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Approximate controllability of time varying measure differential problem of second order with state-dependent delay and non-instantaneous impulse

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Summary

It is comprehended that the systems without any limitation on their Zeno action are enthralled in a vast class of hybrid systems. This article is influenced by a new category of non-autonomous second order measure differential problems with state-dependent delay (SDD) and non-instantaneous impulse (NII). Some new sufficient postulates are created that guarantee solvability and approximate controllability. We employ the fixed point strategy and theory of Lebesgue–Stieltjes integral in the space of piecewise regulated functions. The measure of non-compactness is applied to establish the existence of a solution. Moreover, the measured differential equations generalize the ordinary impulsive differential equations. Thus, our findings are more prevalent than that encountered in the literature. At last, an example is comprised that exhibits the significance of the developed theory.

KEYWORDS:
Measure differential equations, Regulated functions, Non-autonomous second order systems, Lebesgue–Stieltjes integral, Approximate controllability

1 | INTRODUCTION

An evolution equation that describes time-dependent procedures is anointed as a non-autonomous equation. Non-autonomous differential equations emerge in modeling for numerous phenomena which arise in different branches of applied sciences. Thus for more authentic outcomes one must assume that the differential operators depend on the time. The theory of existence of a solution and controllability for the time varying second order differential systems has become the point of discussion among the leading researchers. The existence of solutions to the abstract Cauchy problem described by the non-autonomous second order differential system is primarily associated with the notion of an evolution operator generated by the family $\{A(\sigma) \mid \sigma \in \mathcal{J}\}$ (see Kozak[1], Henríquez[2] and Henríquez et al.[3] among others). One can see Tanabe's book[4] for additional fundamental concepts on such issue.

Impulsive differential equations (IDEs) are raised in the abstract formulation of a physical problem if there are finite number of discontinuities or impulse appear in the finite time period. However, the situation in which infinite number of discontinuities appeared in the limited time period can not be handled by the IDEs. Thus to handel the situation of the Zeno behavior we switch to the measure differential equations (MDEs). Initially, the MDEs were examined by Schmaedeke[5], Sharma[6], and Pandit and Deo[7]. In the literature, it is common to use the fixed point theorem (FPT) to analyze the presence of a solution. Cao and Sun[8]

[0]**Abbreviations:** SDD, state-dependent delay; NII, non-instantaneous impulse; IDEs, Impulsive differential equations; MDEs, measure differential equations; FPT, fixed point theorem

elaborated existence criterion for nonlinear measure differential systems. Further, Cao and Sun[9] used the concept associated to measure of noncompactness and developed sufficient conditions for the existence of a solution to nonlinear MDEs in the space of regulated functions. Kumar and Agarwal[10] proved the existence of a solution for MDEs governed by non-autonomous first order differential systems via the Schauder FPT.

Controllability is a part of qualitative analysis of a dynamical system. This concept was introduced in 1963[11] and carry forwarded by several researchers due to its applications in science and engineering. For infinite-dimensional dynamical control systems, exact and approximate controllability are studied more frequently. However, for such systems, Triggiani[12] pointed out that the control function is not invertible if the operator associated with the unperturbed component is compact. Thus, approximate controllability grab more attention than the exact controllability. Approximate controllability implies that there is a control map that leads the system from an initial point to a neighborhood of the specified state in a finite time duration. The resluts on approximate controllability for time varying dynamical systems is limited. Nagaraj et al.[13] developed approximate controllability for a time-varying differential system governed by second order via the Leray–Schauder nonlinear alternative. In[14,15] the authors utilized Schauder's FPT to establish approximate controllability of a second order non-autonomous systems having finite delay and NII. Zhao et al.[16] discussed approximate controllability for non-autonomous second order evolution hemivariational inequality problem with nonlocal condition. Singh et al.[17] examined approximate controllability to an impulsive control system governed by second order non-autonomous stochastic differential problem. Vijaykumar et al.[18] developed some sufficient conditions for approximate controllability of non-autonomous second order nonlocal neutral differential inclusions.

On the other side, Wan and Sun[19] studied approximate controllability for abstract MDEs in Banach space setting. In 2018, Cao and Sun[20] extended the results for approximate controllability of nonlinear MDEs via Schauder's FPT. Kumar and Abdal[21] discussed some sufficient conditions which grantee the existence of a solution and approximate approximate controllability for MDEs with NII and infinite delay. Recently, Gou and Li[22] established sufficient conditions for approximate controllability of MDEs with nonlocal conditions. Kumar and Abdal[23] examined approximate controllability for non-autonomous second order MDEs involving SDD and nonlocal conditions. In practice, it is worthwhile to include SDD because the modeling of physical marvels may rely on the previous states (see[24,25,26,27]).

However, to the best of our knowledge, there have not been any results concerning approximate controllability of measure differential problems with SDD and NII in Banach spaces. This article will refill this existing gap and extend the discussion for approximate controllability of measuring differential problems with SDD and NII. That is, we will furnish approximate controllability conditions of the concerned system via the Hausdorff measure of non-compactness and Mönch's FPT. The results are developed in the space of piecewise regulated functions because solutions of impulsive MDEs are piecewise regulated. The results of this article will be significant for the impending examination.

The purpose of this article is to examine the solvability and approximate controllability of the following non-autonomous second order MDE with SDD and NII described by the system:

$$\begin{cases} d\xi'(\sigma) = [A(\sigma)\xi(\sigma) + C\mu(\sigma)]d\sigma + \aleph(\sigma, \xi_{\ell(\sigma,\xi_\sigma)})d\lambda(\sigma), \ \sigma \in \bigcup_{k=1}^{m}(r_k, \tau_{k+1}] \subset \mathcal{J} := [0,T], \\ \xi(\sigma) = \alpha_k(\sigma, \xi(\tau_k^-)), \ \sigma \in (\tau_k, r_k], \ 1 \le k \le m, \\ \xi'(\sigma) = \beta_k(\sigma, \xi(\tau_k^-)), \ \sigma \in (\tau_k, r_k], \ 1 \le k \le m, \\ \xi_0 = \psi \in \mathscr{B}, \ \xi'(0) = \xi_1, \end{cases} \tag{1}$$

where $\xi(\cdot)$ is the state variable with values in the Hilbert space $\mathcal{E}$; $\mu(\cdot) \in L^2(\mathcal{J}; \mathcal{U})$ is the control variable, and $\mathcal{U}$ is another Hilbert space; $C : \mathcal{U} \to \mathcal{E}$ is a continuous linear operator; $\aleph : \mathcal{J} \times \mathscr{B} \to \mathcal{E}$, $\ell : \mathcal{J} \times \mathscr{B} \to (-\infty, T]$ are suitable functions; λ is continuous from the left and nondecreasing and has the distributional derivative $d\lambda$[28]; $0 = \tau_0 < r_0 < \tau_1 < r_1 < \cdots < r_m < \tau_{m+1} = T$; $\xi(\tau_k^-)$ denotes the left limit of $\xi(\sigma)$ at $\sigma = \tau_k^-$; α_k, $\beta_k : (\tau_k, r_k] \times \mathcal{E} \to \mathcal{E}$ represent NII for $1 \le k \le m$. Define the history map $\xi_\sigma(\theta) = \xi(\sigma + \theta)$, $\theta \le 0$ for any continuous function ξ specified on $(-\infty, T]$, being $\xi_\sigma \in \mathscr{B}$.

The major findings, significance, and originality are presented downward:

- A set of new sufficient conditions is constructed that grantees the existence of a solution and approximate controllability for System (1).
- In the literature, it is ordinary to assume the uniform boundedness of the nonlinear map during the examination of approximate controllability. We replace this condition with a weaker assumption (see (P6)).
- The theory of measure of non-compactness is applied to show the presence of a solution via the Mönch FPT.

- Impulsive differential equations are analyzed in the space of piecewise continuous functions. Analogously, to negotiate with MDEs involving NII, the space of piecewise regulated functions is mandated.
- The integral of the map $\aleph$ is assumed in the Bochner–Stieltjes sense. Therefore, our developments evolve accessorily more advanced than those existing in the publications.
- The outcomes of this paper generalize the existing results in the liturature for time varying differential systems of second order and MDEs [23].
- An example is also constructed to demonstrate our findings.

The remaining part is framed as: Some fundamental definitions and results are documented in Section 2. We analyze the solvability and approximate controllability of the considered systems in Section 3. The presence of a solution is verified by employing the Mönch FPT. An illustration is formed to demonstrate our major results in Section 4.

