Asymptotic stability of nonlinear diffusion waves for the bipolar quantum Euler-Poisson system with time-dependent damping

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Abstract. We shall investigate the asymptotic behavior of solutions to the Cauchy problem for the one-dimensional bipolar quantum Euler-Poisson system with time-dependent damping effects $\frac{J_i}{(1+t)^\lambda}(i=1,2)$ for $-1<\lambda<1$. Applying the technical time-weighted energy method, we prove that the classical solutions to the Cauchy problem exist uniquely and globally, and time-algebraically converge to the nonlinear diffusion waves. This study generalizes the results in [Y.-P. Li, Nonlinear Anal., 74(2011), 1501-1512] which considered the bipolar quantum Euler-Poisson system with constant coefficient damping.

Keywords. Bipolar quantum Euler-Poisson system; Time-dependent damping; Asymptotic behavior; Smooth solutions; Nonlinear diffusion waves.

AMS subject classifications. 35Q35, 35B40, 76W05.

1 Introduction

To model and simulate the charged particles transport in ultra-small sub-micron semiconductor devices, for instance resonant tunneling diodes, where the quantum effects (such as particle tunneling through potential barriers and build-up in quantum wells [2, 6, 5, 18]) arise and cannot be simulated by classical hydrodynamic models, the quantum hydrodynamic models are driven [6, 7]. The quantum hydrodynamic models usually take the form of compressible Euler-Poisson systems with an additional dispersion term which represent the effect of quantum (Bohn) potential. In this paper, we consider the following one-dimensional isentropic bipolar quantum Euler-Poisson system with time-dependent damping

$$\begin{cases} n_{1t}+J_{1x}=0, \\ J_{1t}+\left(\frac{J_1^2}{n_1}+p(n_1)\right)_x-n_1\left(\frac{(\sqrt{n_1})_{xx}}{\sqrt{n_1}}\right)_x=n_1E-\frac{\mu}{(1+t)^\lambda}J_1, \\ n_{2t}+J_{2x}=0, \\ J_{2t}+\left(\frac{J_2^2}{n_2}+q(n_2)\right)_x-n_2\left(\frac{(\sqrt{n_2})_{xx}}{\sqrt{n_2}}\right)_x=-n_2E-\frac{\mu}{(1+t)^\lambda}J_2, \\ E_x=n_1-n_2, \end{cases} \tag{1.1}$$

for $(x,t)\in\mathbb{R}\times\mathbb{R}_+$, $\mu\in\mathbb{R}_+$, and $\lambda\in\mathbb{R}$. Here, $n_1=n_1(x,t)>0, n_2=n_2(x,t)>0, J_1=J_1(x,t), J_2=J_2(x,t), E=E(x,t)$ are unknowns, represent the electron density, the hole density, the current density of electrons, the current density of holes, and the electric field, respectively. The nonlinear terms $p(n_1)$ and $q(n_2)$ denote the pressures of electrons and holes. The time-dependent damping effects $\frac{\mu}{(1+t)^\lambda}J_i(i=1,2)$ are called under-damping for $\lambda>0$, which are time-gradually-vanishing; and are called over-damping for $\lambda<0$, which are time-gradually-enhancing. The under-damping and over-damping make the structure of the solutions to (1.1) more interesting and complicated.

In recent years, the theoretical theory on the quantum hydrodynamic models for semiconductors has been one of hot spots in mathematical physics because of their physical importance and wide

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range of applications. Due to the mathematical complexity and difficulty, the studies on the bipolar quantum Euler-Poisson system are very limit. Unterreiter [31] first studied the isothermal solutions to the multi-dimensional stationary model in a bounded domain. Liang and Zhang [27] obtained the existence and asymptotic limits of the steady-state solutions in a bounded domain of $\mathbb{R}^d (1 \leq d \leq 3)$. Zhang and Zhang [36] proved the existence and uniqueness of thermal equilibrium solution to the multi-dimensional bipolar quantum Euler-Poisson system in the whole space, and obtain the semi-classical limit and a combined Plank-Debye length limit. Li, Zhang and Zhang [24] established the global existence of the classical solutions to the Cauchy problem for the 3-D bipolar quantum hydrodynamic, and proved the algebraic decay rates of the smooth solutions. Later, they considered the semiclassical limit and the relaxation limit of the solutions to the multi-dimensional model in [37]. Li [20, 21] studied the Global existence and asymptotic behavior of the classical solutions to the 1-D bipolar quantum hydrodynamic model in the whole line and half line, respectively. Hu, Mei and Zhang [14] considered a initial-boundary problem for a 1-D bipolar quantum hydrodynamic model, they first establish the existence and uniqueness of the stationary solution to the corresponding stationary model, then they obtain the exponentially asymptotic stability of the stationary solution and the semi-classical limit. Hu, Li and Liao [15] stuided the stationary solutions to the 1-D bipolar quantum hydrodynamic model with general doping profile in a bounded domain. For classical bipolar hydrodynamic model, we refer the interesting readers, however not limit, to [3, 8, 11, 12, 13, 16, 19, 22, 23, 25] and the reference therein.

Next, to understand the damping effects in the bipolar quantum Euler-Poisson system, it is interesting and challenging to study the asymptotic behavior and regularity of solutions to (1.1). Due to complex and difficulty of the system (1.1), the research is very limit. For the case that $\lambda = -1$, Wu [35] studied the large-time behavior of the classical solutions in the whole space. Moreover, when the effect of quantum potential vanishing, system (1.1) can be reduced to the following classical bipolar Euler-Poisson system with time-dependent damping

$$\begin{cases} n_{1t} + J_{1x} = 0, \\ J_{1t} + \left(\frac{J_1^2}{n_1} + p(n_1)\right)_x = n_1 E - \frac{J_1}{(1+t)^\lambda}, \\ n_{2t} + J_{2x} = 0, \\ J_{2t} + \left(\frac{J_2^2}{n_2} + q(n_2)\right)_x = -n_2 E - \frac{J_2}{(1+t)^\lambda}, \\ E_x = n_1 - n_2. \end{cases}$$

Li, et al [25] first studied the diffusion wave phenomena of global classical solutions for the case $-1 < \lambda < 1$ and $\mu > 0$ when the pressure functions are identical. Furthermore, Wu, Zheng and Luan [33] investigated a more physical, and a challenging case that two pressure functions can be different and with a non-zero doping profile. Later, Wu, Li and Xu [34] studied the stability of the nonlinear diffusion wave in a half line. In this paper, we study the Cauchy problem for the system (1.1) with the initial data and the far field conditions

$$\begin{cases} (n_1, n_2, J_1, J_2)(x, 0) = (n_{10}, n_{20}, J_{10}, J_{20})(x) \to (n_\pm, n_\pm, J_{1\pm}, J_{2\pm}) \quad \text{as} \quad x \to \pm\infty, \\ E(-\infty, t) = E^-, \end{cases} \tag{1.2}$$

where $n_\pm > 0, J_{1\pm}, J_{2\pm}$ and E^- are given constants. Throughout this paper, for simplify, we assume that

$$p(s) = q(s) = s^\gamma, \gamma \geq 1, \, -1 < \lambda < 1, \, \mu = 1.$$

As in [25], the asymptotic profiles of the solutions of IVP (1.1)–(1.2) are expected to be the solutions to the following system

$$\begin{cases} \bar{n}_t + \bar{J}_x = 0, \\ p(\bar{n})_x = -\frac{\bar{J}}{(1+t)^\lambda}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ (\bar{n}, \bar{J}) \to (n_\pm, 0) \quad \text{as} \quad x \to \pm\infty. \end{cases} \tag{1.3}$$

[4] shows that this system possesses a unique self-similar solution in the form of

$$(\bar{n}, \bar{J})(x,t) = (\bar{n}, \bar{J})\left(\frac{x}{\sqrt{(1+t)^{\lambda+1}}}\right),$$

which is usually called the nonlinear diffusion wave.

