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A computational stochastic dynamic model to assess the risk of breakup in a romantic relationship

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Summary

We introduce a new algorithm to find feedback Nash equilibria of a stochastic differential game. Our computational approach is applied to analyze optimal policies to nurture a romantic relationship in the long term. This is a fundamental problem for the applied sciences, which is naturally formulated in this work as a stochastic differential game with nonlinearities. We use our computational model to analyze the risk of marital breakdown. In particular, we introduce the concept of "love at risk" which allows us to estimate the probability of a couple breaking up in the face of possible unfavorable scenarios.

KEYWORDS:
Stochastic Differential Games, Nonlinear problems, Random Differential Equations, Dynamical Analysis, Human Behaviour

1 | INTRODUCTION

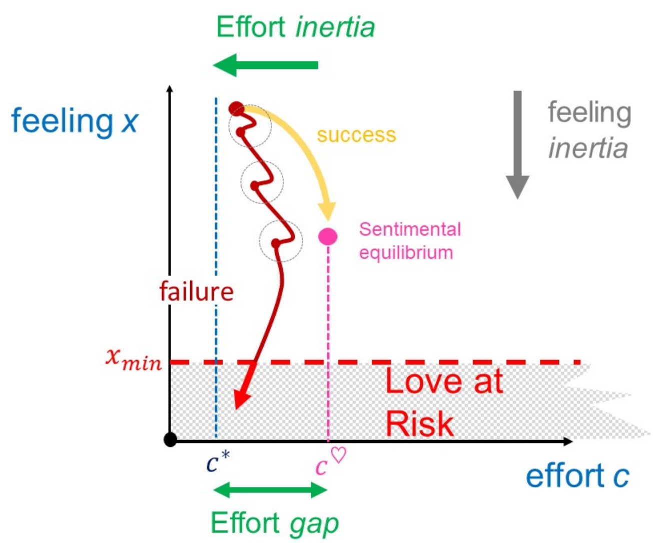
The purpose of this work is twofold. Firstly, we introduce a new algorithm to find feedback Nash equilibria of stochastic differential games and, secondly, we apply our methodology to a problem of significance in the social sciences, related to human behavior.

The numerical analysis of stochastic differential games (SDG) is currently a topic of growing interest (see early publications [1,2] and recent contributions [3,4,5,6,7,8]). Most contributions in the literature on differential games focus on the study of the theoretical properties of certain classes of SDGs. Also, various works that formulate economic and social problems as SDG (see e.g. [9,10,11]) obtain their solutions by using heuristic approximations to avoid solving the stochastic Hamilton-Jacobi-Bellman (HJB) equations of the problem. The most extended approach to dealing with problems with more than two players is the approximation by a linear quadratic problem or using of open-loop solutions (see, for example, [5]). An algorithm has been recently proposed [12] to find feedback Nash equilibrium for a class of finite-horizon stochastic differential games by solving a system of non stationary HJB equations. In this paper, our goal is to solve an infinite-horizon autonomous SDG, which is a common problem in economics and management (see e.g. [13]). Our computational approach below involves solving a stationary HJB system. This seems a useful contribution to the field, given the lack of algorithms to solve directly these kinds of problems.

Our approach here extends the idea in [14], where an algorithm –called RaBVItG (Radial Basis Value Iteration Game)– is introduced to solve the HJB system to find feedback Nash equilibria of deterministic differential games. The core of the algorithm consists of two main loops: value iteration, as in [15,16,17] plus game iteration, as introduced in [14]). More precisely, RaBVItG uses game iteration to find the Nash Equilibrium corresponding to a fixed value of the game and, then, value iteration finds a fixed point solution for the coupled system of value functions (one per player). The feedback Nash equilibrium of the deterministic differential game is found as the convergent solution of both iterations. We introduce below a stochastic version of the RaBVItG algorithm to find feedback Nash equilibria of a SDG.

The second objective of this work is to evaluate the risk of rupture in a dyadic romantic relationship that is intended to last. This is a problem of enormous interest in the social sciences, due to the relevance of long-term romantic relationships -in particular marriage- in most societies[18]. Furthermore, there is an epidemic of failed marriages in the West (see e.g.[19]) which is not well understood in the field of marital psychology[20]. To formulate our problem we model a long-term romantic relationship as an optimal control problem, as originally proposed in[21] and[22], and then extended in[23]. The quality of the relationship is monitored by a state variable $x(t)$ (called *feeling*) whose evolution is controlled by the effort exerted by both partners to keep the relationship alive and well. The couple's problem consists of finding the optimal effort control paths to stay together forever (see[23]). In particular, for the relationship to be viable the feeling must stay above a certain critical value $x_{min} > 0$. Once $x(t)$ drops below the level x_{min}, the relationship enters a risk zone and is in danger of breaking down. It was found in[21] that two effects contribute to hindering the viability of the relationship. First, the feeling is subject to decay as time goes by and, second, there is a tendency to reduce effort below the level required for the relationship to last, thus moving the relationship away from the unique equilibrium path of feeling-effort for the relationship. These two inertial forces can make the feeling approach the risk zone where the breakup is likely. Figure 1 below illustrates the idea of this hindering mechanism to put love at risk (see[21]). Regarding the problem of love at risk, the point of interest in this paper is to estimate the critical value x_{min} under a more realistic

Figure 1 Basic mechanism operating to put love at risk (adapted from[21])



version of the original model[21], where the effort variable is common for both partners and the evolution of the feeling is governed by a deterministic equation. First, we assume here, as in the differential game formulationn[23], that each partner could make effort differently, so there are two different effort variables $c_1(t)$ and $c_2(t)$ controlling the feeling dynamics. Furthermore, we extend both formulations of the couple's relationship by considering that the feeling $x(t)$ is a random variable whose evolution is governed by a stochastic differential equation. We thus introduce a new model formulation of the couple's problem as a SDG. This stochastic generalization allows us to introduce the idea of "probability of rupture" at a certain moment of the relationship, which can be obtained from the probability distribution of $x(t)$ once the threshold value x_{min} is estimated. Using the well-known idea of "value at risk" in finance[24] we provide an estimate of the threshold value x_{min} that will be called *Love at Risk* (LaR) below.

The paper is organized as follows. In Section 2, we present the mathematical model of the couple's sentimental dynamics as a SDG. We pay attention here to the main output of the model solution, namely the stochastic feedback Nash equilibrium, and the feedback mappings that are required for its numerical approximation. In Section 3 we present the computational model. Firstly, we present the discretization of the involved equations and the way to implement the RaBVItG algorithm to solve numerically the couple's problem. In section 4 we analyze several numerical experiments for different types of couples and how our stochastic computational scheme renders new information compared with the deterministic versions of the couple's problem. We also show how the threshold value LaR can be determined using our stochastic structure to estimate the probability of dissolution of a given couple.