2 | PRELIMINARIES

We start this section by listing the definition of phase space and interconnected results. Next, we through light on the concepts of regulated functions and equiregulated set. We also give some other definitions and useful results for further development. We then put few presumptions on $\{A(\sigma) \mid \sigma \in \mathcal{J}\}$ which guarantee the presence of a unique linear evolution family $\{\Theta(\sigma,\tau) \mid 0 \leq \tau \leq \sigma \leq T\}$. Finally, a mild solution for System (1) is specified. We also provide the expression of resolvent operator.

Let $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ be the linear complete seminormed space of mappings $\xi : (-\infty, 0] \to \mathcal{E}$, and the following axioms hold (see [29,30]):

(A) If $\xi : (-\infty, \sigma+\tau] \to \mathcal{E}$, $\sigma \in \mathbb{R}$ and $\tau > 0$, is such that $\xi_\gamma \in \mathscr{B}$ and $\xi|_{[\sigma,\sigma+\tau]} \in C([\sigma,\sigma+\tau];\mathcal{E})$ then for any $s \in [\sigma,\sigma+\tau]$, we have

(i) $\xi_s \in \mathscr{B}$.

(ii) There exists $b \geq 0$ satisfying $\|\xi(s)\| \leq b\|\xi_s\|_{\mathscr{B}}$.

(iii) There is a continuous map $K : \mathbb{R}_+ \to [1,\infty)$, and a locally bounded map $M : \mathbb{R}_+ \to [1,\infty)$ with the property

$$\|\xi_\sigma\|_{\mathscr{B}} \leq K(\sigma - s)\sup\{\|\xi(r)\| \mid s \leq r \leq \sigma\} + M(\sigma - s)\|\xi_s\|_{\mathscr{B}},$$

$K(\cdot)$, $R(\cdot)$ and b does not depend on $\xi(\cdot)$.

(B) For $\xi(\cdot)$ in (A), $s \mapsto \xi_s$ is a $\mathscr{B}$-valued continuous map on $[\sigma,\sigma+\tau]$.

We will denote by $K_T := \max\limits_{s\in\mathcal{J}} K(s)$ and $M_T := \sup\limits_{s\in\mathcal{J}} R(s)$.

Set $\mathrm{R}(\ell^-) = \{\ell(\sigma,\psi) \mid (\sigma,\psi) \in \mathcal{J}\times\mathscr{B},\ \ell(\sigma,\psi) \leq 0\}$. Suppose that $\ell : \mathcal{J}\times\mathscr{B} \to (-\infty, T]$ is continuous and the following hypothesis holds:

(H_ψ) The map $\sigma \mapsto \psi_\sigma$ is continuous from $\mathrm{R}(\ell^-)$ into $\mathscr{B}$, and there is a bounded and continuous function $L^\psi : \mathrm{R}(\ell^-) \to \mathbb{R}^+$ satisfying

$$\|\psi_\sigma\|_{\mathscr{B}} \leq L^\psi(\sigma)\|\psi\|_{\mathscr{B}}, \text{ for every } \sigma \in \mathrm{R}(\ell^-).$$

Denote by $L^\psi := \sup\{L^\psi(\sigma) \mid \sigma \in \mathrm{R}(\ell^-)\}$. The following result will be required in computation:

Lemma 1 ([30])**.** Let $\xi : (-\infty, T] \to \mathcal{E}$ be continuous and $\xi_0 = \psi$. If (H_ψ) holds, then

$$\|\xi_\sigma\|_{\mathscr{B}} \leq (L^\psi + M_T)\|\psi\|_{\mathscr{B}} + K_T \sup\left\{\|\xi(\vartheta)\| \mid \vartheta \in [0,\max\{0,\sigma\}]\right\},\ \sigma \in \mathrm{R}(\ell^-)\bigcup\mathcal{J}.$$

Next, we switch to the notion of equiregulated functions and related consequences.

Definition 1. A map $\zeta : [r,s] \to \mathcal{E}$ is called regulated, if both the limits

$$\lim_{\gamma\to\sigma^-}\zeta(\gamma) = \zeta(\sigma^-),\ \sigma \in (r,s], \text{ and } \lim_{\gamma\to\sigma^+}\zeta(\gamma) = \zeta(\sigma^+),\ \sigma \in [r,s)$$

exist. Let $\mathscr{R}([r,s];\mathcal{E}) = \{\zeta \mid [r,s] \to \mathcal{E} \mid \zeta \text{ is regulated}\}$. Then, from Honig [31], $(\mathscr{R}([r,s];\mathcal{E}), \|\cdot\|_\infty)$ is a Banach space under the norm $\|\zeta\|_\infty = \sup\limits_{r\leq\sigma\leq s}\|\zeta(\sigma)\|$.

Lemma 2. ([32, Proposition 3]). For given $\zeta : [r,s] \to \mathcal{E}$, suppose that $\int_r^s \zeta\, d\lambda$ exists for some $\lambda \in \mathscr{R}([r,s];\mathbb{R})$. Then $\Lambda(\sigma) = \int_\tau^\sigma \zeta\, d\lambda$ is regulated for $r \le \tau \le \sigma \le s$.

Definition 2. ([33]). A set $\mathscr{P} \subset \mathscr{R}([r,s];\mathcal{E})$ is said to be equiregulated, if for every $\varepsilon > 0$ and $r_0 \in [r,s]$, there exists $\delta > 0$ satisfying

(a) if $\zeta \in \mathscr{P}$, $\tau \in [r,s]$ and $r_0 - \delta < \tau < r_0$, then $\|\zeta(r_0^-) - \zeta(\tau)\| < \varepsilon$,

(b) if $\zeta \in \mathscr{P}$, $\tau \in [r,s]$ and $r_0 < \tau < r_0 + \delta$, then $\|\zeta(\tau) - \zeta(r_0^+)\| < \varepsilon$.

Lemma 3. ([33]). Let $\{\zeta^l\}_{l\in\mathbb{N}} \subset \mathscr{R}([r,s];\mathcal{E})$ be an equiregulated sequence. If $\zeta^l(\sigma)$ converges to $\zeta_0(\sigma)$ in $\mathcal{E}$ as $l \to \infty$ for every $r \le \sigma \le s$, then ζ^l converges uniformly to ζ_0.

Let X be a bounded subset of $\mathcal{E}$. The Hausdorff measure of non-compactness is defined by

$$\chi(X) = \inf\{\epsilon > 0 \mid X \subset \cup_{i=1}^n B(\xi_i, r_i),\ \xi_i \in X,\ r_i < \epsilon,\ 1 \le i \le n\},$$

and $B(\xi_i, r_i)$ represents the open ball of radius r_i with center at ξ_i.

Lemma 4 ([34])**.** Let X, Y be bounded subset of $\mathcal{E}$ and $k \in \mathbb{R}$. Then

(i) $\chi(X) = 0$ if and only if X is relatively compact.

(ii) $X \subset Y$ implies $\chi(X) \le \chi(Y)$.

(iii) $\chi(\overline{X}) = \chi(X)$, and $\chi(\{\xi\} \cup X) = \chi(X)$ for every $\xi \in \mathcal{E}$.

(iv) $\chi(X \cup Y) = \max\{\chi(X), \chi(Y)\}$.

(v) $\chi(\kappa X) = |k|\chi(X)$, where $kX = \{\zeta = k\xi \mid \xi \in X\}$.

(vi) $\chi(X + Y) \le \chi(X) + \chi(Y)$, where $X + Y = \{\zeta = \eta + \xi \mid \eta \in X,\ \xi \in Y\}$.

(vii) $\chi(co(X)) = \chi(X)$, where $co(X)$ is the convex hull of X.

Lemma 5 ([35])**.** Let $S \subseteq \mathscr{R}(\mathcal{J};\mathcal{E})$ be equiregulated and bounded. Then $\overline{co}(S)$ is also equiregulated and bounded.

Lemma 6 ([36])**.** For every bounded and equiregulated subset S of $\mathscr{R}(\mathcal{J};\mathcal{E})$, the function $\chi(S(\sigma))$ is regulated for every $\sigma \in \mathcal{J}$. Moreover,

$$\chi(S) = \sup\{\chi(S(\sigma)) \mid \sigma \in \mathcal{J}\}.$$

Suppose that the collection of all maps $F : \mathcal{J} \to \mathcal{E}$ which are Lebesgue–Stieltjes integrable with respect to λ is represented by $LS_\lambda(\mathcal{J};\mathcal{E})$. Let ν_λ be the Lebesgue–Stieltjes measure on $\mathcal{J}$ induced by λ. The fact that the Lebesgue–Stieltjes measure is a regular Borel measure yields the accompanying result:

Lemma 7 ([37])**.** Let N_0 be a countable subset of $LS_\lambda(\mathcal{J};\mathcal{E})$ and $a(\cdot) \in LS_\lambda(\mathcal{J};\mathbb{R}^+)$ satisfying $\|\Lambda(\sigma)\| \le a(\sigma)$ ν_λ-a.e. for all $\Lambda \in N_0$. Then the following inequality hold:

$$\chi\left(\int_{\mathcal{J}} N_0(\sigma) d\lambda(\sigma)\right) \le 2\int_{\mathcal{J}} \chi(N_0(\sigma)) d\lambda(\sigma).$$

For the measure λ, the collection of λ-integrable maps is denoted by $L^1_\lambda(\mathcal{J};\mathcal{E})$.