Before stating our main results, we first give some notations which will be used throughout this paper. The symbol $C > 0$ stands for a generic constant, which is independent of time, and $C_i > 0 (i = 1, 2, \cdots)$ represents some specific constant. $L^p(\mathbb{R})\,(1 \le p < +\infty)$ are the spaces of measurable functions whose p-powers are integrable on $\mathbb{R}$, with the norm $\|\cdot\|_{L^p(\mathbb{R})} = (\int_{L^p(\mathbb{R})} |\cdot|^p dx)^{\frac{1}{p}}$. For the case that $p = 2$, we simply denote $\|\cdot\|_{L^2(\mathbb{R})} = \|\cdot\|$. And $L^\infty(\mathbb{R})$ is the space of essentially bounded measurable functions on $\mathbb{R}$, with the norm $\|\cdot\|_{L^\infty(\mathbb{R})} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |\cdot|$. Moreover, for a nonnegative integer m, $H^m(\mathbb{R})$ denotes the Hilbert space with the norm $\|\cdot\|_m$, especially $\|\cdot\|_0 = \|\cdot\|$. We also denote $\|(f_1, f_2, \cdots, f_k)\|_m^2 := \|f_1\|_m^2 + \|f_2\|_m^2 + \cdots + \|f_k\|_m^2$. Furthermore, letting $T > 0$, we denote by $C^k([0,T]; H^m(\mathbb{R}))$ (resp. $L^2(0,T; H^m(\mathbb{R}))$) the space of k-times continuously differentiable (resp. square integrable) functions on $[0,T]$ with values taken in $H^m(\mathbb{R})$.

The main purpose in this paper is to study the asymptotic behavior of solutions to the IVP (1.1)–(1.2), and the main results are stated as follows.

Theorem 1.1 *(Case of $-1 < \lambda < \frac{1}{7}$) Assume that $(\phi_{10}, \phi_{20}, \psi_{10}, \psi_{20}) \in H^5(\mathbb{R}) \times H^5(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3(\mathbb{R})$. Set $\Phi_0 := \|(\phi_{10}, \phi_{20})\|_5 + \|(\psi_{10}, \psi_{20})\|_3$ and $\delta_1 := |n_+ - n_-| + \delta_0$ with $\delta_0 := |J_{1+}| + |J_{1-}| + |J_{2+}| + |J_{2-}| + |E_+|$. Then, there exists a sufficiently small constant $\varepsilon_0 > 0$ such that if $\Phi_0 + \delta_1 \le \varepsilon_0$, then IVP (1.1)–(1.2) has a unique global classical solution $(n_1, n_2, J_1, J_2, E)(x,t)$, which satisfies*

$$\begin{aligned}
& \sum_{k=0}^{3}(1+t)^{(k+1)(\lambda+1)}\|\partial_x^k(n_1-\bar{n}, n_2-\bar{n})(t)\|^2 + (1+t)^{4\lambda+4}\|\partial_x^4(n_1-\bar{n}, n_2-\bar{n})(t)\|^2 \\
+ & \sum_{k=0}^{2}(1+t)^{k(\lambda+1)+2}\|\partial_x^k(J_1-\hat{J}_1-\bar{J}, J_2-\hat{J}_2-\bar{J})(t)\|^2 \\
+ & (1+t)^{2\lambda+4}\|\partial_x^3(J_1-\hat{J}_1-\bar{J}, J_2-\hat{J}_2-\bar{J})(t)\|^2 \le C(\Phi_0^2+\delta_1),
\end{aligned}$$

and

$$\|(n_1-n_2)(t)\|_3^2 + \|(J_1-\hat{J}_1-J_2+\hat{J}_2)(t)\|_2^2 + \|(E-\hat{E})(t)\|_4^2 \le C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}$$

for some constant $\nu_0 > 0$. Here, E_+ is given in (2.1), $(\phi_{10}, \phi_{20}, \psi_{10}, \psi_{20})(x)$ is given by

$$\phi_{i0}(x) := \int_{-\infty}^{x} (n_{i0}(y) - \hat{n}_i(y,0) - \bar{n}(y+x_0,0))\,dy,\ \ \psi_{i0}(x) := J_{i0}(x) - \hat{J}_i(x,0) - \bar{J}(x+x_0,0)$$

for $i = 1, 2$, with x_0 defined in (3.1), and $(\hat{n}_1, \hat{n}_2, \hat{J}_1.\hat{J}_2, \hat{E})(x,t)$ is defined in (2.11)–(2.13).

Theorem 1.2 *(Case of $\lambda = \frac{1}{7}$) Under the assumption of Theorem 1.1, there exists a sufficiently small constant $\varepsilon_0 > 0$ such that if $\Phi_0 + \delta_1 \le \varepsilon_0$, then the IVP (1.1)–(1.2) has a unique global classical solution $(n_1, n_2, J_1, J_2, E)(x,t)$, which satisfies*

$$\begin{aligned}
& \sum_{k=0}^{3}(1+t)^{\frac{8}{7}(k+1)}\ln^{-2}(2+t)\|\partial_x^k(n_1-\bar{n}, n_2-\bar{n})(t)\|^2 \\
+ & (1+t)^{\frac{32}{7}}\ln^{-2}(2+t)\|\partial_x^4(n_1-\bar{n}, n_2-\bar{n})(t)\|^2 \\
+ & \sum_{k=0}^{2}(1+t)^{\frac{8}{7}k+2}\ln^{-2}(2+t)\|\partial_x^k(J_1-\hat{J}_1-\bar{J}, J_2-\hat{J}_2-\bar{J})(t)\|^2 \\
+ & (1+t)^{\frac{30}{7}}\ln^{-2}(2+t)\|\partial_x^3(J_1-\hat{J}_1-\bar{J}, J_2-\hat{J}_2-\bar{J})(t)\|^2 \le C(\Phi_0^2+\delta_1),
\end{aligned}$$

and

$$\|(n_1-n_2)(t)\|_3^2+\|(J_1-\hat{J}_1-J_2+\hat{J}_2)(t)\|_2^2+\|(E-\hat{E})(t)\|_4^2\leq C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}$$

for some constant $\nu_0>0$.

Theorem 1.3 *(Case of $\frac{1}{7}<\lambda<1$) Under the assumptions of Theorem 1.1, there exists a sufficiently small constant $\varepsilon_0>0$ such that if $\Phi_0+\delta_1\leq\varepsilon_0$, then the IVP (1.1)–(1.2) has a unique global classical solution $(n_1,n_2,J_1,J_2,E)(x,t)$, which satisfies*

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^{(k+1)(\lambda+1)+\frac{1}{2}-\frac{7}{2}\lambda}\|\partial_x^k(n_1-\bar{n},n_2-\bar{n})(t)\|^2+(1+t)^{\frac{9}{2}+\frac{\lambda}{2}}\|\partial_x^4(n_1-\bar{n},n_2-\bar{n})(t)\|^2\\
+&\sum_{k=0}^{2}(1+t)^{k(\lambda+1)+\frac{5}{2}-\frac{7}{2}\lambda}\|\partial_x^k(J_1-\hat{J}_1-\bar{J},J_2-\hat{J}_2-\bar{J})(t)\|^2\\
+&(1+t)^{\frac{9}{2}-\frac{3\lambda}{2}}\|\partial_x^3(J_1-\hat{J}_1-\bar{J},J_2-\hat{J}_2-\bar{J})(t)\|^2\leq C(\Phi_0^2+\delta_1),
\end{aligned}$$

and

$$\|(n_1-n_2)(t)\|_3^2+\|(J_1-\hat{J}_1-J_2+\hat{J}_2)(t)\|_2^2+\|(E-\hat{E})(t)\|_4^2\leq C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}$$

for some constant $\nu_0>0$.

Furthermore, according to Theorem 1.1–1.3, employing the Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{R})}\leq C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}},\tag{1.4}$$

and noting the time-exponential decay rates of the correction functions $(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(x,t)$ (see Lemma 2.2), we have the following estimates:

Corollary 1.4 *Under the assumptions of Theorem 1.1, it holds*

$$\|n_1-\bar{n},n_2-\bar{n})(t)\|_{L^\infty(\mathbb{R})}\leq\begin{cases}C(1+t)^{-\frac{3(\lambda+1)}{4}}, & -1<\lambda<\frac{1}{7},\\ C(1+t)^{-\frac{6}{7}}\ln(2+t), & \lambda=\frac{1}{7},\\ C(1+t)^{\lambda-1}, & \frac{1}{7}<\lambda<1,\end{cases}$$

$$\|J_1-\bar{J},J_2-\bar{J})(t)\|_{L^\infty(\mathbb{R})}\leq\begin{cases}C(1+t)^{-\frac{\lambda+5}{4}}, & -1<\lambda<\frac{1}{7},\\ C(1+t)^{-\frac{9}{7}}\ln(2+t), & \lambda=\frac{1}{7},\\ C(1+t)^{-\frac{3(1-\lambda)}{2}}, & \frac{1}{7}<\lambda<1,\end{cases}$$

and

$$\|E(t)\|_{L^\infty(\mathbb{R})}\leq Ce^{-Ct^{\nu_0}}$$

for some constant $\nu_0>0$.

