $a = 3\,\text{cm}$, $b = 2.99\,\text{cm}$, $E_0 = 100$ kV/cm, $E_k = 32.75$ kV/cm a value of $10^{-12}\,\text{cm}$ is found for the imaginary part. These negligibly small imaginary contributions, which are found for any a

and b and remain negligibly small, are a results of numerical noise in the machine number calculations in Mathematica (or other numerical software packages).

Possibly due to this 'Casus irreduciblilis' issue, $r_{max}(\theta = \pi/2)$ has no solution at $a = b$, so for the reduction to a sphere. However, when b approaches a, $r_{max}(\theta = \pi/2)$ approaches the solution of a sphere. For example, taking again $a = 3\,\text{cm}$, $b = 2.99\,\text{cm}$, $E_0 = 100$ kV/cm, $E_k = 32.75$ kV/cm, the real part of $r_{max}(\theta = \pi/2)$ is $4.27758\,\text{cm}$, while the r_{max} of a sphere at $\theta = \pi/2$ is $4.29894\,\text{cm}$. Instead taking $b = 2.9999999\,\text{cm}$ (seven decimals) gives $r_{max}(\theta = \pi/2) = 4.29894\,\text{cm}$ for the ellipsoid. It can thus be concluded that $r_{max}(\theta = \pi/2)$ approaches the correct solution for a sphere and can be safely used, as the goal is to implement the model for an ellipsoid and not a sphere, for which simpler expressions are already known. Writing out the found solution for r_{max} at $\theta = \pi/2$ for a conducting ellipsoid gives:

$$r_{max}(\theta = \pi/2) = \frac{1}{2\sqrt{3}}\sqrt{\frac{F}{G}}, \tag{19}$$

where F and G are given by:

$$\begin{aligned}
F \;=\; & 32\sqrt[3]{2}(1-i\sqrt{3})a^8E_k^4+12\sqrt[3]{2}i(\sqrt{3}+i)a^2b^6E_k^2\Big(E_0^2+3E_k^2\Big)\\
& +24a^2b^2E_k^2(\sqrt[3]{b^4\Big(3b^4E_k\Big(a^4C_5E_k-C_1E_0^2+2C_1E_k^2\Big)+6a^4E_k^3(2a^4C_7E_k+C_1)-18a^2b^6C_4E_k^2+3a^2b^2C_1E_k\Big(E_0^2-4E_k^2\Big)-b^8C_3\Big)+C_2}\\
& +4\sqrt[3]{2}i(\sqrt{3}+i)a^4E_k^2)\\
& +b^4\Big(4\Big(E_0^2-2E_k^2\Big)\sqrt[3]{b^4\Big(12a^8C_7E_k^4+3a^4b^4C_5E_k^2-18a^2b^6C_4E_k^2+3a^2C_1C_6E_k-b^8C_3-3b^2C_1C_6E_k\Big)+C_2}+2\sqrt[3]{2}(1-i\sqrt{3})a^4E_k^2\Big(4E_0^2+49E_k^2\Big)\Big)\\
& -16a^4E_k^2\sqrt[3]{b^4\Big(12a^8C_7E_k^4+3a^4b^4C_5E_k^2-18a^2b^6C_4E_k^2+3a^2C_1C_6E_k-b^8C_3-3b^2C_1C_6E_k\Big)+C_2}\\
& +(1+i\sqrt{3})(256a^{12}E_k^6-1152a^{10}b^2E_k^6-6b^6\Big(6a^6E_k^4\Big(8E_0^2+49E_k^2\Big)+C_1C_6E_k\Big)\\
& +6a^4b^8C_5E_k^2-36a^2b^{10}C_4E_k^2+6b^4\Big(4a^8C_7E_k^4+a^2C_1C_6E_k\Big)-2b^{12}C_3)^{2/3}+2\sqrt[3]{2}(1-i\sqrt{3})b^8\Big(E_0^2+E_k^2\Big)^2,
\end{aligned}$$