Lemma 8. [38, Corollary 2.6]**.** Suppose that $\mathscr{D} \subset L^1_\lambda(\mathcal{J};\mathcal{E})$ is bounded and there is $\Upsilon(\cdot) \in L^1_\lambda(\mathcal{J};\mathbb{R}_+)$ satisfying $\|\wp(\sigma)\| \le \Upsilon(\sigma)$ λ-a.e. $\sigma \in \mathcal{J}$ for all $\wp \in \mathscr{D}$. If for every $\wp \in \mathscr{D}$, $\wp(\sigma) \in \mathscr{W}(\sigma)$ for λ-a.e. $\sigma \in \mathcal{J}$, where, for every $\sigma \in \mathcal{J}$, $\mathscr{W}(\sigma) \subset \mathcal{E}$ is weakly relatively compact, then $\mathscr{D}$ is weakly relatively compact in $L^1_\lambda(\mathcal{J};\mathcal{E})$.

Finally, we assume the subsequent inhomogeneous differential equation

$$\begin{aligned} \xi''(\sigma) &= A(\sigma)\xi(\sigma) + \tilde{F}(\sigma),\ 0 \le \sigma, \tau \le T, \\ \xi(\tau) &= \xi_0,\ \xi'(\tau) = \xi_1, \end{aligned} \tag{2}$$

where $A(\sigma) : D(A(\sigma)) \subset \mathcal{E} \to \mathcal{E},\ \sigma \in \mathcal{J}$, is a closed dense operator and $\tilde{F} : \mathcal{J} \to \mathcal{E}$ is a suitable mapping[39]. To write the solution of System (2), we first deliberate the existence of an evolution family $\Theta(\sigma,\tau)$ to the homogeneous system

$$\xi''(\sigma) = A(\sigma)\xi(\sigma),\ 0 \le \sigma, \tau \le T. \tag{3}$$

Suppose that the domain D of $A(\sigma)$ does not dependend on σ but it is dense in $\mathcal{E}$ and for each $\xi \in D$, the function $\sigma \mapsto A(\sigma)\xi$ is continuous. Let us give the approach of the evolution operator presented by Kozak[1] as follows:

Definition 3. A family $\Theta(\cdot,\cdot) : \mathcal{J} \times \mathcal{J} \to \mathcal{L}(\mathcal{E})$ is referred to be a fundamental solution of equation (3) if:

$(A1)$ For each $\xi \in \mathcal{E}$, $(\sigma,\tau) \mapsto \Theta(\sigma,\tau)\xi$ from $\mathcal{J} \times \mathcal{J} \to \mathcal{E}$ is of class C^1 and

(i) For every $\sigma \in \mathcal{J}$, $\Theta(\sigma,\sigma) = 0$.

(ii) For every $\sigma, \tau \in \mathcal{J}$, and each $\xi \in \mathcal{E}$,

$$\frac{\partial}{\partial\sigma}\Theta(\sigma,\tau)\xi\bigg|_{\sigma=\tau} = \xi,\quad \frac{\partial}{\partial\tau}\Theta(\sigma,\tau)\xi\bigg|_{\sigma=\tau} = -\xi.$$

$(A2)$ For any $\sigma, \tau \in \mathcal{J}$, $\Theta(\sigma,\tau)\xi \in D(A)$ if $\xi \in D(A)$, and the map $(\sigma,\tau) \mapsto \Theta(\sigma,\tau)\xi$ from $\mathcal{J} \times \mathcal{J} \to \mathcal{E}$ is of class C^2 and

(i) $\frac{\partial^2}{\partial\sigma^2}\Theta(\sigma,\tau)\xi = A(\sigma)\Theta(\sigma,\tau)\xi$.

(ii) $\frac{\partial^2}{\partial\tau^2}\Theta(\sigma,\tau)\xi = \Theta(\sigma,\tau)A(\tau)\xi$.

(iii) $\frac{\partial^2}{\partial\tau\partial\sigma}\Theta(\sigma,\tau)\Big|_{\sigma=\tau}\xi = 0$.

$(A3)$ For all $\sigma, \tau \in \mathcal{J}$, and $\xi \in D(A)$, we have $\frac{\partial}{\partial\tau}\Theta(\sigma,\tau)\xi \in D(A)$. Furthermore, $\frac{\partial^3}{\partial\sigma^2\partial\tau}\Theta(\sigma,\tau)\xi$ and $\frac{\partial^3}{\partial\tau^2\partial\sigma}\Theta(\sigma,\tau)\xi$ exist and

(i) $\frac{\partial^3}{\partial\sigma^2\partial\tau}\Theta(\sigma,\tau)\xi = A(\sigma)\frac{\partial}{\partial\tau}\Theta(\sigma,\tau)\xi$.

(ii) $\frac{\partial^3}{\partial\tau^2\partial\sigma}\Theta(\sigma,\tau)\xi = \frac{\partial}{\partial\sigma}\Theta(\sigma,\tau)A(\tau)\xi$,
and for each $\xi \in D(A)$, the map $(\sigma,\tau) \mapsto A(\sigma)\frac{\partial}{\partial\tau}\Theta(\sigma,\tau)\xi$ is continuous.

For convince, we set $\Lambda(\sigma,\tau) := -\frac{\partial\Theta(\sigma,\tau)}{\partial\tau}$. Also, there exits $\hat{N} > 0$ such that

$$\|\Theta(\sigma+\omega,\tau) - \Theta(\sigma,\tau)\| \le \hat{N}|\omega|$$

for all $\omega, \sigma, \sigma+\omega \in \mathcal{J}$. Thus, for an integrable function $\tilde{F} : \mathcal{J} \to \mathcal{E}$, the mild solution of (2) has the form

$$\xi(\sigma) = \Lambda(\sigma,\tau)\xi_0 + \Theta(\sigma,\tau)\xi_1 + \int_\tau^\sigma \Theta(\sigma,\omega)\tilde{F}(\omega)d\omega.$$

In the literature, several techniques have been operated to demonstrate the existence of the evolution family $\Theta(\sigma,\tau)$. The most common technique is that in which the generator A associated with the cosine family is perturbed to $A(\sigma)$ (see[40,41,42,43]).

Therefore, a solution of System (1) is defined as:

Definition 4. For given $\mu(\cdot) \in L^2(\mathcal{J};\mathcal{U})$, $\xi(\cdot) : (-\infty, T] \to \mathcal{E}$ is called a mild solution of System (1), if $\xi(\sigma) = \psi(\sigma),\ \sigma \in (-\infty, 0]$, $\xi'(0) = \xi_1$, $\xi(\sigma) = \alpha_k(\sigma, \xi(\tau_k^-))$, $\xi'(\sigma) = \beta_k(\sigma, \xi(\tau_k^-))$, $\sigma \in (\tau_k, r_k]$, $1 \le k \le m$ hold and ξ satisfies the following integral equation

$$\xi(\sigma) = \begin{cases} \Lambda(\sigma,0)\psi(0) + \Theta(\sigma,0)\xi_1 + \int_0^\sigma \Theta(\sigma,\omega)C\mu(\omega)d\omega \\ + \int_0^\sigma \Theta(\sigma,\omega)\aleph(\omega,\xi_{\ell(\omega,\xi_\omega)})d\lambda(\omega),\ \sigma \in [0,\tau_1], \\ \Lambda(\sigma,r_k)\alpha_k(r_k,\xi(\tau_k^-)) + \Theta(\sigma,r_k)\beta_k(r_k,\xi(\tau_k^-)) + \int_{r_k}^\sigma \Theta(\sigma,\omega)C\mu(\omega)d\omega \\ + \int_{r_k}^\sigma \Theta(\sigma,\omega)\aleph(\omega,\xi_{\ell(\omega,\xi_\omega)})d\lambda(\omega),\ \sigma \in \cup_{k=1}^m [r_k,\tau_{k+1}]. \end{cases}$$

Definition 5. The System (1) is called approximately controllable on $\mathcal{J}$ if $\overline{\mathfrak{R}(T)} = \mathcal{E}$, where $\mathfrak{R}(T) = \{\xi(T,\mu) \mid \mu(\cdot) \in L^2(\mathcal{J};\mathcal{U})\}$ and $\xi(T,\mu)$ represents a solution of System (1) at $\sigma = T$.