Remark 1.5 *Theorem 1.1–1.3 show that for the case $-1<\lambda<1,\mu>0$, the solutions to the Cauchy problem for the one-dimensional time-dependent damped bipolar quantum Euler-Poisson system with two identical pressure functions and without the doping profile time-algebraic converge to the nonlinear diffusion waves. However, we mention that for the critical case $\lambda=1$, the system (1.1) exhibit different mathematical structures, and the asymptotic behavior of the solutions may different from what we study here. For time-dependent damped Euler and classical bipolar Euler-Poisson equations with $\lambda=1$, one can refer to [1, 9, 26, 29, 30]. Next, we also mention that for the corresponding general case that two pressure functions are different and the doping profile is non-zero, the asymptotic profiles of the solutions are expected to be the corresponding stationary waves rather than the nonlinear diffusion waves, which is more physical, but more challenging. Moreover, it is also interesting to study the same kind of problems for the non-isentropic bipolar quantum Euler-Poisson system with time-dependent damping and the multi-dimensional bipolar quantum Euler-Poisson system with time-dependent damping. These are expected to be studied in the forthcoming papers.*

Remark 1.6 *We shall point out that compared to the results in [25] for the classical bipolar Euler-Poisson system with time-dependent damping, it is highly nontrivial, due to the additional dispersion terms, in establishing the a priori estimates. More precisely, first, after introducing perturbation variables, we reformulate the original problem into a system of forth-order hyperbolic equations rather than second-order, and we need more regularity for the solutions. Second, the nonlinear terms* f_{14x} *and* f_{24x} *shown in* $(3.8)_1$ *and* $(3.8)_2$ *are third-order terms, and they can not be controlled by the left-hand side terms of (3.8) when applying the energy method. For this, we introduce the new variables (3.12), (3.16) and* $(3.17)_1$ *and* $(3.17)_2$*, and reformulated the original problem into (3.18)–(3.19), then* f_{14x} *and* f_{24x} *can be controlled by the left-hand side terms of (3.18), and the a priori estimates can be established by the relation between the variables (3.4)–(3.5) and (3.17), (3.20) (see (3.23)). Moreover, the results in this paper are also totally different from that in [35] which studied the case* $\lambda = -1, \mu > 0$*. When* $-1 < \lambda < 1$*, the time-dependent damping effects lead the decay rates of the nonlinear diffusion waves are no longer in logarithmic form, they are in algebraic form, which enforces us to decompose the range of* λ *into three parts,* $-1 < \lambda < \frac{1}{7}$*,* $\lambda = \frac{1}{7}$ *and* $\frac{1}{7} < \lambda < 1$*, to obtain the algebraic convergence rates of the solutions toward the nonlinear diffusion waves.*

The rest of this paper is organized as follows. In Section 2, we make some preliminaries. In Section 3, we reformulate the original problem in terms of the perturbation variables. Section 4 is the key part of this article, in which we first establish the a priori estimates by means of the energy method, then complete the proofs of Theorem 1.1–1.3.

2 Preliminaries

In this section, we first show the behavior of solutions at far fields and analyze the difference between the original solutions and the nonlinear diffusion waves at $x = \pm\infty$, then, to delete the difference, we construct some correction functions. To begin with, without loss of generality, we assume that

$$E^- = E(-\infty, t) = 0.$$

Set

$$f^\pm(t) := f(\pm\infty, t), \qquad f \in \{n_1, n_2, J_1, J_2, E\}.$$

On the one hand, integrating $(1.1)_5$ with respect to x over $\mathbb{R}$, then taking $t = 0$, we have

$$E^+(0) = \int_{-\infty}^{+\infty} (n_{10}(x) - n_{20}(x))dx =: E_+. \tag{2.1}$$

On the other hand, integrating $(1.1)_5$ with respect to x over $\mathbb{R}$ and differentiating the resultant equation with respect to t gives

$$\frac{d}{dt}E^+(t) = -(J_1^+(t) - J_2^+(t)) + (J_1^-(t) - J_2^-(t)), \tag{2.2}$$

which together with (2.1) leads to

$$E^+(t) = \int_{-\infty}^{+\infty} (n_{10}(x) - n_{20}(x))\, dx + \int_0^t \left[-(J_1^+(\tau) - J_2^+(\tau)) + (J_1^-(\tau) - J_2^-(\tau))\right] d\tau. \tag{2.3}$$

Then, as $x \to \pm\infty$, (1.1) and (1.2) can be reduced to

$$\begin{cases} \frac{d}{dt}n_1^\pm(t) = 0, \text{ i.e., } n_1^\pm(t) = n_\pm, \\ \frac{d}{dt}J_1^\pm(t) = n_\pm E^\pm(t) - \frac{J_1^\pm(t)}{(1+t)^\lambda}, \\ \frac{d}{dt}n_2^\pm(t) = 0, \text{ i.e., } n_2^\pm(t) = n_\pm, \\ \frac{d}{dt}J_2^\pm(t) = -n_\pm E^\pm(t) - \frac{J_2^\pm(t)}{(1+t)^\lambda}, \\ E^+(t) = \int_{-\infty}^{+\infty} (n_{10}(x) - n_{20}(x))\, dx \\ \qquad\qquad + \int_0^t \left[-(J_1^+(\tau) - J_2^+(\tau)) + (J_1^-(\tau) - J_2^-(\tau))\right] d\tau, \\ (J_1^\pm, J_2^\pm, E^+)(0) = (J_{1\pm}, J_{2\pm}, E_+). \end{cases}$$

Further, analyzing similar to [32], we have for $-1<\lambda<1$

$$|J_1^-(t)|+|J_2^-(t)|+\Big|\frac{d}{dt}J_1^-(t)\Big|+\Big|\frac{d}{dt}J_2^-(t)\Big|\le C\delta_0e^{-Ct^{1-\lambda}}, \tag{2.4}$$

for $-1<\lambda<0$

$$|E^+(t)|+\Big|\frac{d}{dt}E^+(t)\Big|+|J_1^+(t)|+|J_2^+(t)|+\Big|\frac{d}{dt}J_1^+(t)\Big|+\Big|\frac{d}{dt}J_2^+(t)\Big|\le C\delta_0e^{-Ct^{1+\lambda}}, \tag{2.5}$$

and for $0\le\lambda<1$

$$|E^+(t)|+\Big|\frac{d}{dt}E^+(t)\Big|+|J_1^+(t)|+|J_2^+(t)|+\Big|\frac{d}{dt}J_1^+(t)\Big|+\Big|\frac{d}{dt}J_2^+(t)\Big|\le C\delta_0e^{-Ct^{1-\lambda}}, \tag{2.6}$$

where δ_0 is defined in Theorem 1.1. Therefore, from (2.4)–(2.6), we obtain that for $-1<\lambda<1$

$$\begin{cases} n_i(\pm\infty,t)-\bar n(\pm\infty,t)=0,\ i=1,2,\\ |J_i(\pm\infty,t)-\bar J(\pm\infty,t)|\le C\delta_0e^{-Ct^{\nu_0}},\ i=1,2,\\ |E(+\infty,t)|\le C\delta_0e^{-Ct^{\nu_0}}\end{cases} \tag{2.7}$$

for some positive constant ν_0.

Moreover, we notice that there are some gaps between the original solutions and the nonlinear diffusion waves at far fields $x=\pm\infty$, more precisely, $J_i-\bar J, E\notin L^2(\mathbb{R})$, $i=1,2$. To delete these gaps, we need to construct some correction functions. For this, we choose the correction functions $(\hat n_1,\hat n_2,\hat J_1,\hat J_2,\hat E)(x,t)$ such that

$$\begin{cases} \hat n_{1t}+\hat J_{1x}=0,\\ \hat J_{1t}=\breve n\hat E-\frac{\hat J_1}{(1+t)^\lambda},\\ \hat n_{2t}+\hat J_{2x}=0,\\ \hat J_{2t}=-\breve n\hat E-\frac{\hat J_2}{(1+t)^\lambda},\\ \hat E_x=\hat n_1-\hat n_2,\end{cases} \quad\text{with}\quad \begin{cases} \breve n(x)\to n_\pm \quad \text{as} \quad x\to\pm\infty\\ \hat n_i(x,t)\to 0 \quad \text{as} \quad x\to\pm\infty,\ i=1,2,\\ \hat J_i(x,t)\to J_i^\pm(t) \quad \text{as} \quad x\to\pm\infty,\ i=1,2,\\ \hat E(x,t)\to E^+(t) \quad \text{as} \quad x\to+\infty,\\ \hat E(x,t)\to 0 \quad \text{as} \quad x\to-\infty,\end{cases} \tag{2.8}$$

and $\breve n(x)$ and the initial data $(\hat n_1,\hat n_2,\hat J_1,\hat J_2,\hat E)(x,0)$ are chosen as follows

$$\begin{cases} \breve n(x)=n_-+(n_+-n_-)\int_{-\infty}^{x+2L_0}m_0(y)dy,\\ \hat n_i(x,0)=C_im_0(x),\ i=1,2\\ \hat J_i(x,0)=J_{i-}+(J_{i+}-J_{i-})\int_{-\infty}^{x}m_0(y)dy,\ i=1,2,\\ \hat E(x,0)=E_+\int_{-\infty}^{x}m_0(y)dy.\end{cases} \tag{2.9}$$