$$G=E_k^2\left(a^2-b^2\right)\sqrt[3]{b^4\left(12a^8C_7E_k^4+3a^4b^4C_5E_k^2-18a^2b^6C_4E_k^2+3a^2C_1C_6E_k-b^8C_3-3b^2C_1C_6E_k\right)+C_2},$$

and where C_1, C_2, C_3, C_4, C_5, C_6 and C_7 are defined as follows:

$$\begin{aligned}
C_1^2 \;=\; & -12a^8E_k^4\left(8E_0^2+E_k^2\right)+36a^6b^2E_k^4\left(7E_0^2+E_k^2\right)-3a^4b^4E_k^2\left(13E_0^4+72E_0^2E_k^2+12E_k^4\right)\\
& +6a^2b^6E_k^2\left(7E_0^4+10E_0^2E_k^2+2E_k^4\right)-3b^8E_0^4\left(4E_0^2+E_k^2\right),
\end{aligned}$$

$$C_2=128a^{12}E_k^6-576a^{10}b^2E_k^6-18a^6b^6E_k^4\left(8E_0^2+49E_k^2\right),$$

$$C_3=2E_0^6-21E_0^4E_k^2+6E_0^2E_k^4+2E_k^6,$$

$$C_4=2E_0^4+2E_0^2E_k^2+3E_k^4,$$

$$C_5 = 5E_0^4 + 46E_0^2E_k^2 + 122E_k^4,$$

$$C_6 = 2a^2E_k^2 + b^2\left(E_0^2 - 2E_k^2\right),$$

$$C_7 = 4E_0^2 + 85E_k^2.$$

Now that analytical expressions are found for $r_{max}(\theta = 0)$ and $r_{max}(\theta = \pi/2)$, the ellipse approximation for r_{max} at arbitrary θ can be applied. Noting that the origin is placed at the tip of the ellipsoidal conductor, the equation for the ellipse where $E = E_k$ is given by:

$$\frac{(z' + a)^2}{a'^2} + \frac{x'^2}{b'^2} = 1, \tag{20}$$

where (z',x') is a point on the ellipse $E = E_k$, a' is its major semi-axis and b' its minor semi-axis. This configuration is depicted in Figure S4, where the grey region represents the photon absorption region. Here it is seen that $a' = a + r_{max}(\theta = 0)$, $z' = r_{max}\cos\theta$, and $x' = r_{max}\sin\theta$.

Setting $z' = 0$ in equation (20), which corresponds to $x' = \pm r_{max}(\theta = \pi/2)$ as can be seen from Figure S4, gives

$$\frac{a^2}{a'^2} + \frac{r_{max}(\theta = \pi/2)^2}{b'^2} = 1. \tag{21}$$

which can be solved for the minor semi-axis b' of the ellipse $E = E_k$:

$$b' = \frac{r_{max}(\theta = \pi/2)}{\sqrt{1 - \frac{a^2}{a'^2}}}. \tag{22}$$

Substituting the found expressions for a', b', z' and x' into equation (20) results in the final expression for $r_{max}(\theta)$ in the ellipse approximation:

$$r_{max}(\theta)=\frac{r_{max}(\theta=\pi/2)\left(\sqrt{r_{max}(\theta=0)^2(2a+r_{max}(\theta=0))^2\sin^2(\theta)+r_{max}(\theta=\pi/2)^2(a+r_{max}(\theta=0))^2\cos^2(\theta)}-ar_{max}(\theta=\pi/2)\cos(\theta)\right)}{r_{max}(\theta=0)(2a+r_{max}(\theta=0))\sin^2(\theta)+r_{max}(\theta=\pi/2)^2\cos^2(\theta)}. \tag{23}$$