For every $\gamma > 0$, we set

$$\mathcal{R}(\gamma, -\Upsilon_{r_k}^{\tau_{k+1}}) = (\gamma I + \Upsilon_{r_k}^{\tau_{k+1}})^{-1},\ 0 \leq k \leq m,$$

where $r_0 = 0$ and $\tau_{m+1} = T$, and

$$\Upsilon_{r_k}^{\tau_{k+1}} = \int_{r_k}^{\tau_{k+1}} \Theta(\tau_{k+1}, s) CC^* \Theta^*(\tau_{k+1}, s) ds,\ 0 \leq k \leq m,$$

Θ^* and C^* are the adjoint of Θ and C, respectively. Observe that for each k, the operator $\Upsilon_{r_k}^{\tau_{k+1}}$ is positive. Thus, $\mathcal{R}(\gamma, -\Upsilon_{r_k}^{\tau_{k+1}})$ is well defined.

Introduce the space of piecewise regulated functions as

$$\mathcal{P}_R(\mathcal{J}; \mathcal{E}) = \{\xi : \mathcal{J} \to \mathcal{E} \mid \xi \in \mathscr{R}(\cup_{k=0}^{m}(r_k, \tau_{k+1}]; \mathcal{E}),\ \text{and } \xi(r_k^-),\ \xi(r_k^+) \text{ exist with } \xi(r_k^-) = \xi(r_k),\ 1 \leq k \leq m\}.$$

Obviously, $(\mathcal{P}_R(\mathcal{J}; \mathcal{E}); \|\cdot\|_\infty)$ is a Banach space, where $\|\xi\|_\infty := \sup_{\sigma \in \mathcal{J}} \|\xi(\sigma)\|$.

Lemma 9 ([44], Theorem 2.2)**.** Let $\mathcal{X}$ be a Banach space, $\mathcal{Y} \subset \mathcal{X}$ open, $0 \in \mathcal{Y}$, and $P : \overline{\mathcal{Y}} \to \mathcal{X}$ is continuous. Moreover, P satisfies the Mönch's conditions:

(i) $S \subset \overline{\mathcal{Y}}$ is countable, $S \subset \overline{co}(\{0\} \cup P(S))$ imply S is relatively compact.

(ii) Boundary condition: $\xi \in \overline{\mathcal{Y}}$, $\kappa \in (0, 1)$, $\xi = \kappa P(\xi)$ imply $\xi \notin \partial \mathcal{Y}$.

Then there exists an element $\zeta \in \overline{\mathcal{Y}}$ such that $P(\zeta) = \zeta$.

3 | MAJOR DEVELOPMENTS

This section is reserved for establishing the solvability and approximate controllability of the measure-driven control system involving SDD and NII. We require the following assumptions to produce our major results:

(P1) The evolution operator $\Theta(\sigma, \omega),\ \omega \leq \sigma$ is compact.

(P2) The function $\aleph : \mathcal{J} \times \mathscr{B} \to \mathcal{E}$ satisfies:

- (i) Let $\xi : (-\infty, T] \to \mathcal{E}$ be such that $\xi_0 = \psi$ and $\xi|_{\mathcal{J}} \in \mathscr{R}(\mathcal{J}; \mathcal{E})$. Furthermore, for each $\psi \in \mathcal{E}$, the map $\sigma \mapsto \aleph(\sigma, \psi)$ is ν_λ-measurable on $\mathcal{J}$, and $\psi \mapsto \aleph(\cdot, \psi)$ is continuous.
- (ii) There is a map $g(\cdot) \in LS_\lambda(\mathcal{J}; \mathbb{R}_+)$ and a continuous nondecreasing map $h : \mathbb{R}_+ \to \mathbb{R}^+$ such that for each $\psi \in \mathscr{B}$
$$\|\aleph(\sigma, \psi)\| \leq g(\sigma) h(\|\psi\|_{\mathscr{B}}),\ \text{for } \nu_\lambda\text{-a.e. } \sigma \in \mathcal{J}.$$
- (iii) For every bounded subset $S \subset \mathcal{E}$, $\chi(\aleph(\sigma, S)) \leq g_\aleph(\sigma)\chi(S)$, for a.e. $\sigma \in \mathcal{J}$, for some $g_\aleph(\cdot) \in LS_\lambda(\mathcal{J}; \mathbb{R}^+)$.

(P3) For each $1 \leq k \leq m$, the map $\alpha_k : (\tau_k, r_k] \times \mathcal{E} \to \mathcal{E}$ is continuous. Also, for each $1 \leq k \leq m$, there exist constants $\hbar_{\alpha_k}$, $\tilde{\hbar}_{\alpha_k} > 0$ such that

$$\|\alpha_k(\sigma, \xi) - \alpha_k(\tilde{\sigma}, \tilde{\xi})\| \leq \hbar_{\alpha_k}[|\sigma - \tilde{\sigma}| + \|\xi - \tilde{\xi}\|],\ \|\alpha_k(\sigma, \xi)\| \leq \tilde{\hbar}_{\alpha_k}(1 + \|\xi\|),\ \sigma \in \bigcup_{k=1}^{m}(\tau_k, r_k] \text{ and } \xi, \tilde{\xi} \in \mathcal{E}.$$

(P4) For each $1 \leq k \leq m$, the map $\beta_k : (\tau_k, r_k] \times \mathcal{E} \to \mathcal{E}$ is continuous. Also, for each $1 \leq k \leq m$, there exist constants $\hbar_{\beta_k}$, $\tilde{\hbar}_{\beta_k} > 0$ such that

$$\|\beta_k(\sigma, \xi) - \beta_k(\sigma, \tilde{\xi})\| \leq \hbar_{\beta_k}\|\xi - \tilde{\xi}\|,\quad \|\beta_k(\sigma, \xi)\| \leq \tilde{\hbar}_{\beta_k}(1 + \|\xi\|),\ \sigma \in \bigcup_{k=1}^{m}(\tau_k, r_k] \text{ and } \xi, \tilde{\xi} \in \mathcal{E}.$$

(P5) For each $0 \leq k \leq m$, $\gamma \mathcal{R}(\gamma, -\Upsilon_{r_k}^{\tau_{k+1}}) \to 0$ as $\gamma \downarrow 0$ in the strong operator topology.

(P6) $\sup\{\|\aleph(\sigma, \xi)\| \mid \xi \in \mathscr{B}\} \leq \mathrm{b}(\sigma)$ for ν_λ-a.e. $\sigma \in \mathcal{J}$, for some $\mathrm{b}(\cdot) \in L^1_\lambda(\mathcal{J}; \mathbb{R}_+)$.

Lemma 10 ([45,46])**.** The next statements are equivalent:

(a) If $B^*\Theta^*(\sigma,\tau)\zeta = 0$ for all $\sigma,\tau \in \mathcal{J}$, then $\zeta = 0$.

(b) Hypothesis (P4) holds.

(c) The linear control system associated with problem (1) is approximately controllable on $\mathcal{J}$.

For given $\xi^d \in \mathcal{E}$ and $\gamma > 0$, define the control map

$$\mu^\gamma(\sigma,\xi) = C^*\Theta^*(\tau_{k+1},\sigma)\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})\big[\xi^{\tau_{k+1}} - \Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi(\tau_k^-)) - \Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi(\tau_k^-)) - \int_{r_k}^{\tau_{k+1}} \Theta(\tau_{k+1},\omega)\aleph(\omega,\xi_{\ell(\omega,\xi_\omega)})d\lambda(\omega)\big],\ 0 \le k \le m, \tag{4}$$

where $\alpha_0(0,\cdot) = \psi(0)$, $\beta_0(0,\cdot) = \xi_1$ and $\xi^{\tau_{m+1}} = \xi^d$.

For further development, we set $\sup\limits_{0\le\sigma,\tau\le T}\|\Theta(\sigma,\tau)\| \le N$, $\sup\limits_{0\le\sigma,\tau\le T}\|\Lambda(\sigma,\tau)\| \le M$, $\|C\| \le c$, and $\left(1 + \frac{c^2N^2T}{\gamma}\right) = \widetilde{N}$.