Here $L_0>0$ is some constant, $m_0(x)$ is chosen as

$$m_0(x)\ge0,\quad m_0(x)\in C_0^\infty(\mathbb{R}),\quad \operatorname{supp} m_0(x)\subset[-L_0,L_0],\quad \int_{-\infty}^{+\infty}m_0(x)dx=1, \tag{2.10}$$

and C_1,C_2 are constants such that $C_i+\int_0^t(J_i^-(\tau)-J_i^+(\tau))d\tau (i=1,2)$ have no constant terms and

$$C_1-C_2=\int_{-\infty}^{+\infty}(n_{10}(x)-n_{20}(x))dx.$$

Remark 2.1 *$|E^+(t)|\le C\delta_0e^{-Ct^{\nu_0}}$ and (2.3) imply that $\int_{-\infty}^{+\infty}(n_{10}(x)-n_{20}(x))dx+\int_0^t\big[(J_1^-(\tau)-J_1^+(\tau))-(J_2^-(\tau)-J_2^+(\tau))\big]d\tau$ has no constant terms, then there exist some constants C_1 and C_2 such that $C_i+\int_0^t(J_i^-(\tau)-J_i^+(\tau))d\tau(i=1,2)$ have no constant terms and $C_1-C_2=\int_{-\infty}^{+\infty}(n_{10}(x)-n_{20}(x))dx$.*

Then, as in [32], we define

$$\hat{n}_i(x,t) = (C_i + \int_0^t (J_i^-(\tau) - J_i^+(\tau))d\tau)m_0(x),\ i=1,2, \tag{2.11}$$

$$\hat{J}_i(x,t) = J_i^-(t) + (J_i^+(t) - J_i^-(t)) \int_{-\infty}^x m_0(y)dy,\ i=1,2, \tag{2.12}$$

$$\hat{E}(x,t) = E^+(t) \int_{-\infty}^x m_0(y)dy. \tag{2.13}$$

One can verify that (2.11)–(2.13) satisfies (2.8)–(2.9). Noting the decay rates of $J_1^\pm(t), J_2^\pm(t), E^+(t)$ (see (2.4)–(2.6)), and the definition of $m_0(x)$ (see (2.10)), we have the following Lemma.

Lemma 2.2 *There holds*

$$\|\partial_t^l \partial_x^k (\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(t)\|_{L^\infty(\mathbb{R})} \le C\delta_0 e^{-Ct^{\nu_0}},\ k \ge 0,\ l=0,1 \tag{2.14}$$

for some positive constant ν_0*, and*

$$\begin{gathered}
(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(+\infty,t) = (0,0,J_1^+(t), J_2^+(t), E^+(t)),\\
(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(-\infty,t) = (0,0,J_1^-(t), J_2^-(t), 0),\\
\int_{-\infty}^{+\infty} (\hat{n}_1(x,0) - \hat{n}_2(x,0))dx = \int_{-\infty}^{+\infty} (n_{10}(x) - n_{20}(x))dx = E_+.
\end{gathered} \tag{2.15}$$

3 Reformulation of the original problem

Since it is convenient to regard the solution $(n_1, n_2, J_1, J_2, E)(x,t)$ to the IVP (1.1)–(1.2) as the perturbation of $(\bar{n}, \bar{n}, \bar{J}, \bar{J}, 0)$, we are going to reformulate the original problem in terms of the perturbation variables in this section. To begin with, we define

$$x_0 := \frac{1}{n_+ - n_-} \int_{-\infty}^{+\infty} (n_{10}(x) - \hat{n}_1(x,0) - \bar{n}(x,0))dx. \tag{3.1}$$

Then by the same method as in [13, 25], we can prove that

$$\int_{-\infty}^{+\infty} (n_i(x,t) - \hat{n}_i(x,t) - \bar{n}(x+x_0,t))dx = 0,\ i=1,2, \tag{3.2}$$

the details are omitted. Next, from (1.1), (1.3) and (2.8), we have

$$\begin{cases}
(n_1 - \hat{n}_1 - \bar{n})_t + (J_1 - \hat{J}_1 - \bar{J})_x = 0,\\
(J_1 - \hat{J}_1 - \bar{J})_t + \left(\frac{J_1^2}{n_1} + p(n_1) - p(\bar{n})\right)_x - n_1 \left(\frac{(\sqrt{n_1})_{xx}}{\sqrt{n_1}}\right)_x = n_1 E - \check{n}\hat{E} - \frac{J_1 - \hat{J}_1 - \bar{J}}{(1+t)^\lambda} + f_0,\\
(n_2 - \hat{n}_2 - \bar{n})_t + (J_2 - \hat{J}_2 - \bar{J})_x = 0,\\
(J_2 - \hat{J}_2 - \bar{J})_t + \left(\frac{J_2^2}{n_2} + p(n_2) - p(\bar{n})\right)_x - n_2 \left(\frac{(\sqrt{n_2})_{xx}}{\sqrt{n_2}}\right)_x = -n_2 E + \check{n}\hat{E} - \frac{J_2 - \hat{J}_2 - \bar{J}}{(1+t)^\lambda} + f_0,\\
(E - \hat{E})_x = (n_1 - \hat{n}_1 - \bar{n}) - (n_2 - \hat{n}_2 - \bar{n}).
\end{cases} \tag{3.3}$$

Here $f_0 := (1+t)^\lambda p(\bar{n})_{xt} + \lambda(1+t)^{\lambda-1} p(\bar{n})_x$, and $(\bar{n}, \bar{J}) = (\bar{n}, \bar{J})(x+x_0,t)$ is the shifted diffusion wave. Now we take the perturbation variables $(\phi_1, \phi_2, \psi_1, \psi_2, \mathcal{H})(x,t)$ as

$$(\phi_i, \psi_i) := \big(\int_{-\infty}^x (n_i(y,t) - \hat{n}_i(y,t) - \bar{n}(y+x_0,t))\,dy, J_i(x,t) - \hat{J}_i(x,t) - \bar{J}(x+x_0,t)\big),\ i=1,2, \tag{3.4}$$

and

$$\mathcal{H}(x,t) := E(x,t) - \hat{E}(x,t). \tag{3.5}$$

Next, using the fact that

$$n_i\left(\frac{(\sqrt{n_i})_{xx}}{\sqrt{n_i}}\right)_x=\frac{1}{2}n_{ixxx}-\frac{1}{2}\left(\frac{n_{ix}^2}{n_i}\right)_x,\ i=1,2,$$

we get the following equations for perturbation variables $(\phi_1,\phi_2,\psi_1,\psi_2,\mathcal{H})$

$$\begin{cases}\phi_{1t}+\psi_1=0,\\ \psi_{1t}+\left(\frac{(\psi_1+\hat{J}_1+\bar{J})^2}{\phi_{1x}+\hat{n}_1+\bar{n}}+p(\phi_{1x}+\hat{n}_1+\bar{n})-p(\bar{n})\right)_x-\frac{1}{2}(\phi_{1xxxx}+\hat{n}_{1xxx}+\bar{n}_{xxx})\\ \quad+\frac{1}{2}\left(\frac{(\phi_{1xx}+\hat{n}_{1x}+\bar{n}_x)^2}{\phi_{1x}+\hat{n}_1+\bar{n}}\right)_x=(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}+(\phi_{1x}+\hat{n}_1+\bar{n}-\breve{n})\hat{E}-\frac{\psi_1}{(1+t)^\lambda}+f_0,\\ \phi_{2t}+\psi_2=0,\\ \psi_{2t}+\left(\frac{(\psi_2+\hat{J}_2+\bar{J})^2}{\phi_{2x}+\hat{n}_2+\bar{n}}+p(\phi_{2x}+\hat{n}_2+\bar{n})-p(\bar{n})\right)_x-\frac{1}{2}(\phi_{2xxxx}+\hat{n}_{2xxx}+\bar{n}_{xxx})\\ \quad+\frac{1}{2}\left(\frac{(\phi_{2xx}+\hat{n}_{2x}+\bar{n}_x)^2}{\phi_{2x}+\hat{n}_2+\bar{n}}\right)_x=-(\phi_{2x}+\hat{n}_2+\bar{n})\mathcal{H}-(\phi_{2x}+\hat{n}_2+\bar{n}-\breve{n})\hat{E}-\frac{\psi_2}{(1+t)^\lambda}+f_0,\\ \mathcal{H}=\phi_1-\phi_2,\end{cases}\tag{3.6}$$

with the initial data

$$(\phi_1,\phi_2,\psi_1,\psi_2,\mathcal{H})(x,0)=(\phi_{10},\phi_{20},\psi_{10},\psi_{20},\phi_{10}-\phi_{20}).\tag{3.7}$$