2 | MATHEMATICAL MODEL

Our model is a stochastic two–person generalization of the optimal control model for a long-term romantic relationship introduced in [21]. A deterministic differential game model was considered in [23]. Also, a mean-field stochastic version of the original model was considered in [25]. In this paper, the state of the relationship at time $t \geq 0$ is described by $x(t)$ –the *feeling* variable–, which is modeled by a stochastic process $\{x(t)\}_{t\geq 0}$, with $x : [0, \infty) \to X \subseteq \mathbb{R}^+$, being X the state space. The feeling evolves according to a stochastic differential equation

$$\mathrm{d}x(t) = \left[-rx(t) + a_1 c_1(t) + a_2 c_2(t)\right] \mathrm{d}t + \sigma(x(t))\,\mathrm{d}w, \tag{1}$$

where $r, a_1, a_2 > 0$ and, for $i = 1, 2$, $c_i : [0, \infty) \to \mathbb{R}^+$ is a (piece-wise continuous) function that measures the effort put into the relationship by partner i at time t, and $w(t)$ is a Wiener process. Equation (1) thus is a general version of the differential equation presented in [23], called the "second law of thermodynamics for sentimental relationships". Here the time evolution of the *feeling* includes a random term, due to the fact that the couple's evaluation of the state of the relationship may be subjected to some observational error or uncertainty at any time t. The total well-being W_i of each partner i is defined as the conditional expectation

$$W_i\left(c_i\right) = \mathbb{E}\left(\int_0^\infty e^{-\rho_i t}\left(U_i(x(t)) - D_i\left(c_i(t)\right)\right)\mathrm{d}t \,|\, x(0) = y\right),\ i = 1, 2, \tag{2}$$

where U_i and D_i are, respectively, the utility of feeling and disutility of effort, while $\rho_i > 0$ is the individual rate of temporal preference. These objective functionals are stochastic versions of the deterministic well-being functionals considered in [23]. The functions U and D are assumed to satisfy the same properties as in [23], namely $U_i'(x) > 0$, $U_i''(x) < 0$, and $U_i'(x) \to 0$ as $x \to +\infty$, and also $D_i''(c_i) > 0$, $D_i'(c_i^*) = 0$ for some $c_i^* \geq 0$, $D_i'(c_i) \to +\infty$ as $c_i \to +\infty$, for $i = 1, 2$.. Notice that c_i^* gives the effort level preferred (myopically) by partner i. The underlying psychological rationale behind these assumptions is explained in detail in [21].

The couple's problem considered in this paper can thus be stated as follows. Given the feeling dynamics (1), and the initial feeling level $x(0) = x_0$, find the effort trajectories $c_1^*(t), c_2^*(t)$ such that each individual well-being integral (2) is maximal. This is an infinite-horizon stochastic differential two-person game. Notice that the relationship will be viable as long as that the state of relationship $x(t)$ remains above a certain value $x_{min} > 0$ (see Figure 1). Let us define the pair $\left(c_1^\heartsuit(t), c_2^\heartsuit(t)\right)$ that solves the couple's problem. We aim to find a Nash equilibrium for this differential game. The differential game is autonomous so we consider stationary feedback solutions of the problem, that are of defined as $c_i = S_i(x)$, being $S_i : X \to \mathbb{R}^+$ the *feedback* map that provides the effort by player i for the feeling x. We look for a couple of optimal strategies $\left(S_1^\heartsuit(\cdot), S_2^\heartsuit(\cdot)\right)$, such that $S_i^\heartsuit : X \to \mathbb{R}^+$ is a stationary feedback Nash equilibrium of the stochastic differential game. Indeed, this equilibrium is attained if $S_1^\heartsuit(x(t))$ solves

$$\max_{c_1(t)} \mathbb{E}\left(\int_0^\infty e^{-\rho_1 t}\left(U_1(x(t)) - D_1\left(c_1(t)\right)\right)\mathrm{d}t \,|\, x(0) = y\right) \tag{3}$$

with $\mathrm{d}x(t) = \left[-rx(t) + a_1 c_1(t) + a_2 S_2^\heartsuit(x(t))\right] \mathrm{d}t + \sigma(x(t))\,\mathrm{d}w$, and also $S_2^\heartsuit(x(t))$ solves

$$\max_{c_2(t)} \mathbb{E}\left(\int_0^\infty e^{-\rho_2 t}\left(U_2(x(t)) - D_2\left(c_2(t)\right)\right)\mathrm{d}t \,|\, x(0) = y\right) \tag{4}$$

with $\mathrm{d}x(t) = \left[-rx(t) + a_1 S_1^\heartsuit(x(t)) + a_2 c_2(t)\right] \mathrm{d}t + \sigma(x(t))\,\mathrm{d}w$, where $y = x_0$, and $c_i(t) \in \mathbb{R}^+$ for $t \geq 0$.

Assume that there exists a stochastic feedback Nash equilibrium $S^\heartsuit = \left(S_1^\heartsuit, S_2^\heartsuit\right)$ for the couple's problem. Let $v_i^\heartsuit : X \to \mathbb{R}$ be the *value function* of partner i, defined by

$$v_i^\heartsuit\left(x_0\right) = W_i\left(S_i^\heartsuit(x(t))\right),\ i = 1, 2.$$

where $S_i^{\heartsuit}(x(t))$ is the optimal feedback control problem for partner i with initial state $x(0) = x_0$. The value functions $v_i^{\heartsuit}$ must satisfy the stochastic Hamilton-Jacobi-Bellman (HJB) equations, which in this case are given by

$$\begin{cases} \rho_1 v_1(x) = \max_{c_1\in\mathbb{R}^+} \left\{ U_1(x) - D_1(c_1) + v_1'(x)\left(-rx + a_1c_1 + a_2S_2^{\heartsuit}(x)\right) + \frac{1}{2}v_1''(x)\sigma^2(x) \right\}, \\ \rho_2 v_2(x) = \max_{c_2\in\mathbb{R}^+} \left\{ U_2(x) - D_2(c_2) + v_2'(x)\left(-rx + a_1S_1^{\heartsuit}(x) + a_2c_2\right) + \frac{1}{2}v_2''(x)\sigma^2(x) \right\}. \end{cases} \tag{5}$$

The solution of (5) gives the stochastic feedback maps $S_i^{\heartsuit} : X \to \mathbb{R}^+$, $i = 1, 2$, defined as

$$\begin{cases} S_1^{\heartsuit}(x) = arg\max_{c_1\in\mathbb{R}^+} \left\{ U_1(x) - D_1(c_1) + v_1'(x)\left(-rx + a_1c_1 + a_2S_2^{\heartsuit}(x)\right) + \frac{1}{2}v_1''(x)\sigma^2(x) \right\}, \\ S_2^{\heartsuit}(x) = arg\max_{c_2\in\mathbb{R}^+} \left\{ U_2(x) - D_2(c_2) + v_2'(x)\left(-rx + a_1S_1^{\heartsuit}(x) + a_2c_2\right) + \frac{1}{2}v_2''(x)\sigma^2(x) \right\}, \end{cases} \tag{6}$$

which constitute a a feedback Nash stocashtic equilibirum of the problem. Given $x(0) = x_0$, inserting $S_i^{\heartsuit}(x(t))$, $i = 1, 2$, into (1), we obtain

$$\mathrm{d}x(t) = \left[-rx(t) + a_1S_1^{\heartsuit}(x(t)) + a_2S_2^{\heartsuit}(x(t))\right]\mathrm{d}t + \sigma(x(t))\,\mathrm{d}w,$$

that is, the optimal evolution of the stochastic process $\left\{x^{\heartsuit}(t)\right\}_{t\geq 0}$ which solves the (stochastic) couple's problem with initial state x_0.

3 | A COMPUTATIONAL MODEL

General existence or uniqueness results for feedback Nash equilibria for differential games are not available in the literature[26], except for some particular cases, namely the so-called Linear Quadratic models[5]. Thus, a computational approach is required to find a solution. As far as we know, the following method, that can be considered as a generalization of[14], is novel. We are not aware of other available similar algorithm to solve an infinite horizon SDG in feedback Nash equilibrium. While it can be applied to a general class on n-player SDG, we present the algorithm adapted to the SDG model for the couple's problem described in the preceding section.