Looking back at Figure S3, the ellipse fit of the data points gives $r_{max}(\theta = 0) = (5.743 - 5)\,\text{cm} = 0.743\,\text{cm}$ in the z-direction, and $r_{max}(\theta = \pi/2) = 1.408\,\text{cm}$ in the x-direction. Equation (23) gives for the same input, $a = 5\,\text{cm}$, $b = 2\,\text{cm}$, $E_0 = 95.2$ kV/cm and $E_k = 32$ kV/cm, the values of $r_{max}(\theta = 0) = 0.736\,\text{cm}$ and $r_{max}(\theta = \pi/2) = 1.397\,\text{cm}$. The discrepancy between these values is very small (relative error of about 1%) and caused by equation (23) being derived using the data points of $E = E_k$ at $\theta = 0$ and $\theta = \pi/2$ and basing the ellipse shape on that, while the ellipse in Figure S3 is based on more data points. Hence, the ellipse of equation (23) is formulated such that the two computed r_{max} at $\theta = 0$ and $\theta = \pi/2$ lie on the ellipse, while for the fit the optimal fit does not necessarily go through these two points precisely. Equation (23) can also be compared to Figure S3 for arbitrary $0 \leq \theta \leq \pi/2$. For example, $\theta = 0.9497$ rad gives $r_{max} = 0.928\,\text{cm}$ for the ellipse fit, and $r_{max} = 0.919\,\text{cm}$ for equation (23). Moreover, $\theta = 0.5120$ rad gives $r_{max} = 0.792\,\text{cm}$ and $r_{max} = 0.784\,\text{cm}$, respectively. We thus conclude that equation (23) is a fair approximation of the actual r_{max} of a conducting ellipsoid.

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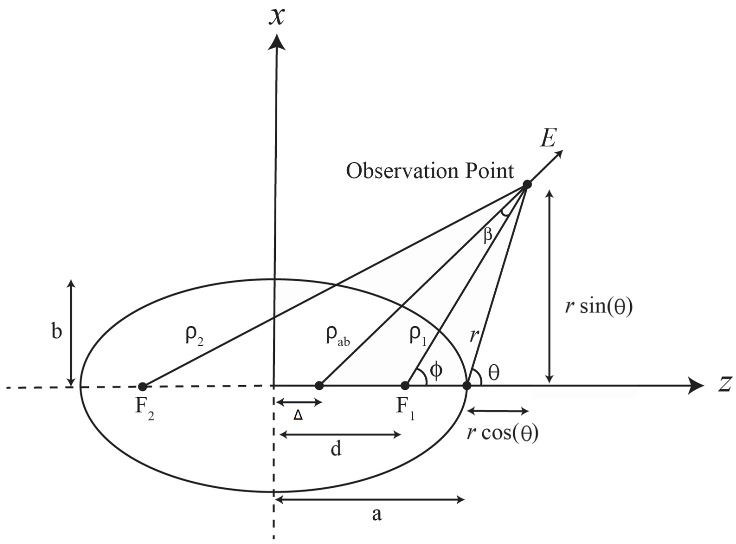

Figure S1. Schematic of the bisector giving the electric field direction outside a conducting ellipsoid.