Theorem 1. Suppose that $(P1)-(P4)$ hold. Then System (1) possesses at least one solution on $(-\infty,T]$ provided

$$\max_{1\le k\le m}\left[\tilde{\hbar}_{\alpha_k} + \widetilde{N}(M\tilde{\hbar}_{\alpha_k} + N\tilde{\hbar}_{\beta_k})\right] + N\widetilde{N}\liminf_{\kappa\to+\infty}\frac{h(\kappa)}{\kappa}\int_0^T g(\omega)d\lambda(\omega) \le 1, \tag{5}$$

and

$$\max_{1\le k\le m}\left\{\left(1 + \frac{2c^2N^2T}{\gamma}\right)\left(M\hbar_{\alpha_k} + N\hbar_{\beta_k}\right) + \hbar_{\alpha_k}\right\} + 2N\widetilde{N}\int_0^T g_\aleph(\omega)d\lambda(\omega) := M_0 < 1.$$

Proof. Let $Y := \{\zeta \in \mathcal{P}_R(\mathcal{J};\mathcal{E}) \mid \zeta(0) = \psi(0)\}$ be endowed with the uniform convergence topology. Our goal is to apply Lemma 9 to the operator $P^\gamma : Y \to Y$ defined by

$$(P^\gamma\xi)(\sigma) = \begin{cases} \Lambda(\sigma,0)\psi(0) + \Theta(\sigma,0)\xi_1 + \int_0^\sigma \Theta(\sigma,\omega)C\mu^\gamma(\omega,\xi)d\omega + \int_0^\sigma \Theta(\sigma,\omega)\aleph(\omega,\bar{\xi}_{\ell(\omega,\bar{\xi}_\omega)})d\lambda(\omega),\ \sigma \in [0,\tau_1], \\ \alpha_k(\sigma,\xi(\tau_k^-)),\ \sigma \in (\tau_k,r_k],\ 1 \le k \le m, \\ \Lambda(\sigma,r_k)\alpha_k(r_k,\xi(\tau_k^-)) + \Theta(\sigma,r_k)\beta_k(r_k,\xi(\tau_k^-)) \\ + \int_{r_k}^\sigma \Theta(\sigma,\omega)C\mu^\gamma(\omega,\xi)d\omega + \int_{r_k}^\sigma \Theta(\sigma,\omega)\aleph(\omega,\bar{\xi}_{\ell(\omega,\bar{\xi}_\omega)})d\lambda(\omega),\ \sigma \in [r_k,\tau_{k+1}],\ 1 \le k \le m. \end{cases}$$

where $\bar{\xi} : (-\infty,T] \to \mathcal{E}$ verifies $\xi_0 = \psi$ and $\bar{\xi} = \xi$ on $\mathcal{J}$.

The remaining part is splitted as:

Step I: There is a $q > 0$ such that $P^\gamma(\overline{B}_q) \subseteq \overline{B}_q$, where $\overline{B}_q = \{\xi \in Y \mid \|\xi\| \le q\}$.
Suppose it does not hold, then for every $q > 0$, there is $\xi^q(\cdot) \in \overline{B}_q$ and $\sigma^q \in \mathcal{J}$ satisfying $\|P^\gamma\xi^q(\sigma^q)\| > q$. Before proceeding, using Lemma 1, we get

$$\|\bar{\xi}_{\ell(\omega,\bar{\xi}_\omega)}\|_{\mathscr{B}} \le K_Tq + (L^\psi + M_T)\|\psi\|_{\mathscr{B}} := q^*.$$

Thus, for $\sigma^q \in [0,\tau_1]$, hypothesis (P2) and equation (4) yield that

$$\begin{aligned}\|\mu^\gamma(\sigma,\xi)\| &= \Big\|C^*\Theta^*(\tau_1,\sigma)\mathcal{R}(\gamma,-\Upsilon_0^{\tau_1})\big[\xi^{\tau_1} - \Lambda(\tau_1,0)\psi(0) - \Theta(\tau_1,0)\xi_1 - \int_0^{\tau_1}\Theta(\tau_1,\omega)\aleph(\omega,\xi_{\ell(\omega,\xi_\omega)})d\lambda(\omega)\big]\Big\| \\ &\le \frac{cN}{\gamma}\left\{\|\xi^{\tau_1}\| + Mb\|\psi\|_{\mathscr{B}} + N\|\xi_1\| + Nh(q^*)\int_0^T g(\omega)d\lambda(\omega)\right\},\end{aligned}$$

and for $\sigma^q \in [r_k,\tau_{k+1}]$, (P3) and (P4) yield that

$$\|\mu^\gamma(\sigma,\xi)\| \le \frac{cN}{\gamma}\left\{\|\xi^{\tau_{k+1}}\| + (M\tilde{\hbar}_{\alpha_k} + N\tilde{\hbar}_{\beta_k})(1+q) + Nh(q^*)\int_0^T g(\omega)d\lambda(\omega)\right\}.$$

Now, for any $\sigma^q \in [0, \tau_1]$, we compute

$$\begin{aligned} q &< \|(P^\gamma \xi^q)(\sigma^q)\| \\ &\le Mb\|\psi\|_{\mathscr{B}} + N\|\xi_1\| + \frac{c^2N^2T}{\gamma}\left(\|\xi^{\tau_1}\| + N\|\xi_1\|\right) + N\widetilde{N}h(q^*)\int_0^T g(\omega)d\lambda(\omega). \end{aligned}$$

For $\sigma^q \in (\tau_k, r_k]$, $1 \le k \le m$, it follows from (P3) that

$$q < \|P^\gamma \xi^q(\sigma^q)\| \le \tilde{\hbar}_{\alpha_k}(1+q).$$

Finally, for $\sigma^q \in [r_k, \tau_{k+1}]$, $1 \le k \le m$, we get

$$\begin{aligned} q &< \|(P^\gamma \xi^q)(\sigma^q)\| \\ &\le \frac{c^2N^2T}{\gamma}\|\xi^{\tau_{k+1}}\| + \widetilde{N}(M\tilde{\hbar}_{\alpha_k} + N\tilde{\hbar}_{\beta_k})(1+q) + N\widetilde{N}h(q^*)\int_0^T g(\omega)d\lambda(\omega). \end{aligned}$$

Thus, for all $\sigma^q \in \mathcal{J}$, we obtain

$$\begin{aligned} q &< \|(P^\gamma \xi^q)(\sigma^q)\| \\ &\le Mb\|\psi\|_{\mathscr{B}} + N\|\xi_1\| + \frac{c^2N^2T}{\gamma}\left[\|\xi^{\tau_1}\| + \|\xi^{\tau_{k+1}}\| + N\|\xi_1\|\right] \\ &\quad + \left[\tilde{\hbar}_{\alpha_k} + \widetilde{N}(M\tilde{\hbar}_{\alpha_k} + N\tilde{\hbar}_{\beta_k})\right](1+q) + N\widetilde{N}h(q^*)\int_0^T g(\omega)d\lambda(\omega), \end{aligned}$$

and hence

$$\begin{aligned} 1 &\le \left[\tilde{\hbar}_{\alpha_k} + \widetilde{N}(M\tilde{\hbar}_{\alpha_k} + N\tilde{\hbar}_{\beta_k})\right] + N\widetilde{N}\liminf_{\kappa\to+\infty}\frac{h(\kappa)}{\kappa}\int_0^T g(\omega)d\lambda(\omega) \\ &\le \max_{1\le k\le m}\left[\tilde{\hbar}_{\alpha_k} + \widetilde{N}(M\tilde{\hbar}_{\alpha_k} + N\tilde{\hbar}_{\beta_k})\right] + N\widetilde{N}\liminf_{\kappa\to+\infty}\frac{h(\kappa)}{\kappa}\int_0^T g(\omega)d\lambda(\omega), \end{aligned}$$

that contradicts inequality (5). So, let $q > 0$ be such that $P^\gamma(\overline{B}_q) \subseteq \overline{B}_q$.

Step II: $P^\gamma : \overline{B}_q \to \overline{B}_q$ satisfies condition (ii) of Lemma 9. For any $\xi \in \overline{B}_q$, $0 < \kappa < 1$, let us consider $\xi = \kappa P^\gamma \xi$. Then Step I yields that $\|\xi\| = \|\kappa P^\gamma \xi\| < \|P^\gamma \xi\| \le q$. Thus, $\xi \notin \overline{B}_q$.