The model is discretized in a Semi-Lagrangian way (see, for instance,[27]). This implies first discretizating in time and space and the using numerical interpolation (in this case, using radial base functions- see[14]). The discretization of (2) is performed through the trapezoid rule, taking $h > 0$ as a time step. Given a value of the state variable $y \in X$, consider the following discrete version of (2):

$$W_i^h\left(c_i^h\right) = \mathbb{E}\left\{ h\sum_{k=0}^{\infty} e^{-\rho_i k}\left(U_i\left(x_k\right) - D_i\left(c_{i,k}\right)\right) | x_0 = y \right\},\ i = 1.2, \tag{7}$$

where $c_i^h = \left\{c_{i,k}\right\}_{k\geq 0}$ is a sequence of (feasible) controls for partner i, defined by the piece-wise constant function $c_i^h(\tau) = c_{i,k},\ \tau \in [t_k, t_{k+1})$, where $t_k = hk, k \in \mathbb{N} \cup \{0\}$. Furthermore, the sequence $x_k = x(t_k)$ is obtained by time discretization of (1) using the Euler-Maruyama scheme (see, for instance,[28]), that is,

$$x_{k+1} = x_k + hf\left(x_k, c_{1,k}, c_{2,k}\right) + \sigma\left(x_k\right)\xi_k, \tag{8}$$

with $f(x, c_1, c_2) = -rx + a_1c_1 + a_2c_2$, $x_0 = y$, and ξ_k denotes the increment of a standard Brownian motion $w(t)$ in the interval $[t_k, t_{k+1})$. Then, the corresponding discrete value function is for partner $i = 1, 2$ is given by

$$v_i^h(y) = \max_{c_i^h} W_i^h\left(c_i^h\right).$$

It can be proven (see[29]) that, given that (2) is defined as an expected value, the Gaussian variable ξ_k can be replaced –in a computationally efficient way– by a discrete variable with probability distribution

$$\mathbb{P}\left(\xi_k = \sqrt{h}\right) = \mathbb{P}\left(\xi_k = -\sqrt{h}\right) = \frac{1}{2}.$$

Therefore, we can redefine (8) as a set of two displacements,

$$x_{k+1} = x_k + \delta_d,\ d = 1, 2,$$

where

$$\begin{cases} \delta_1 = hf(x_k, c_{1,k}, c_{2,k}) + \sigma(x_k)\sqrt{h}, \\ \delta_2 = hf(x_k, c_{1,k}, c_{2,k}) - \sigma(x_k)\sqrt{h}. \end{cases}$$

Next, for the sake of simplicity, we will write

$$\delta_d = hf\left(x_k, c_{1,k}, c_{2,k}\right) + \sigma\left(x_k\right)\left(\pm\sqrt{h}\right).$$

The Dynamic Programming Principle (DPP) in discrete time implies that the discrete value functions satisfy (see[27])

$$\begin{cases} v_1^h(y) = \max_{c_1\in\mathbb{R}^+}\left\{h\left(U_1(y) - D_1\left(c_1\right)\right) + \frac{(1-\rho_1 h)}{2}\sum_{d=1}^2 v_1^h\left(y + \delta_d\left(y, c_1, S_2^h(y)\right)\right)\right\}, \\ v_2^h(y) = \max_{c_2\in\mathbb{R}^+}\left\{h\left(U_2(y) - D_2\left(c_2\right)\right) + \frac{(1-\rho_2 h)}{2}\sum_{d=1}^2 v_2^h\left(y + \delta_d\left(y, S_1^h(y), c_2\right)\right)\right\}, \end{cases} \tag{9}$$

together with the corresponding discrete version of (6), namely

$$\begin{cases} S_1^h(y) = arg\max_{c_1\in\mathbb{R}^+}\left\{h\left(U_1(y) - D_1\left(c_1\right)\right) + \frac{(1-\rho_1 h)}{2}\sum_{d=1}^2 v_1^h\left(y + \delta_d\left(y, c_1, S_2^h(y)\right)\right)\right\}, \\ S_2^h(y) = arg\max_{c_2\in\mathbb{R}^+}\left\{h\left(U_2(y) - D_2\left(c_2\right)\right) + \frac{(1-\rho_2 h)}{2}\sum_{d=1}^2 v_2^h\left(y + \delta_d\left(y, S_1^h(y), c_2\right)\right)\right\}. \end{cases} \tag{10}$$

To obtain an numerical approximation of the functions v_i^h, satisfying (9), we consider a spatial discretization of the state space. Let us define $\tilde{X} = \left\{y_j\right\}_{j=1,\dots,Q} \subset X$ a set of arbitrary Q points. Notice that, in general, the points $y_i^\sharp = y_j + \delta_d\left(y_j, c_1, c_2\right)$ in (9) do not belong to $\tilde{X}$. To find approximate values $\tilde{v}_i^h\left(y_j\right)$ of $v_i^h\left(y_j\right)$ for $y_j \in \tilde{X}, i = 1, 2$, the values $v_i^h\left(y^\sharp\right)$ in (9) are calculated through a collocation mesh-free algorithm using the set of scattered nodes $\tilde{X}$[30], by means of

$$\begin{cases} \tilde{v}_1^h\left(y_j\right) = \max_{c_1\in\mathbb{R}^+}\left\{h\left(U_1\left(y_j\right) - D_1\left(c_1\right)\right) + \left(1 - \rho_1 h\right)\overline{RBF}\left[V_1\right]\left(y_1^\#\right)\right\}, \\ \tilde{v}_2^h\left(y_j\right) = \max_{c_2\in\mathbb{R}^+}\left\{h\left(U_2\left(y_j\right) - D_2\left(c_2\right)\right) + \left(1 - \rho_2 h\right)\overline{RBF}\left[V_2\right]\left(y_2^\#\right)\right\}, \end{cases} \tag{11}$$

with discrete feedback strategies

$$\begin{cases} \tilde{S}_1^h\left(y_j\right) = arg\max_{c_1\in\mathbb{R}^+}\left\{h\left(U_1\left(y_j\right) - D_1\left(c_1\right)\right) + \left(1 - \rho_1 h\right)\overline{RBF}\left[V_1\right]\left(y_1^\#\right)\right\}, \\ \tilde{S}_2^h\left(y_j\right) = arg\max_{c_2\in\mathbb{R}^+}\left\{h\left(U_2\left(y_j\right) - D_2\left(c_2\right)\right) + \left(1 - \rho_1 h\right)\overline{RBF}\left[V_1\right]\left(y_2^\#\right)\right\}. \end{cases} \tag{12}$$

where

$$\begin{cases} y_1^\sharp = y_j + hf\left(y_j, c_1, \tilde{S}_2^h\left(y_j\right)\right) + \sigma\left(y_j\right)\left(\pm\sqrt{h}\right), \\ y_2^\sharp = y_j + hf\left(y_j, \tilde{S}_1^h\left(y_j\right), c_2\right) + \sigma\left(y_j\right)\left(\pm\sqrt{h}\right), \end{cases}$$

and $\overline{RBF}[V_i]$, $i = 1, 2$, denoting the average of the i-th value function's approximation by radial basis functions[31]. Specifically, for $y^\sharp$ that does not belong to $\tilde{X}$, we have, for $i = 1, 2$,