Further, substituting $(3.6)_1$ into $(3.6)_2$, and $(3.6)_3$ into $(3.6)_4$, respectively, we obtain the following system for ϕ_1 and ϕ_2 that

$$\begin{cases}\phi_{1tt}+\frac{\phi_{1t}}{(1+t)^\lambda}+\frac{1}{2}\phi_{1xxxx}-(p'(\bar{n})\phi_{1x})_x=-(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}-f_0-f_{11}+f_{12x}+f_{13x}+f_{14x},\\ \phi_{2tt}+\frac{\phi_{2t}}{(1+t)^\lambda}+\frac{1}{2}\phi_{2xxxx}-(p'(\bar{n})\phi_{2x})_x=(\phi_{2x}+\hat{n}_2+\bar{n})\mathcal{H}-f_0-f_{21}+f_{22x}+f_{23x}+f_{24x},\end{cases}\tag{3.8}$$

with

$$(\phi_1,\phi_2)(x,0)=(\phi_{10},\phi_{20})(x),\quad (\phi_{1t},\phi_{2t})(x,0)=(-\psi_{10},-\psi_{20})(x).\tag{3.9}$$

Here for $i=1,2$

$$f_{i1}:=(-1)^{i+1}(\phi_{ix}+\hat{n}_i+\bar{n}-\breve{n})\hat{E}+\frac{1}{2}(\hat{n}_{ixxx}+\bar{n}_{xxx}),\quad f_{i2}:=\frac{(-\phi_{it}+\hat{J}_i+\bar{J})^2}{\phi_{ix}+\hat{n}_i+\bar{n}},$$
$$f_{i3}:=p(\phi_{ix}+\hat{n}_i+\bar{n})-p(\bar{n})-p'(\bar{n})\phi_{ix},\quad f_{i4}:=\frac{1}{2}\frac{(\phi_{ixx}+\hat{n}_{ix}+\bar{n}_x)^2}{\phi_{ix}+\hat{n}_i+\bar{n}}.$$

Moreover, from (3.8)–(3.9), we have

$$\begin{aligned}\mathcal{H}_{tt}+\frac{\mathcal{H}_t}{(1+t)^\lambda}+\frac{1}{2}\mathcal{H}_{xxxx}-(p'(\bar{n})\mathcal{H}_x)_x+2\bar{n}\mathcal{H}&=-(\phi_{1x}+\phi_{2x}+\hat{n}_1+\hat{n}_2)\mathcal{H}-(f_{11}+f_{21})\\ +\quad(f_{12}-f_{22})_x+(f_{13}-f_{23})_x+(f_{14}-f_{24})_x,&\end{aligned}\tag{3.10}$$

with

$$\mathcal{H}(x,0)=\phi_{10}(x)-\phi_{20}(x),\quad \mathcal{H}_t(x,0)=-\psi_{10}(x)+\psi_{20}(x).\tag{3.11}$$

Furthermore, due to the additional dispersion term, it is difficult to estimate $(\phi_1,\phi_2,\mathcal{H})$ from (3.8)–(3.9) and (3.10)–(3.11) directly. More precisely, the nonlinear terms f_{14x} and f_{24x} in (3.8) and (3.10) can not be controlled by the left hand side terms. To overcome this difficulty, we introduce the new variables

$$w_1=\sqrt{n_1},\ w_2=\sqrt{n_2}.\tag{3.12}$$

Then, from (1.1)–(1.2), we have

$$\begin{cases}2w_1w_{1t}+J_{1x}=0,\\ J_{1t}+\left(\frac{J_1^2}{w_1^2}+p(w_1^2)\right)_x-w_1^2\left(\frac{w_{1xx}}{w_1}\right)_x=w_1^2E-\frac{J_1}{(1+t)^\lambda},\\ 2w_2w_{2t}+J_{2x}=0,\\ J_{2t}+\left(\frac{J_2^2}{w_2^2}+p(w_2^2)\right)_x-w_2^2\left(\frac{w_{2xx}}{w_2}\right)_x=-w_2^2E-\frac{J_2}{(1+t)^\lambda},\\ E_x=w_1^2-w_2^2,\end{cases}\tag{3.13}$$

with the initial data

$$(w_1, w_2, J_1, J_2) = (\sqrt{n_{10}}, \sqrt{n_{20}}, J_{10}, J_{20})(x). \tag{3.14}$$

Further, substituting $(3.13)_1$ into $(3.13)_2$, and $(3.13)_3$ into $(3.13)_4$, respectively, we have

$$\begin{cases} w_{1tt} + \frac{w_{1t}}{(1+t)^\lambda} + \frac{1}{2}w_{1xxxx} = -\frac{w_{1t}^2}{w_1} - \frac{1}{2w_1}(w_1^2 E)_x + \frac{1}{2w_1}\left(\frac{J_1^2}{w_1^2} + p(w_1^2)\right)_{xx} + \frac{w_{1xx}^2}{2w_1}, \\ w_{2tt} + \frac{w_{2t}}{(1+t)^\lambda} + \frac{1}{2}w_{2xxxx} = -\frac{w_{2t}^2}{w_2} + \frac{1}{2w_2}(w_2^2 E)_x + \frac{1}{2w_2}\left(\frac{J_2^2}{w_2^2} + p(w_2^2)\right)_{xx} + \frac{w_{2xx}^2}{2w_2}. \end{cases} \tag{3.15}$$

Moreover, denoting

$$\bar{w} = \sqrt{\bar{n}}, \tag{3.16}$$

and setting

$$z_1(x,t) = w_1(x,t) - \bar{w}(x+x_0,t), \quad z_2(x,t) = w_2(x,t) - \bar{w}(x+x_0,t), \tag{3.17}$$

which together with (3.15) leads to the following system for z_1 and z_2

$$\begin{cases} z_{1tt} + \frac{z_{1t}}{(1+t)^\lambda} + \frac{1}{2}z_{1xxxx} = -g_1 - g_{11} - g_{12} + g_{13} + g_{14}, \\ z_{2tt} + \frac{z_{2t}}{(1+t)^\lambda} + \frac{1}{2}z_{2xxxx} = -g_1 - g_{21} + g_{22} + g_{23} + g_{24}, \end{cases} \tag{3.18}$$

with the initial data

$$\begin{cases} z_1(x,0) = z_{10}(x) := \sqrt{n_{10}(x)} - \sqrt{\bar{n}(x+x_0,0)},\ z_2(x,0) = z_{20}(x) := \sqrt{n_{20}(x)} - \sqrt{\bar{n}(x+x_0,0)}, \\ z_{1t}(x,0) = z_{11}(x) := -\frac{J_{10x}(x)}{2\sqrt{n_{10}(x)}} + \frac{\bar{J}_x(x+x_0,0)}{2\sqrt{\bar{n}(x+x_0,0)}},\ z_{2t}(x,0) = z_{21}(x) := -\frac{J_{20x}(x)}{2\sqrt{n_{20}(x)}} + \frac{\bar{J}_x(x+x_0,0)}{2\sqrt{\bar{n}(x+x_0,0)}}, \end{cases} \tag{3.19}$$

where

$$g_1 := \bar{w}_{tt} + \frac{1}{2}\bar{w}_{xxxx},$$

and for $i = 1, 2$

$$g_{i1} := \frac{(z_{it} + \bar{w}_t)^2}{z_i + \bar{w}}, \quad g_{i2} := \frac{1}{2(z_i + \bar{w})}\left((z_i + \bar{w})^2(\mathcal{H} + \hat{E})\right)_x,$$

$$g_{i3} := \frac{p((z_i + \bar{w})^2)_{xx}}{2(z_i + \bar{w})} - \frac{p(\bar{w}^2)_{xx}}{2\bar{w}} + \frac{1}{2(z_i + \bar{w})}\left(\frac{(-\phi_{it} + \hat{J}_i + \bar{J})^2}{(z_i + \bar{w})^2}\right)_{xx}, \quad g_{i4} := \frac{(z_{ixx} + \bar{w}_{xx})^2}{2(z_i + \bar{w})}.$$