Step III: $P^\gamma(\overline{B}_q)$ is equiregulated on $\mathcal{J}$. For $\sigma_0 \in [0, \tau_1)$, we get

$$\begin{aligned} \|(P^\gamma\xi)(\sigma) - (P^\gamma\xi)(\sigma_0^+)\| &\le \|(\Lambda(\sigma,0) - \Lambda(\sigma_0^+,0))\|\|\psi(0)\| + \|\Theta(\sigma,0) - \Theta(\sigma_0^+,0)\|\|\xi_1\| \\ &\quad + c\int_0^{\sigma_0^+}\|\Theta(\sigma,\omega) - \Theta(\sigma_0^+,\omega)\|\|\mu^\gamma(\omega)\|d\omega + cN\int_{\sigma_0^+}^{\sigma}\|\mu^\gamma(\omega)\|d\omega \\ &\quad + h(q^*)\int_0^{\sigma_0^+}\|\Theta(\sigma,\omega) - \Theta(\sigma_0^+,\omega)\|g(\omega)d\lambda(\omega) \\ &\quad + Nh(q^*)\int_{\sigma_0^+}^{\sigma} g(\omega)d\lambda(\omega) \\ &= \sum_{l=1}^{6} S_l. \end{aligned}$$

The compactness of $\Theta(\sigma,\omega)$ for $\sigma - \omega > 0$ and its strong continuity yields that $\Theta(\sigma,\omega)$ is continuous in the uniform operator topology. Therefore, S_1, S_2, S_3, $S_5 \to 0$ as $\sigma \to \sigma_0^+$. Next, the absolute continuity of Lebesgue integral yields that $S_4 \to 0$ as

$\sigma \to \sigma_0^+$. Now, let $\tilde{g}(\sigma) = \int_0^\sigma g(\omega)d\lambda(\omega)$. Then by Lemma 2, $\tilde{g}(\sigma)$ is regulated on $\mathcal{J}$. Thus,

$$S_6 \leq Nh(q^*)\int_{\sigma_0^+}^{\sigma} g(\omega)d\lambda(\omega) = Nh(q^*)(\tilde{g}(\sigma) - \tilde{g}(\sigma_0^+)) \to 0 \text{ as } \sigma \longrightarrow \sigma_0^+,$$

and independent of ξ. Similarly, we can show that $\|(P^\gamma\xi)(\sigma_0^-) - (P^\gamma\xi)(\sigma)\| \to 0$ as $\sigma \to \sigma_0^-$ for each $\sigma_0 \in (0, \tau_1]$.

For $\sigma_0 \in [\tau_k, r_k)$, $1 \leq k \leq m$

$$\begin{aligned}\|(P^\gamma\xi)(\sigma) - (P^\gamma\xi)(\sigma_0^+)\| =& \|\alpha_k(\sigma, \xi(\tau_k^-)) - \alpha_k(\sigma_0^+, \xi(\tau_k^-))\|\\ \leq& \hbar_{\alpha_k}|\sigma - \sigma_0^+| \to 0 \text{ as } \sigma \to \sigma_0^+.\end{aligned}$$

Similarly, $\|(P^\gamma\xi)(\sigma_0^-) - (P^\gamma\xi)(\sigma)\| \to 0$ as $\sigma \to \sigma_0^-$ for $\sigma_0 \in (\tau_k, r_k]$, $1 \leq k \leq m$.

For $\sigma_0 \in [r_k, \tau_{k+1})$, $1 \leq k \leq m$, we get

$$\begin{aligned}\|(P^\gamma\xi)(\sigma) - (P^\gamma\xi)(\sigma_0^+)\| \leq& \|(\Lambda(\sigma, r_k) - \Lambda(\sigma_0^+, r_k))\|\|\alpha_k(r_k, \xi(\tau_k^-))\| + \|\Theta(\sigma, 0) - \Theta(\sigma_0^+, 0)\|\|\alpha_k(r_k, \xi(\tau_k^-))\|\\ &+ c\int_{r_k}^{\sigma_0^+} \|\Theta(\sigma, \omega) - \Theta(\sigma_0^+, \omega)\|\|\mu^\gamma(\omega)\|d\omega + cN\int_{\sigma_0^+}^{\sigma} \|\mu^\gamma(\omega)\|d\omega\\ &+ h(q^*)\int_{r_k}^{\sigma_0^+} \|\Theta(\sigma, \omega) - \Theta(\sigma_0^+, \omega)\|g(\omega)d\lambda(\omega)\\ &+ Nh(q^*)\int_{\sigma_r^+}^{\sigma} g(\omega)d\lambda(\omega)\\ =& \sum_{l=1}^{6} S_l.\end{aligned}$$

Now using the same arguments as in the previous case, we infer that $\|(P^\gamma\xi)(\sigma) - (P^\gamma\xi)(\sigma_0^+)\| \to 0$ as $\sigma \to \sigma_0^+$. Similarly, one can show that $\|(P^\gamma\xi)(\sigma_0^-) - (P^\gamma\xi)(\sigma)\| \to 0$ as $\sigma \to \sigma_0^-$ and independent of ξ. Thus, by Definition 2, the conclusion follows.

Step IV: We will assert that P^γ is continuous.

Let $\{\xi^l\}_{l=1}^\infty \subset \overline{B}_q$ be a sequence converges to ξ in $\overline{B}_q$. From Axiom (B), we find that $\ell(\omega, \overline{\xi^l}_\omega) \to \ell(\omega, \overline{\xi}_\omega)$ as $l \to \infty$ for every $\omega \in \mathcal{J}$. One can easily compute that

$$\begin{aligned}\|\mu^\gamma(\omega, \xi^l) - \mu^\gamma(\omega, \xi)\| \leq& \frac{cN}{\gamma}\left[M\hbar_{\alpha_k}\|\xi^l(\tau_k^-) - \xi(\tau_k^-)\| + N\hbar_{\beta_k}\|\xi^l(\tau_k^-) - \xi(\tau_k^-)\|\right]\\ &+ N\int_{r_k}^{\tau_{k+1}} \|\aleph(\omega, \overline{\xi^l}_{\ell(\omega, \overline{\xi^l}_\omega)}) - \aleph(\omega, \overline{\xi}_{\ell(\omega, \overline{\xi}_\omega)})\|d\lambda(\omega).\end{aligned}$$

For any $\sigma \in \bigcup_{k=0}^m [r_k, \tau_{k+1}]$, we have

$$\begin{aligned}\|(P^\gamma\xi^l)(\sigma) - (P^\gamma\xi)(\sigma)\| \leq& \left[M\hbar_{\alpha_k}\|\xi^l(\tau_k^-) - \xi(\tau_k^-)\| + N\hbar_{\beta_k}\|\xi^l(\tau_k^-) - \xi(\tau_k^-)\|\right]\\ &+ cN\int_{r_k}^{\sigma} \|\mu^\gamma(\omega, \xi^l) - \mu^\gamma(\omega, \xi)\|d\omega\\ &+ N\int_{r_k}^{\sigma} \|\aleph(\omega, \overline{\xi^l}_{\ell(\omega, \overline{\xi^l}_\omega)}) - \aleph(\omega, \overline{\xi}_{\ell(\omega, \overline{\xi}_\omega)})\|d\lambda(\omega)\\ \leq& \widetilde{N}\left[M\hbar_{\alpha_k}\|\xi^l(\tau_k^-) - \xi(\tau_k^-)\| + N\hbar_{\beta_k}\|\xi^l(\tau_k^-) - \xi(\tau_k^-)\|\right]\\ &+ N\widetilde{N}\int_{r_k}^{\sigma} \|\aleph(\omega, \overline{\xi^l}_{\ell(\omega, \overline{\xi^l}_\omega)})) - \aleph(\omega, \overline{\xi}_{\ell(\omega, \overline{\xi}_\omega)})\|d\lambda(\omega),\end{aligned}$$

that approaches to 0 as $l \to \infty$ by (P2) and the dominated convergence theorem. Next, for $\sigma \in \bigcup_{k=1}^{m}(\tau_k, r_k]$, $\|(P^{\gamma}\xi^l)(\sigma) - (P^{\gamma}\xi)(\sigma)\| \to 0$ as $l \to \infty$ due to the continuity of α_k. By Step III, the family $\{(P^{\gamma}\xi^l)(\sigma)\}_{l=1}^{\infty}$ is equiregulated. This fact and the overhead justification with Lemma 3 exhibit that $P^{\gamma}\xi^l$ converges uniformly to $P^{\gamma}\xi$ as $l \to \infty$. That is,

$$\|P^{\gamma}\xi^l - P^{\gamma}\xi\|_{\infty} = \sup_{\sigma\in\mathcal{J}}\|(P^{\gamma}\xi^l)(\sigma) - (P^{\gamma}\xi)(\sigma)\| \to \infty,$$

as $l \to \infty$. This verifies our claim.

Step V: The Mönch condition holds. Suppose $B \subset \overline{B}_q$ is countable and $B \subset \overline{co}(\{0\} \cup P^{\gamma}(B))$, we assert that $\chi(B) = 0$. We may assume that $B = \{\xi^l\}_{l=1}^{\infty}$. Clearly, $P^{\gamma}(B)$ is equiregulated on $\mathcal{J}$. Thus, in view of Lemma 5 and $B \subset \overline{co}(\{0\} \cup P^{\gamma}(B))$, one can concude that B is equiregulated on $\mathcal{J}$.