$$\tilde{v}_i^h\left(y_{i,d}^\#\right) \approx \overline{RBF}\left[\mathrm{V}_i\right] = \frac{1}{2}\sum_{d=1}^{2}\sum_{j=1}^{Q}\lambda_{i,j}\Phi\left(\left\|y_{i,d}^\# - y_j\right\|\right),$$

where

$$y_{i,d}^\sharp = y_j + hf\left(y_j, \left[c_1, c_2\right]\right) \pm \sigma\left(y_j\right)\sqrt{h},$$

and $\lambda_{i,j} \in \mathbb{R}$ are weighting coefficients, with $\Phi\left(\left\|y - y_j\right\|\right) = \exp\left(-\frac{\|y-y_j\|^2}{\sigma^2}\right)$, and $\sigma > 0$ (see[30]). In addition, for $i = 1, 2$, and $j = 1, ..., Q$, the parameters $\lambda_{i,j}$, are obtained by solving

$$\mathrm{A}\bar{\lambda}_i = \mathrm{V}_i,$$

where A is the matrix with entries $\mathrm{A}_{j,l} = \Phi\left(\left\|y_l - y_j\right\|\right)$ for $j = 1, ..., Q$, and $\bar{\lambda}_i = \left[\lambda_{i,1}, ..., \lambda_{i,Q}\right]^T$.

Algorithm pseudocode

The algorithm to produce a solution of the discretized problem of the previous section is called RaBVItG, which refers to Radial Basis approximations, Value Iteration and Game Iteration. It essentially consists of two main loops: game iteration, to find a Nash Equilibrium for a given value function, and value iteration, to improve the approximation of the value function, given a previously obtained equilibrium. Both iterations are sequentially interspersed until convergence is reached. We provide the details below.

Let V_i and C_i be $Q-$dimensional arrays of real values for the value functions and for the effort controls of each partner $i = 1, 2$ evaluated at the points $y_j \in \tilde{X}$:

$$\mathrm{V}_i = \left[\tilde{v}_i^h\left(y_1\right), ..., \tilde{v}_i^h\left(y_Q\right)\right]^T,\ \mathrm{C}_i = \left[\tilde{c}_i^h\left(y_1\right), ..., \tilde{c}_i^h\left(y_Q\right)\right]^T,\ i = 1, 2.$$

Let $\mathrm{V} = \left[\mathrm{V}_1, \mathrm{V}_2\right]$ and $\mathrm{C} = \left[\mathrm{C}_1, \mathrm{C}_2\right]$, denote the arrays storing the information for both partners. Let $T_i = \left[T_{i,1}, ..., T_{i,Q}\right] : \mathbb{R}^Q \to \mathbb{R}^Q$ and $G_i = \left[G_{i,1}, ..., G_{i,Q}\right] : \mathbb{R}^Q \to \mathbb{R}^Q$ be two operators defined component-wise by

$$T_{i,j}\left(\mathrm{V}_i\right) = h\left(U_i\left(y_j\right) - D_i\left(c_i\right)\right) + \left(1 - \rho_i h\right)\overline{RBF}\left[\mathrm{V}_i\right]\left(y_j + \delta_d\right),\ j = 1, ..., Q,$$

and

$$G_{i,j}\left(\mathrm{V}_i\right) = arg\max_{c_i \in \mathbb{R}^+}\left\{h\left(U_i\left(y_j\right) - D_i\left(c_i\right)\right) + \left(1 - \rho_i h\right)\overline{RBF}\left[\mathrm{V}_i\right]\left(y_j + \delta_d\right)\right\},\ j = 1, ..., Q, \tag{13}$$

with δ_d, $d = 1, 2$, the set of displacements:

$$\delta_d = hf\left(y_j, \left[c_1, c_2\right]\right) \pm \sigma\left(y_j\right)\sqrt{h}.$$

Next we explain the two main loops of RaBVItG.

1. *Game Iteration.* Given $s = 0, 1, , ...,$ we generate a candidate C_i^{s+1} to optimal control policy at step $s + 1$ for partner i, as follows:

$$\mathrm{C}_i^{s+1} = \theta\mathrm{C}_i^s + (1 - \theta) G_i\left(\mathrm{C}^s, \mathrm{V}_i^r\right),\ i = 1, 2,$$

where G_i is defined in (13), $\theta \in (0, 1)$ is a weighting coefficient –see [32], and V_i^r is defined below. The Game Iteration loop follows the scheme

$$\begin{cases} \tilde{c}_{1,j}^{s+1} \equiv \theta\tilde{c}_{1,j}^s + (1 - \theta)\, arg\max_{c_1 \in \mathbb{R}^+}\left\{h\left(U_1\left(y_j\right) - D_1\left(c_1\right)\right) + \left(1 - \rho_1 h\right)\overline{RBF}\left[\mathrm{V}_1^r\right]\left(y_1^\sharp\right)\right\}, \\ \tilde{c}_{2,j}^{s+1} \equiv \theta\tilde{c}_{2,j}^s + (1 - \theta)\, arg\max_{c_2 \in \mathbb{R}^+}\left\{h\left(U_2\left(y_j\right) - D_2\left(c_2\right)\right) + \left(1 - \rho_2 h\right)\overline{RBF}\left[\mathrm{V}_2^r\right]\left(y_2^\sharp\right)\right\}, \\ y_1^\sharp = y_j + hf\left(y_j, \left[c_1, \tilde{c}_{2,j}^s\right]\right) \pm \sigma\left(y_j\right)\sqrt{h}, \\ y_2^\sharp = y_j + hf\left(y_j, \left[\tilde{c}_{1,j}^s, c_2\right]\right) \pm \sigma\left(y_j\right)\sqrt{h}, \end{cases}$$

for $s = 0, 1, ...,$ where $\tilde{c}_{i,j}^s \equiv \tilde{c}_{i,j}^s\left(y_j\right)$. This scheme is iterated until a convergence criterion is satisfied, that is, $\left\|\mathrm{C}^{s+1} - \mathrm{C}^s\right\| < \epsilon_1$, for a given $\epsilon_1 > 0$ ($\|\cdot\|$ is the Euclidean norm). Given value functions V_i^r, $i = 1, 2$, a candidate for feedback Nash equilibrium is thus obtained:

$$\mathrm{C}^{s+1} = \left[\mathrm{C}_1^{s+1}, \mathrm{C}_2^{s+1}\right].$$

This is the input for the next loop, to produce an estimate of the functions V_i^{r+1}, $i = 1, 2$.

2. *Value Iteration.* Given a candidate C^{s+1} for the feedback Nash equilibrium, obtained from the previous loop, the value functions at step $r + 1$ are updated as follows:

$$\mathrm{V}_i^{r+1} = T_i\left(\mathrm{V}_i^r; \mathrm{C}^{s+1}\right),\ i = 1, 2,$$

where $T_i = \left[T_{i,j}\right]$ are defined component-wise, for $j = 1, ..., Q$, by

$$\begin{cases} T_{1,j} \equiv h\left(U_1\left(y_j\right) - D_1\left(\tilde{c}_{1,j}^{s+1}\right)\right) + \left(1 - \rho_1 h\right)\overline{RBF}\left[\mathrm{V}_1^r\right]\left(y_1^\sharp\right), \\ T_{2,j} \equiv h\left(U_2\left(y_j\right) - D_2\left(\tilde{c}_{2,j}^{s+1}\right)\right) + \left(1 - \rho_2 h\right)\overline{RBF}\left[\mathrm{V}_2^r\right]\left(y_2^\sharp\right), \\ y_1^\sharp \equiv y_2^\sharp = y_j + hf\left(y_j, \left[\tilde{c}_{1,j}^{s+1}, \tilde{c}_{2,j}^{s+1}\right]\right) \pm \sigma\left(y_j\right)\sqrt{h}. \end{cases}$$