Here we have used the fact that

$$-\frac{\bar{w}_t}{(1+t)^\lambda} = -\frac{1}{(1+t)^\lambda}\frac{\bar{n}_t}{2\bar{w}} = -\frac{1}{(1+t)^\lambda}\frac{(1+t)^\lambda p(\bar{n})_{xx}}{2\bar{w}} = -\frac{p(\bar{w}^2)_{xx}}{2\bar{w}}.$$

Finally, setting

$$\chi = z_1 - z_2, \tag{3.20}$$

and subtracting $(3.18)_2$ from $(3.18)_1$, we obtain

$$\chi_{tt} + \frac{\chi_t}{(1+t)^\lambda} + \frac{1}{2}\chi_{xxxx} = -(g_{11} - g_{21}) - (g_{12} + g_{22}) + (g_{13} - g_{23}) + (g_{14} - g_{24}), \tag{3.21}$$

with

$$\chi(x,0) = z_{10}(x) - z_{20}(x), \quad \chi_t(x,0) = z_{11}(x) - z_{21}(x). \tag{3.22}$$

Remark 3.1 *The definition of z_1, z_2 and χ imply that*

$$z_i = \frac{\phi_{ix} + \hat{n}_i}{\sqrt{n_i} + \sqrt{\bar{n}}},\ i = 1, 2,\ \chi = \frac{\mathcal{H}_x + \hat{E}_x}{\sqrt{n_1} + \sqrt{n_2}}. \tag{3.23}$$

4 A priori estimates

To prove Theorem 1.1–1.3, we only need to study the global existence of classical solution $(\phi_1,\phi_2,\mathcal{H},z_1,z_2,\chi)(x,t)$ to IVP (3.8)–(3.9), (3.10)–(3.11), (3.18)–(3.19) and (3.21)–(3.22). To prove this, we shall employ the standard continuation argument based on the local existence and the a priori estimates. The local existence of the classical solution can be obtained by Galerkin method together with iterative method (refer to [10] for details). Thus, the main effort in the rest of this article is to establish the a priori estimates for the solution $(\phi_1,\phi_2,\mathcal{H},z_1,z_2,\chi)(x,t)$. To begin with, we list the decay estimates of the nonlinear diffusion wave $(\bar n,\bar J)$ and $\bar w$ as follows (see [25]).

Lemma 4.1 *For $-1<\lambda<1$, it holds that*

$$\min\{n_+,n_-\}\le \bar n(x,t)\le \max\{n_+,n_-\},\min\{\sqrt{n_+},\sqrt{n_-}\}\le \bar w(x,t)\le \max\{\sqrt{n_+},\sqrt{n_-}\},$$

$$\|\partial_t^l\partial_x^k(\bar n,\bar w)(t)\|\le C|n_+-n_-|(1+t)^{-\frac{(2k-1)(\lambda+1)}{4}-l},\ k+l\ge 1,\ k,l\ge 0,$$

and

$$\|\partial_t^l\partial_x^k\bar J(t)\|\le C|n_+-n_-|(1+t)^{-\frac{(2k-3)(\lambda+1)}{4}-(l+1)},\ k+l\ge 0,\ k,l\ge 0.$$

Furthermore, it holds that

$$\|\partial_t^l\partial_x^k(\bar n,\bar w)(t)\|_{L^\infty(\mathbb{R})}\le C|n_+-n_-|(1+t)^{-\frac{k(\lambda+1)}{2}-l},\ k+l\ge 1,\ k,l\ge 0,$$

and

$$\|\partial_t^l\partial_x^k\bar J(t)\|_{L^\infty(\mathbb{R})}\le C|n_+-n_-|(1+t)^{-\frac{(k-1)(\lambda+1)}{2}-(l+1)},\ k+l\ge 0,\ k,l\ge 0.$$

Next, for $T>0$, we define the solution space as

$$\begin{aligned}X(T):=\{&(\phi_1,\phi_2,\mathcal{H},z_1,z_2,\chi)(x,t)\mid \phi_i\in C^j([0,T];H^{5-2j}(\mathbb{R})),z_i\in C^j([0,T];H^{4-2j}(\mathbb{R})),\\ &\mathcal{H}\in C^j([0,T];H^{5-2j}(\mathbb{R})),\chi\in C^j([0,T];H^{4-2l}(\mathbb{R})),\ i=1,2,\ j=0,1,2\}.\end{aligned}$$

It is convenience to introduce

$$\begin{aligned}N(T)^2:=&\sup_{0\le t\le T}\Big\{\sum_{i=1}^2\Big[\|\phi_i(t)\|^2+(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\lambda+3}\|\phi_{itt}(t)\|^2\\ +&\sum_{k=0}^3(1+t)^{(k+1)(\lambda+1)}\|\partial_x^k z_i(t)\|^2+(1+t)^{4\lambda+4}\|\partial_x^4 z_i(t)\|^2+\sum_{k=0}^1(1+t)^{(k+1)(\lambda+1)+2}\|\partial_x^k z_{it}(t)\|^2\\ +&(1+t)^{2\lambda+4}\|\partial_x^2 z_{it}(t)\|^2+(1+t)^{2\lambda+4}\|z_{itt}(t)\|^2\Big]\Big\},\quad -1<\lambda<\frac{1}{7},\end{aligned}\tag{4.1}$$

$$\begin{aligned}N(T)^2:=&\sup_{0\le t\le T}\Big\{\ln^{-2}(2+t)\sum_{i=1}^2\Big[\|\phi_i(t)\|^2+(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\frac{22}{7}}\|\phi_{itt}(t)\|^2\\ +&\sum_{k=0}^3(1+t)^{\frac{8}{7}(k+1)}\|\partial_x^k z_i(t)\|^2+(1+t)^{\frac{32}{7}}\|\partial_x^4 z_i(t)\|^2+\sum_{k=0}^1(1+t)^{\frac{8}{7}(k+1)+2}\|\partial_x^k z_{it}(t)\|^2\\ +&(1+t)^{\frac{30}{7}}\|\partial_x^2 z_{it}(t)\|^2+(1+t)^{\frac{30}{7}}\|z_{itt}(t)\|^2\Big]\Big\},\quad \lambda=\frac{1}{7},\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}N(T)^2:=&\sup_{0\le t\le T}\Big\{(1+t)^{\frac12-\frac{7\lambda}{2}}\sum_{i=1}^2\Big[\|\phi_i(t)\|^2+(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\lambda+3}\|\phi_{itt}(t)\|^2\\ +&\sum_{k=0}^3(1+t)^{(k+1)(\lambda+1)}\|\partial_x^k z_i(t)\|^2+(1+t)^{4\lambda+4}\|\partial_x^4 z_i(t)\|^2+\sum_{k=0}^1(1+t)^{(k+1)(\lambda+1)+2}\|\partial_x^k z_{it}(t)\|^2\\ +&(1+t)^{2\lambda+4}\|\partial_x^2 z_{it}(t)\|^2+(1+t)^{2\lambda+4}\|z_{itt}(t)\|^2\Big\},\quad \frac{1}{7}<\lambda<1.\end{aligned}\tag{4.3}$$

Then, to prove Theorem 1.1–1.3, it is sufficient to prove the following a priori estimates:

Proposition 4.2 *For $T>0$, let $(\phi_1,\phi_2,\mathcal{H},z_1,z_2,\chi)\in X(T)$ be the solution to IVP (3.8)–(3.9),(3.10)–(3.11),(3.18)–(3.19) and (3.21)–(3.22) in the time interval $[0,T]$. There exists a sufficiently small constant $\varepsilon>0$ such that if $N(T)+\delta_1\le\varepsilon$, then the following estimates hold for any $t\in[0,T]$*

$$\begin{aligned}
&\sum_{i=1}^{2}\Big[\|\phi_i(t)\|^2+(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\lambda+3}\|\phi_{itt}(t)\|^2\\
+\ &\sum_{k=0}^{3}(1+t)^{(k+1)(\lambda+1)}\|\partial_x^k z_i(t)\|^2+(1+t)^{4\lambda+4}\|\partial_x^4 z_i(t)\|^2+\sum_{k=0}^{1}(1+t)^{(k+1)(\lambda+1)+2}\|\partial_x^k z_{it}(t)\|^2\\
+\ &(1+t)^{2\lambda+4}\|\partial_x^2 z_{it}(t)\|^2+(1+t)^{2\lambda+4}\|z_{itt}(t)\|^2\Big]\le C(\Phi_0^2+\delta_1),\quad -1<\lambda<\frac{1}{7},
\end{aligned}$$