Now, by the virtue of Lemma 7, for $\sigma \in [0, \tau_1]$, we have

$$\begin{aligned}
\chi((P^{\gamma}(B))(\sigma)) \le& \frac{2c^2N^3}{\gamma}\int_0^{\tau_1}\chi(\{\aleph(\omega, \overline{\xi^l}_{\ell(\omega,\overline{\xi^l}_{\omega})})\})d\lambda(\omega) + 2N\int_0^{\tau_1}\chi(\{\aleph(\omega, \overline{\xi^l}_{\ell(\omega,\overline{\xi^l}_{\omega})})\})d\lambda(\omega)\\
\le& 2N\widetilde{N}\int_0^{\tau_1} g_{\aleph}(\omega)\sup_{-\infty<\vartheta\le 0}\chi(B(\omega+\vartheta))d\lambda(\omega)\\
\le& 2N\widetilde{N}\int_0^{\tau_1} g_{\aleph}(\omega)\sup_{0\le r\le\tau_1}\chi(B(r))d\lambda(\omega)\\
\le& \left[2N\widetilde{N}\int_0^{\tau_1} g_{\aleph}(\omega)d\lambda(\omega)\right]\chi(B).
\end{aligned}$$

For $\sigma \in (\tau_k, r_k]$, $1 \le k \le m$,

$$\begin{aligned}
\chi((P^{\gamma}(B))(\sigma)) \le& \chi(\{\alpha_k(\sigma, \xi^l(\tau_k^-))\})\\
\le& \hbar_{\alpha_k}\sup_{\tau_k<r\le r_k}\chi(B(r))\\
\le& \hbar_{\alpha_k}\chi(B).
\end{aligned}$$

For $\sigma \in [r_k, \tau_{k+1}]$, $1 \le k \le m$,

$$\begin{aligned}
\chi((P^{\gamma}(B))(\sigma)) \le& \chi(\{\Lambda(\sigma, r_k)\alpha_k(\sigma, \xi^l(\tau_k^-))\}) + \chi(\{\Theta(\sigma, r_k)\beta_k(\sigma, \xi^l(\tau_k^-))\})\\
&+ \chi(\{\int_{r_k}^{\sigma}\Theta(\sigma,\omega)\mu^{\gamma}(\omega, \xi^l)d\omega + \chi(\{\int_{r_k}^{\sigma}\Theta(\sigma,\omega)\aleph(\omega, \overline{\xi^l}_{\ell(\omega,\overline{\xi^l})})d\lambda(\omega)\}\\
\le& \left(1 + \frac{2c^2N^2T}{\gamma}\right)\left(M\hbar_{\alpha_k} + N\hbar_{\beta_k}\right)\sup_{r_k<r\le\tau_{k+1}}\chi(B(r))\\
&+ 2N\widetilde{N}\int_{r_k}^{\tau_{k+1}} g_{\aleph}(\omega)\sup_{-\infty<\vartheta\le 0}\chi(B(\omega+\vartheta))d\lambda(\omega)\\
\le& \left(1 + \frac{2c^2N^2T}{\gamma}\right)\left(M\hbar_{\alpha_k} + N\hbar_{\beta_k}\right)\sup_{r_k<r\le\tau_{k+1}}\chi(B(r))\\
&+ 2N\widetilde{N}\int_{r_k}^{\tau_{k+1}} g_{\aleph}(\omega)\sup_{r_k<r\le\tau_{k+1}}\chi(B(r))d\lambda(\omega)\\
\le& \left[\left(1 + \frac{2c^2N^2T}{\gamma}\right)\left(M\hbar_{\alpha_k} + N\hbar_{\beta_k}\right) + 2N\widetilde{N}\int_{r_k}^{\tau_{k+1}} g_{\aleph}(\omega)d\lambda(\omega)\right]\chi(B).
\end{aligned}$$

Thus, for any $\sigma \in [0,T]$, we obtain

$$\chi((P^{\gamma}(B))(\sigma)) \le \left[\max_{1\le k\le m}\left\{\left(1+\frac{2c^2N^2T}{\gamma}\right)\left(M\hbar_{\alpha_k}+N\hbar_{\beta_k}\right)+\hbar_{\alpha_k}\right\}+2N\widetilde{N}\int_0^{\tau} g_{\aleph}(\omega)d\lambda(\omega)\right]\chi(B).$$
$$=M_0\chi(B).$$

Then, by the hypothesis in Mönch condition, it follows that

$$\chi(B) \le \chi(\overline{co}(\{0\}\cup P^{\gamma}(B))) = \chi(P^{\gamma}(B)) \le M_0\chi(B).$$

The fact $M_0 < 1$ yields that $(1-M_0)\chi(B) \le 0$. Hence, $\chi(B)=0$. Therefore, B is relatively compact in $\mathscr{R}(\mathcal{J};\mathcal{E})$. Consequently, P^{γ} has a fixed point in $\overline{B}_q$ due to Lemma 9. □

Theorem 2. Under the hypotheses $(P1)-(P6)$, System (1) is approximately controllable on $\mathcal{J}$.

Proof. Let $\xi^{\gamma} \in \overline{B}_q$ be such that $P^{\gamma}\xi^{\gamma} = \xi^{\gamma}$. Then ξ^{γ} is a solution of System (1) on $(-\infty, T]$ under the control defined

$$\mu^{\gamma}(\sigma,\xi^{\gamma}) = C^*\Theta^*(\tau_{k+1},\sigma)\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})\big[\xi^{\tau_{k+1}}-\Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi^{\gamma}(\tau_k^-))-\Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi^{\gamma}(\tau_k^-))$$
$$-\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(\omega,\overline{\xi}^{\gamma}_{\omega})})d\lambda(\omega)\big],\ 0\le k\le m, \tag{6}$$

and satisfies

$$\xi^{\gamma}(\tau_{k+1}) = \Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi^{\gamma}(\tau_k^-))+\Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi^{\gamma}(\tau_k^-))+\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)C\mu^{\gamma}(\omega,\xi^{\gamma})d\omega$$
$$+\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(\omega,\overline{\xi}^{\gamma}_{\omega})})d\lambda(\omega).$$

Now, utilizing the control function described by (6), we compute

$$\xi^d-\xi^{\gamma}(\tau_{k+1}) = \xi^d-\Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi^{\gamma}(\tau_k^-))-\Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi^{\gamma}(\tau_k^-))$$
$$-\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(\omega,\overline{\xi}_{\omega})})d\lambda(\omega)$$
$$-\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)CC^*\Theta^*(\tau_{k+1},\omega)\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})[\xi^d-\Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi^{\gamma}(\tau_k^-))$$
$$-\Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi^{\gamma}(\tau_k^-))-\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(r,\overline{\xi}^{\gamma}_{r})})d\lambda(r)]d\omega.$$

In view of $I-\Upsilon_{r_k}^{\tau_{k+1}}\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}}) = \gamma\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})$, $0\le k\le m$, it follows that

$$\xi^d-\xi^{\gamma}(\tau_{k+1}) = \gamma\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})[\xi^d-\Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi^{\gamma}(\tau_k^-))-\Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi^{\gamma}(\tau_k^-))$$
$$-\int_{r_k}^{\tau_{k+1}}\Theta(\tau_{k+1},\omega)\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(\omega,\overline{\xi}^{\gamma}_{\omega})})d\lambda(\omega)].$$

Observe that by (P6) and Lemma 8, $\{\aleph(\cdot,\overline{\xi}^{\gamma}_{\ell(\cdot,\overline{\xi}^{\gamma}_{\cdot})})\}$ is weakly relatively compact in $L^1_\lambda(\mathcal{J};\mathcal{E})$. Thus, we can exert a subsequence, again denoted by $\{\aleph(\cdot,\overline{\xi}^{\gamma}_{\ell(\cdot,\overline{\xi}^{\gamma}_{\cdot})})\}$, that converges weakly to $\widetilde{\aleph}(\cdot)$ in $L^1_\lambda(\mathcal{J};\mathcal{E})$, say. Let

$$\begin{aligned}Q_{\tau_{k+1}} =&\xi^d - \Lambda(\tau_{k+1},r_k)\alpha_k(r_k,\xi^\gamma(\tau_k^-)) - \Theta(\tau_{k+1},r_k)\beta_k(r_k,\xi^\gamma(\tau_k^-))\\ &- \int_{r_k}^{\tau_{k+1}} \Theta(\tau_{k+1},\omega)\widetilde{\aleph}(\omega)d\lambda(\omega).\end{aligned}$$