This loop is iterated until satisfying the convergence criterion $\left\|\mathrm{V}^{r+1}-\mathrm{V}^{r}\right\|<\epsilon_2$, with $\epsilon_2>0$ given. Candidate solution for the value functions are thus obtained,

$$\mathrm{V}^{r+1}=\left[\mathrm{V}_1^{r+1},\mathrm{V}_2^{r+1}\right].$$

Once the convergence conditions are met, the algorithm generates the outputs

$$\mathrm{V}^{\heartsuit}=\left[\mathrm{V}_1^{\heartsuit},\mathrm{V}_2^{\heartsuit}\right],\mathrm{C}^{\heartsuit}=\left[\mathrm{C}_1^{\heartsuit},\mathrm{C}_2^{\heartsuit}\right]$$

as the computational solutions for the value functions and control policies of the couple's problem. Notice that $\mathrm{V}^{\heartsuit}$ and $\mathrm{C}^{\heartsuit}$ is an approximate fixed-point of the numerical scheme

$$\begin{cases}\mathrm{C}^{s+1}=G(\mathrm{C}^{s},\mathrm{V}^{r},\Delta),\\ \mathrm{V}^{r+1}=T(\mathrm{V}^{r},\mathrm{C}^{s+1},\Delta),\end{cases}$$

where Δ denotes the chosen set of spatio-temporal discretization parameters. Once $\left(\mathrm{C}^{\heartsuit},\mathrm{V}^{\heartsuit}\right)$ are obtained, we can recover the corresponding approximated feedback maps defined in (12).

For the purpose of our model analysis below, we take $f\left(y,\left[c_1,c_2\right]\right)=-ry+a_1c_1+a_2c_2$, and $\sigma\left(y_j\right)\equiv\sigma$ constant..

4 | NUMERICAL ANALYSIS

We present here the numerical results for the couple's problem defined in section 3 for the functional and parameter input specification given in Table 1. Notice that our model inputs satisfy all model assumptions specified in section 2. Furthermore, it is a convenient choice for the sake of comparison with previous works[23] and[33] where the same set of inputs is considered. The algorithm code has been written and run in MATLAB. The set of parameter values used in the computational experiments below are $h=1/12$, $\epsilon_1=0.001$, $\epsilon_2=0.0001$, $Q=15$, $x\in X=[0,5]$ and $\theta=0.95$.

Table 1 Model inputs: functions and parameters

	r	a_1	a_2	σ	D_i	c_i^*	U_i	ρ_i
Homogamous	-2	1.75	1	1.75	$\frac{(c_i-c_i^*)}{2}$	0.2	$5\ln(x+1)$	0.1
				1.25				
				0.5				
				0				
Heteterogamous		1	1	1.75				
				1.25				
				0.5				
				0				

4.1 | Preliminary analysis: uncertainty effect

In Figures 2 and 3, we show the effort feedback policies and the value (well-being) functions for each partner, for two types of couples, homogamous and heterogamous, respectively. They differ here only in the effort efficiency of each partner, which is represented by a_1 and a_2. Homogamous couples are formed by partners with $a_1=a_2$, otherwise they are heterogamous. Different implications of this asymmetry are discussed in detail in[23]. The effort and value curves in Figures 2 and 3 correspond to different levels of stochasticity, i.e. $\sigma=0.5,1.25,1.75$. The curves corresponding to the deterministic case ($\sigma=0$) are also provided, so our results can be compared with those in[23], where the non-stochastic case is analyzed. It allows us to analyze the impact of stochasticity on effort policies and well-being compared with the benchmark case of a deterministic feeling dynamics.

It follows from the analysis that, as the uncertainty σ about the true state of the relationship increases, both partners' effort curves monotonically shift upwards and their welfare curves (value functions) shift downwards. As a consequence, in the face

Figure 2 Computational feedback analysis of a homogamous couple ($a_1 = a_2 = 1$) at different σ values

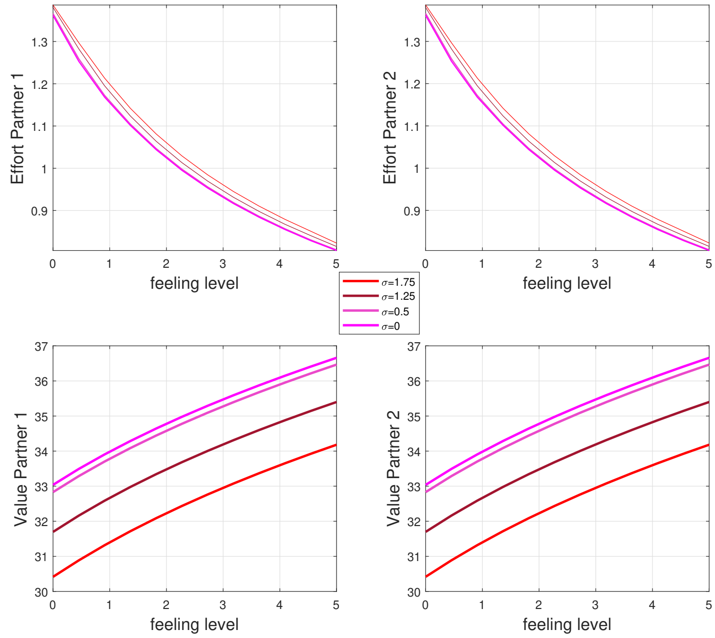
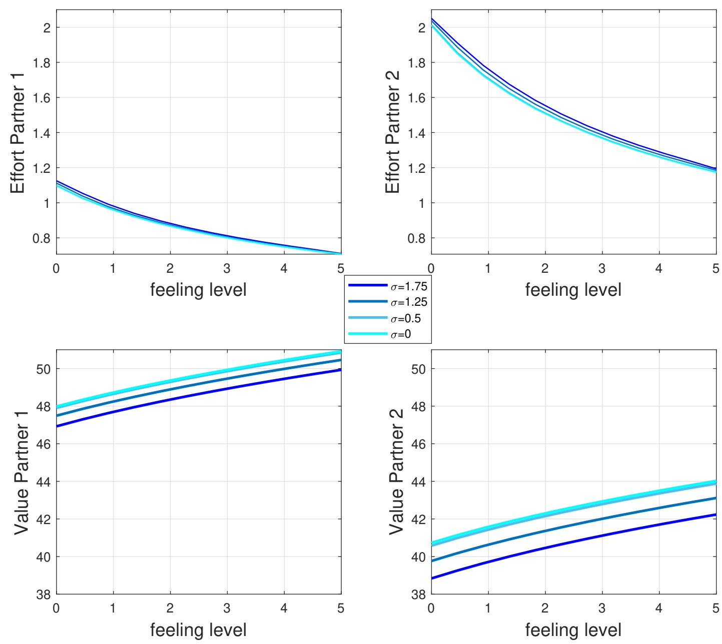


Figure 3 Computational feedback analysis of a heterogamous couple ($a_1 = 1, a_2 = 1.75$) at different σ values



of uncertainty, couples must make more effort and expect less reward in terms of well-being. This qualitative effect holds in general for both homogamous and heterogamous couples, as figures 2 and 3 show.