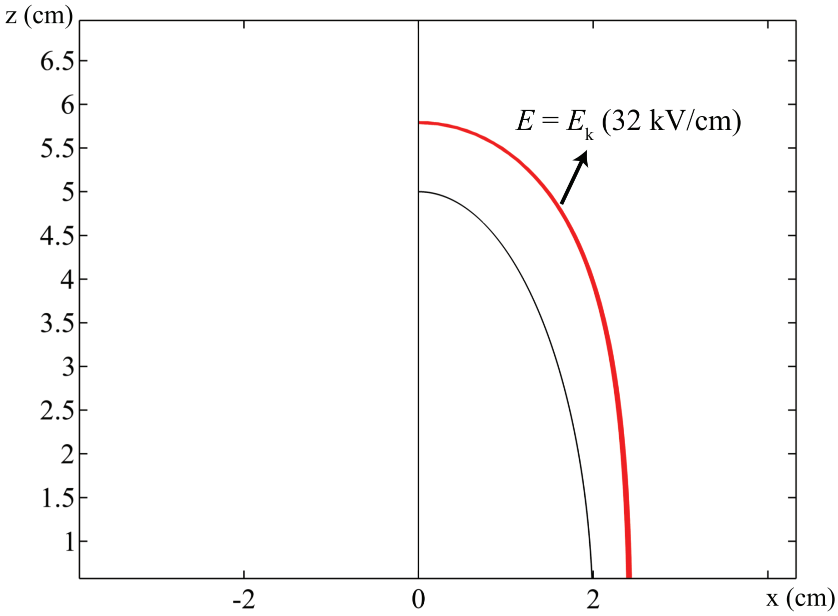

Figure S2. The line of constant $E = E_k$ (red) for an ellipse with semi-axes $a = 5\,\mathrm{cm}$ (vertical) and $b = 2\,\mathrm{cm}$ (horizontal). Here only half of the width of the ellipse is shown and the axis of symmetry is drawn.

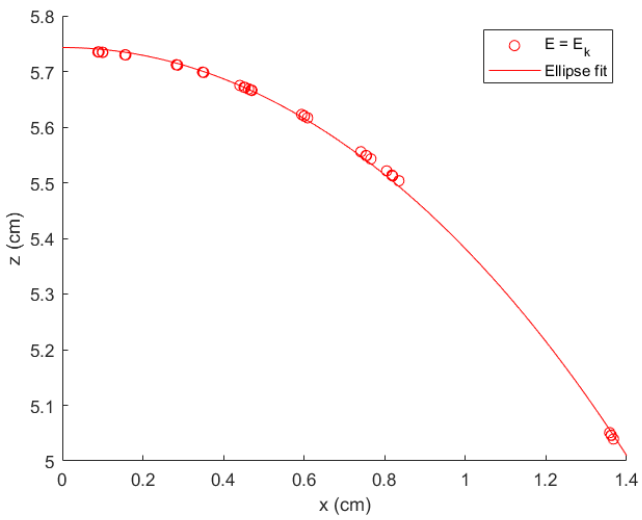

Figure S3. Fit of $E = E_k$ data points using an ellipse fit of $z = a'\sqrt{1-(x/b')^2}$. It is found that $a' = (5.743 \pm 0.002)\,\mathrm{cm}$ and $b' = (2.86 \pm 0.01)\,\mathrm{cm}$.

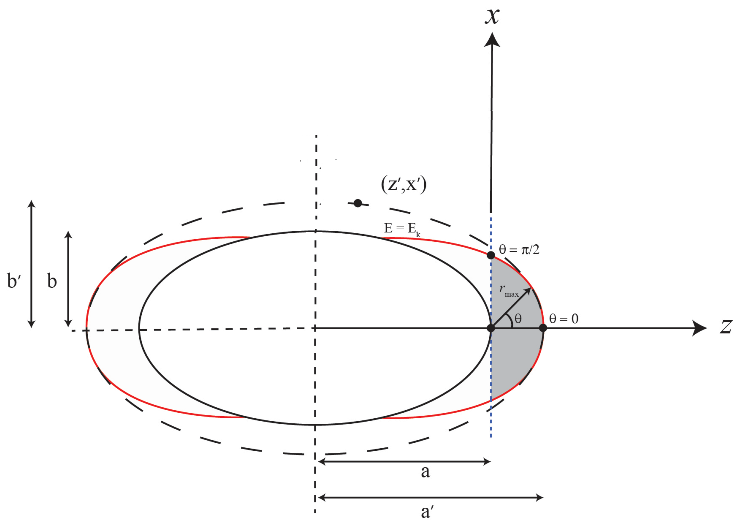

Figure S4. The conducting ellipsoid (solid, black line), with positions of constant $E = E_k$ (solid, red line), the edge of the ionization region, approximated as an ellipse shape (dashed line) with coordinates (z',x') and semi-axes a' and b'.