$$\begin{aligned}
&\ln^{-2}(2+t)\sum_{i=1}^{2}\Big[\|\phi_i(t)\|^2+(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\frac{22}{7}}\|\phi_{itt}(t)\|^2\\
+\ &\sum_{k=0}^{3}(1+t)^{\frac{8}{7}(k+1)}\|\partial_x^k z_i(t)\|^2+(1+t)^{\frac{32}{7}}\|\partial_x^4 z_i(t)\|^2+\sum_{k=0}^{1}(1+t)^{\frac{8}{7}(k+1)+2}\|\partial_x^k z_{it}(t)\|^2\\
+\ &(1+t)^{\frac{30}{7}}\|\partial_x^2 z_{it}(t)\|^2+(1+t)^{\frac{30}{7}}\|z_{itt}(t)\|^2\Big]\le C(\Phi_0^2+\delta_1)\ln^2(2+t),\quad \lambda=\frac{1}{7},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{2}\Big[\|\phi_i(t)\|^2+(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\lambda+3}\|\phi_{itt}(t)\|^2\\
+\ &\sum_{k=0}^{3}(1+t)^{(k+1)(\lambda+1)}\|\partial_x^k z_i(t)\|^2+(1+t)^{4\lambda+4}\|\partial_x^4 z_i(t)\|^2+\sum_{k=0}^{1}(1+t)^{(k+1)(\lambda+1)+2}\|\partial_x^k z_{it}(t)\|^2\\
+\ &(1+t)^{2\lambda+4}\|\partial_x^2 z_{it}(t)\|^2+(1+t)^{2\lambda+4}\|z_{itt}(t)\|^2\Big]\le C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda}{2}-\frac{1}{2}},\quad \frac{1}{7}<\lambda<1.
\end{aligned}$$

Moreover, for $-1<\lambda<1$ and $0<t<T$, there holds

$$\|(\mathcal{H},\mathcal{H}_x,\mathcal{H}_t,\mathcal{H}_{xx},\chi,\chi_x,\chi_t,\chi_{xx},\chi_{xt},\chi_{xxx})(t)\|^2\le C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.$$

The proof of Proposition 4.2 will be completed by some Lemmas for the sake of clarify. To begin with, from the Sobolev inequality (1.4) and the a priori assumptions (4.1)–(4.3) as well as (3.23), we have for $i=1,2$

$$\|\phi_i(t)\|\le\begin{cases}N(t), & -1<\lambda<\frac{1}{7},\\ N(t)\ln(2+t), & \lambda=\frac{1}{7},\\ N(t)(1+t)^{\frac{7\lambda-1}{4}}, & \frac{1}{7}<\lambda<1,\end{cases}\tag{4.4}$$

$$\|(\phi_{ixx},z_{ix},\phi_{ixt},z_{it})(t)\|\le CN(t),\ -1<\lambda<1,\tag{4.5}$$

and for $-1<\lambda<1$

$$\|\phi_{ix}(t)\|_{L^\infty(\mathbb{R})}+\|z_i(t)\|_{L^\infty(\mathbb{R})}\le CN(t),\tag{4.6}$$

$$\|\phi_{it}(t)\|_{L^\infty(\mathbb{R})}\le CN(t)(1+t)^{\frac{\lambda+1}{2}-1},\tag{4.7}$$

$$\|\phi_{ixx}(t)\|_{L^\infty(\mathbb{R})}+\|z_{ix}(t)\|_{L^\infty(\mathbb{R})}\le CN(t)(1+t)^{-\frac{\lambda+1}{2}},\tag{4.8}$$

$$\|\phi_{ixt}(t)\|_{L^\infty(\mathbb{R})}+\|\phi_{itt}(t)\|_{L^\infty(\mathbb{R})}+\|z_{it}(t)\|_{L^\infty(\mathbb{R})}\le CN(t)(1+t)^{-1},\tag{4.9}$$

$$\|\phi_{ixxx}(t)\|_{L^\infty(\mathbb{R})}+\|z_{ixx}(t)\|_{L^\infty(\mathbb{R})}\le CN(t)(1+t)^{-\lambda-1}\tag{4.10}$$

$$\|\phi_{ixxt}(t)\|_{L^\infty(\mathbb{R})}+\|z_{ixt}(t)\|_{L^\infty(\mathbb{R})}\le CN(t)(1+t)^{-\frac{\lambda+1}{4}-1}.\tag{4.11}$$

Furthermore, we also have for $i = 1, 2$ and $-1 < \lambda < 1$

$$\|n_{ix}(t)\| \le C(N(t) + \delta_1), \tag{4.12}$$
$$\|J_i(t)\|_{L^\infty(\mathbb{R})} \le C(N(t) + \delta_1)(1+t)^{\frac{\lambda+1}{2}-1}, \tag{4.13}$$
$$\|n_{ix}(t)\|_{L^\infty(\mathbb{R})} + \|w_{ix}(t)\|_{L^\infty(\mathbb{R})} \le C(N(t) + \delta_1)(1+t)^{-\frac{\lambda+1}{2}}, \tag{4.14}$$
$$\|n_{it}(t)\|_{L^\infty(\mathbb{R})} + \|w_{it}(t)\|_{L^\infty(\mathbb{R})} + \|J_{ix}(t)\|_{L^\infty(\mathbb{R})} + \|J_{it}(t)\|_{L^\infty(\mathbb{R})} \le C(N(t) + \delta_1)(1+t)^{-1}, \tag{4.15}$$
$$\|n_{ixx}(t)\|_{L^\infty(\mathbb{R})} + \|w_{ixx}(t)\|_{L^\infty(\mathbb{R})} \le C(N(t) + \delta_1)(1+t)^{-\lambda-1}, \tag{4.16}$$
$$\|n_{ixt}(t)\|_{L^\infty(\mathbb{R})} + \|w_{ixt}(t)\|_{L^\infty(\mathbb{R})} + \|J_{ixx}(t)\|_{L^\infty(\mathbb{R})} \le C(N(t) + \delta_1)(1+t)^{-\frac{\lambda+1}{4}-1}. \tag{4.17}$$

Since $N(T) + \delta_1 \ll 1$, then from (4.6), there exist some constants $C_3, C_4 > 0$ such that

$$0 < \frac{1}{C_3} \le n_i = \phi_{ix} + \hat{n}_i + \bar{n} \le C_3,\ 0 < \frac{1}{C_4} \le w_i = \sqrt{n_i} \le C_4,\ i = 1, 2. \tag{4.18}$$

We first show the exponential decay estimates of $\mathcal{H}$ and χ in Lemma 4.3 and 4.4.

Lemma 4.3 *There exists a sufficiently small constant $\varepsilon_1 > 0$ such that if $N(T) + \delta_1 \le \varepsilon_1$, then the following estimate holds that for $-1 < \lambda < 1$ and $t \in [0, T]$*

$$\|(\mathcal{H}, \mathcal{H}_x, \mathcal{H}_t, \mathcal{H}_{xx}, \chi, \chi_x, \chi_t, \chi_{xx})(t)\|^2 \le C(\Phi_0^2 + \delta_1)e^{-Ct^{\nu_0}}. \tag{4.19}$$

Proof. First, by performing $\int_{-\infty}^{+\infty}((3.10) \times 2\mathcal{H}_t + (3.21) \times 2\chi_t)dx$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\left(2\bar{n}\mathcal{H}^2 + \mathcal{H}_t^2 + p'(\bar{n})\mathcal{H}_x^2 + \frac{1}{2}\mathcal{H}_{xx}^2 + \chi_t^2 + \frac{1}{2}\chi_{xx}^2\right)dx + \int_{-\infty}^{+\infty} 2(1+t)^{-\lambda}(\mathcal{H}_t^2 + \chi_t^2)dx \\
= &\int_{-\infty}^{+\infty}(p''(\bar{n})\bar{n}_t\mathcal{H}_x^2 + 2\bar{n}_t\mathcal{H}^2)dx + \int_{-\infty}^{+\infty} 2\mathcal{H}_t\Big[-(\phi_{1x} + \phi_{2x} + \hat{n}_1 + \hat{n}_2)\mathcal{H} - (f_{11} + f_{21}) \\
&+(f_{12} - f_{22})_x + (f_{13} - f_{23})_x + (f_{14} - f_{24})_x\Big]dx + \int_{-\infty}^{+\infty} 2\chi_t\Big[-(g_{11} - g_{21}) - (g_{12} + g_{22}) \\
&+(g_{13} - g_{23}) + (g_{14} - g_{24})\Big]dx.
\end{aligned} \tag{4.20}$$