4.2 | Love at Risk

To assess the probability of breakup of a romantic relationship we now pay attention to the model parameter x_{min}, mentioned in section 1, below which the feeling variable must remain to guarantee a sufficiently rewarding relationship. This is the threshold feeling level for the relationship to start facing a risk of breakup (see Figure 1). This parameter can be thought of as a value at risk, which is defined in finance as a probabilistic measure of incurring a given loss[24]. In a similar fashion, for a given probability $\alpha \in (0, 1)$, we define the value *Love at Risk* (LaR) (at a certain time $k > 0$) as the feeling value x_{min} such that

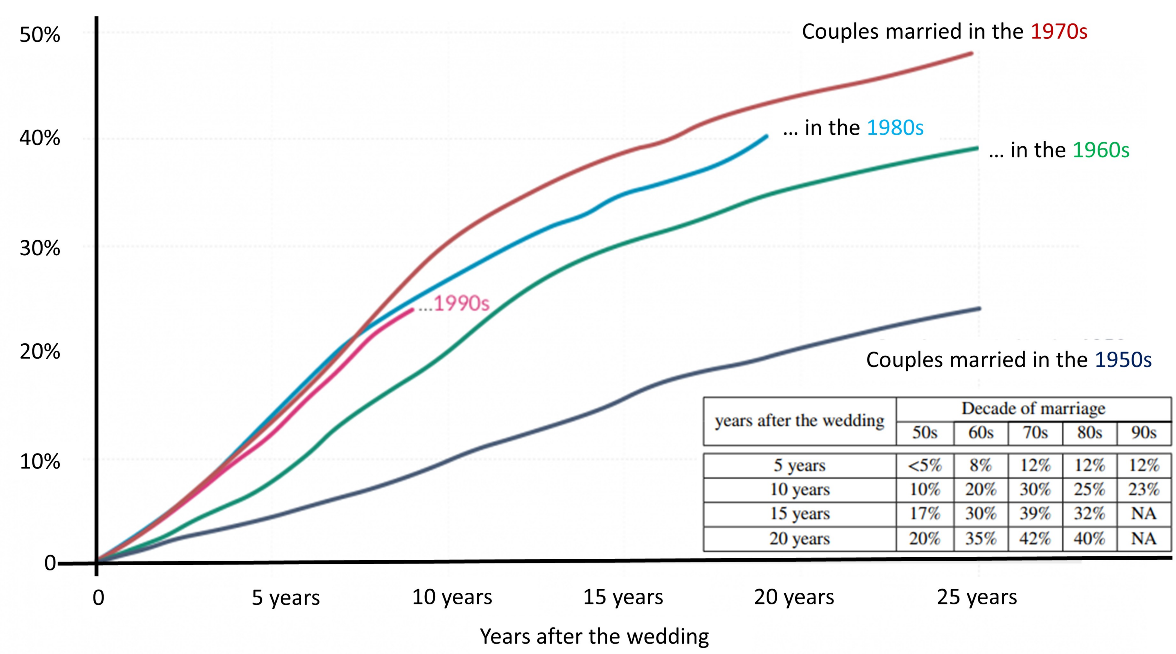
$$\mathbb{P}\left(x_k^{\heartsuit} \le x_{min}\right) = \alpha$$

where $x_k^{\heartsuit}$ is the (optimal) solution of the computational couple's problem defined in section 3, and $\mathbb{P}$ is its probability function, so that x_{min} is the α-percentile of the distribution of $x_k^{\heartsuit}$.

In order to illustrate our methodology, we consider realistic estimates of the probability of divorce in the US. They are shown in Figure 4, where different values of $\alpha = \alpha(k)$ are given, for different cohorts of marriages, k months after the wedding, for $k = 60, 120, 180, 240$.

Without loss of generality, we consider a heterogamous marriage, as specified in Table 1, which may be facing a certain uncertainty σ in the their feeling dynamics (1). To estimate the Love at Risk value for such kind of marriage, five years after the wedding, we proceed as follows. We compute a large sample of realizations of the optimal solution $x^{\heartsuit}(k)$ for the computational stochastic model in section 3, for $\sigma = 0, 0.5, 1.25, 1.75$. Given that the time variable k in our computational model corresponds to months, we generate an estimate of the probability densities of the random variable $x^{*}(60)$ for the different values of σ. They are displayed in Figure 5. According to Figure 4, $\alpha(k = 60) \approx 0.10$, on average, over the marriage cohorts. The LaR level at five years can thus be estimated as the first decile of the feeling distribution corresponding to each σ value in Figure 5.

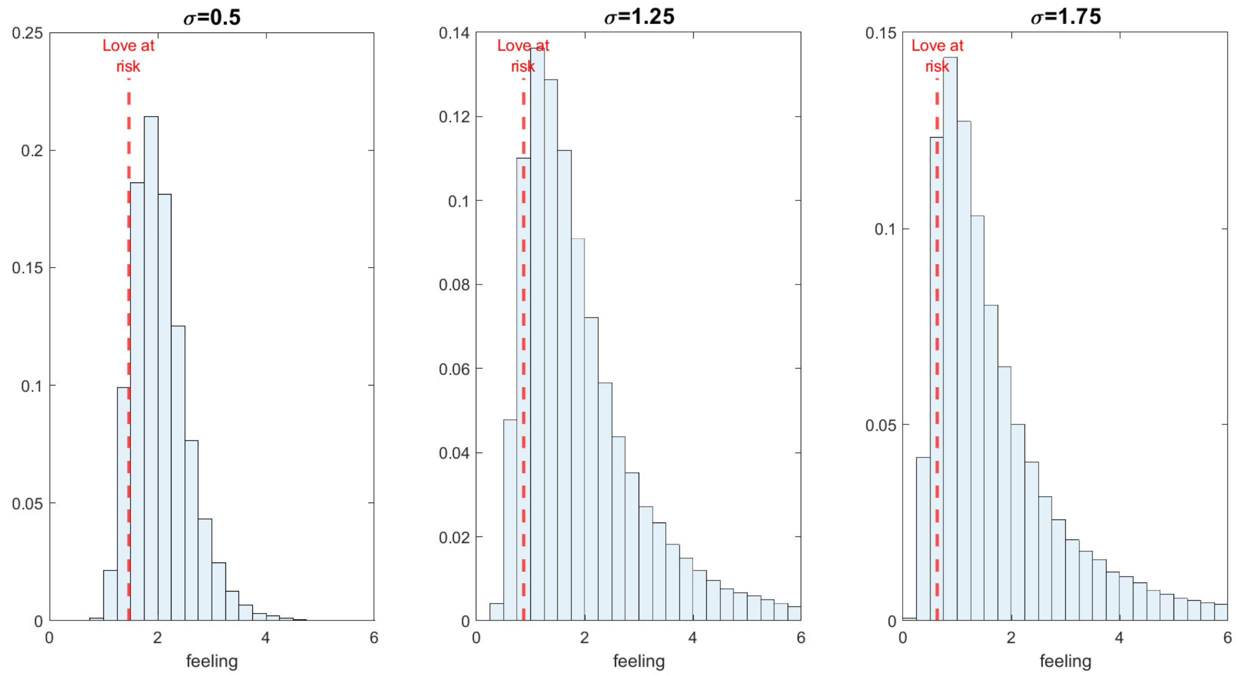
Figure 4 Share of marriages ending in divorce in the US: percentage of straight couples who divorced after a given number of years of marriage (Source: Our World in Data and[34]).



years after the wedding	Decade of marriage				
	50s	60s	70s	80s	90s
5 years	<5%	8%	12%	12%	12%
10 years	10%	20%	30%	25%	23%
15 years	17%	30%	39%	32%	NA
20 years	20%	35%	42%	40%	NA

In general, the LaR level fluctuates with the type of couple, the time after the wedding and the noise term in the feeling dynamics. For the heterogamous couple under consideration here, it is apparent from Figure 5 that the LaR level after five years decreases as σ increases.