Notice that the map $\Phi : L^1_\lambda(\mathcal{J};\mathcal{E}) \to \mathcal{R}([r_k,\tau_{k+1}];\mathcal{E})$ defined by $\Phi\xi(\cdot) = \int_{r_k}^{\cdot} \Theta(\cdot,\omega)\xi(\omega)d\lambda(\omega)$ is compact for each k by the same arguments as used to justify the compactness of P^γ. Therefore,

$$\left\| \int_{r_k}^{\tau k+1} \Theta(\tau_{k+1},\omega)[\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(\omega,\overline{\xi}^{\gamma}_{\omega})}) - \widetilde{\aleph}(\omega)]d\lambda(\omega) \right\| \to 0,$$

as $\gamma \to 0^+$ for each $0 \le k \le m$. From (P5), it follows that

$$\begin{aligned}\|\xi^d - \xi^\gamma(\tau_{k+1})\| = &\left\|\gamma\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})Q_{\tau_{k+1}}\right\|\\ &+\left\|\gamma\mathcal{R}(\gamma,-\Upsilon_{r_k}^{\tau_{k+1}})\right\| \left\| \int_{r_k}^{\tau_{k+1}} \Theta(\tau_{k+1},\omega)[\aleph(\omega,\overline{\xi}^{\gamma}_{\ell(\omega,\overline{\xi}^{\gamma}_{\omega})}) - \widetilde{\aleph}(\omega)]d\lambda(\omega)] \right\|\\ &\to 0 \text{ as } \gamma \to 0^+.\end{aligned}$$

This completes the proof. □

4 | AN APPLICATION

We apply our finding to the following partial differential equation

$$\begin{cases} \partial\Big(\dfrac{\partial z(\sigma,w)}{\partial\sigma}\Big) = \Big[\dfrac{\partial^2}{\partial w^2}z(\sigma,w) + b(\sigma)\dfrac{\partial}{\partial w}z(\sigma,w) + \mu(\sigma,w)\Big]\partial\sigma \\ \quad +\Big(\displaystyle\int_{-\infty}^{\sigma} \mathrm{k}(\sigma,\varkappa)h(\varkappa, z(\varkappa-\varrho_1(\sigma)\varrho_2(\|z(\sigma,w)\|),w))d\varkappa\Big)\partial\lambda(\sigma), \\ \quad \sigma\in[0,1],\ w\in[0,2\pi] \\ z(\sigma,0) = z(\sigma,2\pi),\ 0\le\sigma\le 1, \\ z(\sigma,w) = \alpha_k(\sigma, z(\sigma_k^-,w)),\ \sigma\in(\tau_k,r_k],\ 1\le k\le m, \\ \frac{\partial}{\partial\sigma}z(\sigma,w) = \beta_k(\sigma,z(\sigma_k^-,w)),\ \sigma\in(\tau_k,r_k],\ 1\le k\le m, \\ z(\vartheta,w) = z_0(\vartheta,w),\ \vartheta\in(-\infty,0], \\ \dfrac{\partial}{\partial\sigma}z(\sigma,w)\Big|_{\sigma=0} = z_1(0,w), \end{cases} \tag{7}$$

where the function $b(\cdot)$, $\mathrm{k}(\cdot,\cdot)$, $h(\cdot,\cdot)$ and $\lambda(\cdot)$ will be define as we proceed further.

To model this problem, consider the space $\mathcal{E} = L^2(\mathbb{T};\mathbb{C})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the quotient group (see Henríquez et al.[3]). Denote by $H^2(\mathbb{T};\mathbb{C})$ the Sobolev space of 2π-periodic maps $z : \mathbb{R}\to\mathbb{C}$ with $z''\in\mathcal{E}$.

Let $A(\sigma) = A_0 + \tilde{A}(\sigma)$, where $A_0 z = \frac{d^2z(w)}{dw^2}$ with the domain $D(A_0) = H^2(\mathbb{T};\mathbb{C})$ and $\tilde{A}(\sigma) = b(\sigma)\frac{dz(w)}{dw}$ with the domain $D(\tilde{A}(\sigma)) = H^1(\mathbb{T};\mathbb{C})$. Then it is known from Henríquez[2] that $\{A(\sigma) : \sigma\in[0,1]\}$ generates a compact evolution operator $\{\Theta(\sigma,\tau) \mid \sigma,\ \tau\in[0,1]\}$. Moreover, $e_l(\xi) = \frac{1}{\sqrt{2\pi}}\exp(il\xi),\ l\in\mathbb{Z}$, and $\{e_l \mid l\in\mathbb{Z}\}$ is an orthonormal basis for $\mathcal{E}$.

Let $\mathscr{B} = C_0\times L^p(g,\mathcal{E})$, $1\le p<\infty$ (see[30]) and

$$\|\varphi\|_{\mathscr{B}} = \|\varphi(0)\| + \left(\int_{-\infty}^{0} g(\omega)\|\varphi(\omega)\|^p d\omega\right)^{1/p}.$$

For an appropriate function $g : (-\infty, 0] \to \mathbb{R}$, one can check that the axioms (A) and (B) hold in $\mathscr{B}$. We set the following hypotheses on System (7):

(i) $h : \mathbb{R} \times \mathcal{E} \to \mathbb{R}$ is uniformly bounded and continuous.

(ii) The map $\mathrm{k} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is taken in such a way that $|\mathrm{k}(\sigma, \sigma + r)| < a_{\mathrm{k}}(r)$ and $\int_{-\infty}^{0} a_{\mathrm{k}}(r)dr < \infty$, where a_{k} is define from $(-\infty, 0]$ into $\mathbb{R}_+$.

(iii) For $\varphi \in \mathscr{B}$, set $\varphi(\vartheta)(w) = \varphi(\vartheta, w)$.

Define $\aleph(\cdot, \cdot) : [0, 1] \times \mathscr{B} \to \mathcal{E}$, and $\ell(\cdot, \cdot) : [0, 1] \times \mathscr{B} \to (-\infty, 1]$ by

$$\aleph(\sigma, \varphi)(w) = \aleph(\sigma, \varphi(\cdot, w)) = \int_{-\infty}^{0} \mathrm{k}(\sigma, \sigma + r)h(\sigma + r, \varphi(r, w))dr,$$

$$\ell(\sigma, \varphi) = \varrho_1(\sigma)\varrho_2(\|\varphi(0)\|),$$

for $\varphi \in \mathscr{B}$, and for $j = 1, 2$, $\varrho_j : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous. Thus, System (7) can be written in the abstract form given by System (1).

Take

$$\lambda(\sigma) = \begin{cases} 1 - \frac{1}{2}, & 0 \le \sigma \le 1 - \frac{1}{2}, \\ \cdots \\ 1 - \frac{1}{n}, & 1 - \frac{1}{n-1} < \sigma \le 1 - \frac{1}{n},\ n = 2, 3, \cdots, \\ \cdots \\ 1, & \sigma = 1. \end{cases}$$

Clearly, $\lambda : [0, 1] \to \mathbb{R}$ is the left continuous nondecreasing map.

Let $\mathcal{U} = \left\{ \mu = \sum_{l=2}^{\infty} \mu_l e_l : \sum_{l=2}^{\infty} |\mu_l|^2 < \infty \right\}$, and $\|\mu\| = \left(\sum_{l=2}^{\infty} |\mu_l|^2 \right)^{1/2}$. Set $C = I$. Also, the adjoint of $\Theta(\sigma, s)$ is given by

$$\Theta^*(\sigma, s)y = \sum_{l=1}^{\infty} y_l(\sigma, s)\overline{\langle y, e_l \rangle} e_l,$$

where $y_l(\sigma, s)$ is the solution of the following differential equation

$$\begin{aligned} y''(\sigma) &= -\, l^2 y(\sigma) + ilb(\sigma)y(\sigma), \\ y(s) &= 0,\ y'(s) = y_1. \end{aligned}$$

Furthermore, for any $\mu = \sum_{l=1}^{\infty} \mu_l e_l(\xi) \in \mathcal{E}$, we get

$$C^*\Theta^*(\sigma, s)\mu = \Theta^*(\sigma, s)\mu = \sum_{l=1}^{\infty} y_l(\sigma, s)\bar{\mu}_l e_l, \text{ for all } \sigma \in [0, 1].$$

Now, for every $\sigma \in [0, 1]$, if $\|\Theta^*(\sigma, s)\mu\| = 0$, then

$$\sum_{l=1}^{\infty} |y_l(\sigma, s)|^2 |\mu_l|^2 = 0,$$

which implies that $\mu_l = 0$, $l \in \mathbb{N}$, and hence $\mu = 0$. Thus, in view of Lemma 10, (P5) hold. Therefore, Theorem 2 ensures approximate controllability of System (7) on $[0, 1]$.

5 | CONCLUSIONS

The topic of approximate controllability for an abstract measure differential problem with SDD and NII has been dissected for the first time. We have initially list the existence criteria for a mild solution of the considered system, and the existence result is verified by employing the Mönch fixed point theorem. The approximate controllability is also justified under some simple sufficient conditions. The developments of this article generalize the existing outcomes in the literature for time varying differential systems governed by second order and MDEs [23]. In the future, it is attractive to expand the works of this paper for MDEs with different fractional derivatives.

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Conflict of interest

The authors declare no potential conflict of interests.

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