Figure 5 Love at Risk for a heterogamous couple with $a_1 = 1, a_2 = 1.75$ at $k = 60$ at different σ values. Empirical densities are obtained from a sample of 10 000 feeling trajectories.



Notice that both the probability estimates of rupture $\alpha = \alpha(k)$ in the US, given in Figure 4, and the distribution of the (controlled) feeling variable $x^{\heartsuit} = x_k^{\heartsuit}$ of the couple's problem vary with k. As a consequence, the LaR level $x_{min} = x_{min}(k)$ also varies with k, and it can be estimated in a dynamic fashion using our computational model. To obtain the sequence of LaR levels $x_{min}(k)$ for the heterogamous couple under study and for the different levels of uncertainty σ, we proceed as follows. Given σ, we generate the distribution of the feeling $x_k^{\heartsuit}$ for each k from a sample of 10 000 realizations of the following stochastic numerical scheme obtained in section 3:

$$\text{(SM1)} \quad \begin{cases} c_{i,k} = \tilde{S}_i^h\left(x_k\right),\ i = 1, 2, \\ x_{k+1} = x_k + hf(x_k, c_{1,k}, c_{2,k}) + \sqrt{h}\sigma\xi_k \\ x_0 \in X. \end{cases}$$

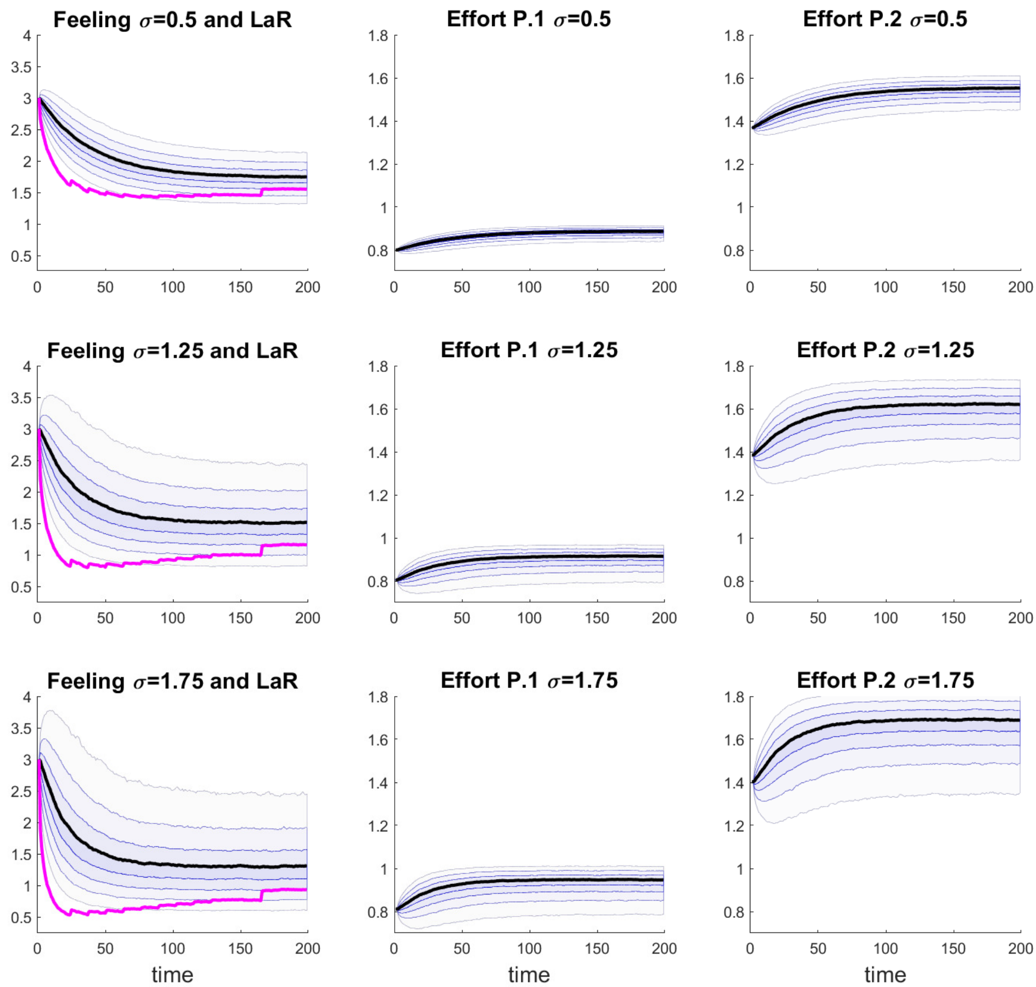
Notice that the scheme above defines a stabilization mechanism for the relationship, since the control policies defined by the stochastic feedback Nash maps allow partners to react optimally to perturbations of the feeling at any time. Once the distribution $x_k^{\heartsuit}$ is simulated, we estimate the sequence of LaR levels at time k, $x_{min}(k)$, from condition $\mathbb{P}(x_k^{\heartsuit} \le x_{min}) = \alpha(k)$, where the probability values $\alpha(k)$ are obtained from the data source of Figure 4.

In figure 6 we show the simulation of the model above for an initial value $x_0 = 3$. The figure displays the different percentile trajectories (from 10 to 80) of the feeling variable $x_k^{\heartsuit}$ for the different levels of σ, as well as the corresponding effort trajectories of each partner. The curve in pink corresponds to the dynamic LaR levels estimated by the computational model. As in the static exercise above ($k = 60$), it can be seen that, for every $k > 0$, the LaR curves are convex, monotonically decreasing as σ increases, and they eventually approach a stationary value.

4.3 | Estimating the probability of breakup in the face of a shock

Regarding the odds of survival of a relationship whose evolution is described by our control model, we analyze how the couple react optimally in the face of a shock. This is a relevant question since relationships are subjected to external shocks over the life course (see e.g.[35]). Notice that the feedback control mechanism provided by our analysis in section 3 is particularly useful here, since it allows partners to adjust their effort levels after a perturbation of the feeling to drive it back to a successful path.

Figure 6 Simulations for the stochastic process of the feeling for different σ values, together with the corresponding effort distributions for both partners of a heterogamous couple with $a_1 = 1$ and $a_2 = 1.75$. The dynamic LaR level is plotted in pink. Values of $\alpha(k)$ are approximated using data from Table (2). Trajectories corresponding to percentiles 10 to 80 are also represented.

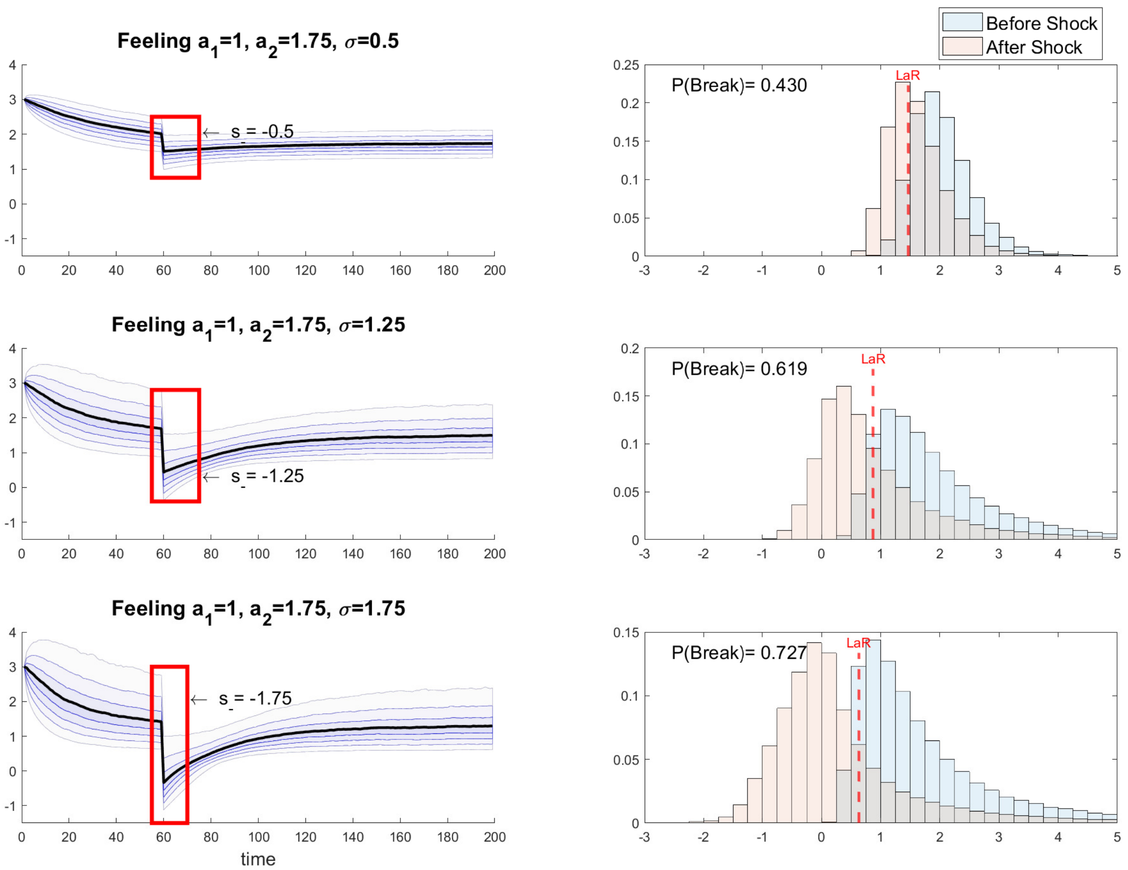


We address the shock problem by estimating the change in the probability of breakup after a shock of the feeling occurs at a given time $k > 0$. Assume that the feeling is affected by a certain sequence of punctual shocks $s_- = \{s_k\}_{k\geq 0}$. Then the stabilization mechanism provided by the feedback analysis reads as follows

$$\text{(SM2)} \begin{cases} c_{i,k} = \tilde{S}_i^h\left(x_k\right),\ i = 1,2, \\ x_{k+1} = x_k + hf(x_k, c_{1,k}, c_{2,k}) + \sqrt{h}\sigma\xi_k - s_-, \\ x_0 \in X. \end{cases}$$

Even though the stabilization mechanism SM2 is working, the perturbed feeling trajectory may enter the zone of risk of rupture at a certain moment k (that is, below the level of LaR $x_{min}(k)$) with some probability, and then remain within the risk zone for some time. This is a critical period that can be painful or even impossible to get through and can make the relationship break up eventually. Thus the probability that a perturbed trajectory controlled by SM2 spends a certain period below the dynamic LaR curve -see Figure 6- serves as a measure of the risk to the survival of the relationship. This probability can be estimated from an ensemble of realizations of the process governed by SM2.

Figure 7 Left: Feedback response to a one-period shock of size $s\_$- proportional to σ five years after the wedding ($k = 60$) for a heterogamous couple with $a_1 = 1, a_2 = 1.75$, and for $\sigma = 0.5, 1.25, 1.75$. Feeling trajectories are obtained using the numerical scheme SM2. Right: Empirical distribution for the feeling values obtained from an sample of 1000 trajectories before and after the shock for the different values of σ. LaR values correspond to the unperturbed process as shown in Figure 5.



To illustrate the probability estimate described above, consider the case that $s\_$ consists of a large one-period shock (of size σ) taking place at time $k = 60$ (five years after the wedding). In Figure 7 (left) we show the percentile trajectories of the stochastic process steered by the stabilization mechanism SM2 for a particular heterogamous couple and for different values of σ. They coincide with the corresponding trajectories of Figure 6 before the shock at $k = 60$ for every σ value. Computing a large ensemble of trajectories, we produce an estimate of the distribution of the feeling values for the perturbed process over a whole year ($k = 60, ..., 72$) after the shock at $k = 60$. In Figure 7 (right) we show the empirical distributions of the feeling variable before the shock and over one year after the shock. Using the the LaR level at $k = 60$ as the benchmark, the probability of breakup over a year after the shock can be estimated from the empirical distribution after the shock (in pink) for different values of σ. As shown in Figure 7 (right), given that the shock at $k = 60$ has the size of the uncertainty σ, the probability of breakup over the year after the event increases as σ increases.

We also analyze how the probability of breakup after a shock varies with respect to the size of the shock and the uncertainty of the feeling dynamics. For the same type of heterogamous couple considered above, Table 2 shows the probabilities of breakup for different values of σ and different sizes of a one-period shock occurring five years after the wedding. Our estimates show that, for any level σ, the probability of breakup increases as the size of the shock increases. Also, a higher level of uncertainty entails a lower LaR level and, in addition, it makes more likely that the level of feeling remains in the secure zone (i.e. over x_{min}) for the relationship.

Table 2 Probability of breakup of a heterogamous relationship with $a_1 = 1, a_2 = 1.75$ for different uncertainty levels σ and different shocks s_-, five years after the wedding ($k = 60$). The simulation is obtained using the scheme SM2.

σ	x_{min}	s_-	$\mathbb{P}\left(break\|s_-,\sigma\right)$
0.5	1.51	-0.1	0.1488
		-0.5	0.4304
		-1.25	0.8636
		-1.75	0.9593
1.25	1.02	-0.1	0.1334
		-0.5	0.3080
		-1.25	0.6195
		-1.75	0.7505
1.75	0.77	-0.1	0.1339
		-0.5	0.3135
		-1.25	0.6114
		-1.75	0.7266

5 | CONCLUSIONS

In this article we have introduced an algorithm to find feedback Nash equilibria for a class of stochastic differential games. The algorithm builds on a combination of two fixed point iterations, a first one to find the Nash equilibrium by fixing the value of the game, and a second iteration to find the value of the game given a Nash equilibrium. The algorithm can be applied to a general class of N-player infinite horizon stochastic games. We have also considered a substantial issue in the applied sciences, namely the design of a long-term rewarding romantic relationship. We formulate this problem as a two-person optimal control problem to steer the feeling of the relationship in a stochastic environment. The algorithm allows us to find approximate solutions of a computational version of the control problem for different stochastic dynamics. In particular, we have focused on estimating the risk of breakup of a long-term relationship at a certain time after the initial commitment. Using divorce data in the US, the proposed algorithm gives an estimate of the feeling level below which the relationship can probably break up -called Love at Risk here-. Also, the computational model allows us to estimate the probability of breaking up in the face of an external shock. The analysis can be applied to different types of couples and different levels of stochasticity in the feeling dynamics.

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