The right hand side of (4.20) can be estimated as follows. First, from the decay rate of $\bar{n}_t$ shown in Lemma 4.1, it is easy to obtain

$$\int_{-\infty}^{+\infty}(p''(\bar{n})\bar{n}_t\mathcal{H}_x^2 + 2\bar{n}_t\mathcal{H}^2)dx \le C\delta_1\int_{-\infty}^{+\infty}(1+t)^{-1}(\mathcal{H}^2 + \mathcal{H}_x^2)dx. \tag{4.21}$$

We next estimate the second term on the right hand side of (4.20). First, noting the exponential decay rates of $\hat{n}_1, \hat{n}_2, \hat{E}$ in Lemma 2.2 and of $\bar{n}_{xxx}$ in Lemma 4.1 , we have

$$\begin{aligned}
&-\int_{-\infty}^{+\infty} 2\mathcal{H}_t\Big[(\phi_{1x} + \phi_{2x} + \hat{n}_1 + \hat{n}_2)\mathcal{H} + (f_{11} + f_{21})\Big]dx \\
\le\ &-\frac{d}{dt}\int_{-\infty}^{+\infty}(\phi_{1x} + \phi_{2x})\mathcal{H}^2dx + \int_{-\infty}^{+\infty}(\phi_{1xt} + \phi_{2xt} - \hat{n}_1 - \hat{n}_2)\mathcal{H}^2dx \\
&+C\delta_1\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\mathcal{H}_t^2dx + C\delta_1e^{-Ct^{\nu_0}} \\
\le\ &-\frac{d}{dt}\int_{-\infty}^{+\infty}(\phi_{1x} + \phi_{2x})\mathcal{H}^2dx + C(N(t) + \delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}\mathcal{H}^2dx \\
&+C\delta_1\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\mathcal{H}_t^2dx + C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned} \tag{4.22}$$

Second, taking integration by parts and using the Young inequality, Lemma 2.2, 4.1, (4.5),(4.8),(4.9), and (4.13)–(4.15), one gets

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} 2\mathcal{H}_t(f_{12}-f_{22})_x dx \\
\leq\ & \int_{-\infty}^{+\infty} 2\mathcal{H}_t\left(-\frac{J_1^2}{n_1^2}\mathcal{H}_{xx}-\frac{2J_1}{n_1}\mathcal{H}_{xt}\right)dx \\
&+C\int_{-\infty}^{+\infty}|\mathcal{H}_t|(|\phi_{2xx}|+|\phi_{2xt}|+|\bar{n}_x|+|\bar{J}_x|)(|\mathcal{H}_t|+|\mathcal{H}_x|+|\hat{n}_1|+|\hat{n}_2|+|\hat{J}_1|+|\hat{J}_2|)dx \\
\leq\ & \frac{d}{dt}\int_{-\infty}^{+\infty}\frac{J_1^2}{n_1^2}\mathcal{H}_x^2dx-\int_{-\infty}^{+\infty}\Big(\frac{2J_1J_{1t}}{n_1^2}-\frac{2J_1^2n_{1t}}{n_1^3}\Big)\mathcal{H}_x^2dx+\int_{-\infty}^{+\infty}\Big(\frac{4J_1J_{1x}}{n_1^2}-\frac{4J_1^2n_{1x}}{n_1^3}\Big)\mathcal{H}_x\mathcal{H}_tdx \\
&+\int_{-\infty}^{+\infty}\Big(\frac{2J_{1x}}{n_1}-\frac{2J_1n_{1x}}{n_1^2}\Big)\mathcal{H}_t^2dx+C(\|\phi_{2xx}(t)\|_{L^\infty(\mathbb{R})}+\|\phi_{2xt}(t)\|_{L^\infty(\mathbb{R})}+\|\bar{n}_x(t)\|_{L^\infty(\mathbb{R})} \\
&+\|\bar{J}_x(t)\|_{L^\infty(\mathbb{R})})\int_{-\infty}^{+\infty}|\mathcal{H}_t|(|\mathcal{H}_t|+|\mathcal{H}_x|+|\hat{n}_1|+|\hat{n}_2|)dx \\
&+C(\|\hat{J}_1(t)\|_{L^\infty(\mathbb{R})}+\|\hat{J}_2(t)\|_{L^\infty(\mathbb{R})})\int_{-\infty}^{+\infty}(\mathcal{H}_t^2+\phi_{2xx}^2+\phi_{2xt}^2+\bar{n}_x^2+\bar{J}_x^2)dx \\
\leq\ & \frac{d}{dt}\int_{-\infty}^{+\infty}\frac{J_1^2}{n_1^2}\mathcal{H}_x^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\mathcal{H}_t^2dx \\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}\mathcal{H}_x^2dx+C\delta_1e^{-Ct^{\nu_0}}. \qquad (4.23)
\end{aligned}
$$

Next, noting that

$$
\begin{aligned}
(f_{13}-f_{23})_x \;=\; & (p'(n_1)-p'(\bar{n}))\mathcal{H}_{xx}+(p'(n_1)-p'(n_2))(\phi_{2xx}+\bar{n}_x) \\
& +p'(n_1)\hat{n}_{1x}-p'(n_2)\hat{n}_{2x}-p''(\bar{n})\bar{n}_x\mathcal{H}_x,
\end{aligned}
$$

then applying the Taylor formula, using Lemma 2.2, 4.1, (4.8) and (4.14), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}2\mathcal{H}_t(f_{13}-f_{23})_xdx \\
\leq\ & -\frac{d}{dt}\int_{-\infty}^{+\infty}(p'(n_1)-p'(\bar{n}))\mathcal{H}_x^2dx+\int_{-\infty}^{+\infty}(p''(n_1)n_{1t}-p''(\bar{n})\bar{n}_t)\mathcal{H}_x^2dx \\
&-\int_{-\infty}^{+\infty}2\mathcal{H}_t\mathcal{H}_x(p''(n_1)n_{1x}-p''(\bar{n})\bar{n}_x)dx+C\int_{-\infty}^{+\infty}(|\phi_{2xx}|+|\bar{n}_x|)|\mathcal{H}_t|(|\mathcal{H}_x|+|\hat{n}_1|+|\hat{n}_2|)dx \\
&+C\int_{-\infty}^{+\infty}|\mathcal{H}_t|(|\hat{n}_1|+|\hat{n}_2|+|\bar{n}_x\mathcal{H}_x|)dx \\
\leq\ & -\frac{d}{dt}\int_{-\infty}^{+\infty}(p'(n_1)-p'(\bar{n}))\mathcal{H}_x^2dx+C\int_{-\infty}^{+\infty}(|\phi_{1xt}|+|\hat{n}_{1t}|+|(\phi_{1x}+\hat{n}_1)\bar{n}_t|)\mathcal{H}_x^2dx \\
&+C\int_{-\infty}^{+\infty}(|\phi_{1xx}|+|\hat{n}_{1x}|+|(\phi_{1x}+\hat{n}_1)\bar{n}_x|)|\mathcal{H}_t\mathcal{H}_x|dx \\
&+C\int_{-\infty}^{+\infty}(|\phi_{2xx}|+|\bar{n}_x|)|\mathcal{H}_t|(|\mathcal{H}_x|+|\hat{n}_1|+|\hat{n}_2|)dx+C\int_{-\infty}^{+\infty}|\mathcal{H}_t|(|\hat{n}_1|+|\hat{n}_2|+|\bar{n}_x\mathcal{H}_x|)dx \\
\leq\ & -\frac{d}{dt}\int_{-\infty}^{+\infty}(p'(n_1)-p'(\bar{n}))\mathcal{H}_x^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\mathcal{H}_t^2dx \\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}\mathcal{H}_x^2dx+C\delta_1e^{-Ct^{\nu_0}}. \qquad (4.24)
\end{aligned}
$$

Finally, using the following identity

$$(f_{14}-g_{14})_x=\frac{n_{1x}n_{1xx}}{n_1}-\frac{n_{1x}^3}{2n_1^2}-\frac{n_{2x}n_{2xx}}{n_2}+\frac{n_{2x}^3}{2n_2^2}=4w_{1x}\chi_{xx}+4w_{2xx}\chi_x,$$

and employing (4.14) and (4.16), we get

$$\begin{aligned}&\int_{-\infty}^{+\infty}2\mathcal{H}_t(f_{14}-f_{24})_x dx\\ \le\;& C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\mathcal{H}_t^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi_x^2+\chi_{xx}^2)dx.\end{aligned}\tag{4.25}$$

Therefore, combining (4.23)–(4.25), we have the following estimate for the second term on the right hand of side of (4.20)

$$\begin{aligned}&\int_{-\infty}^{+\infty}2\mathcal{H}_t\Big[-(\phi_{1x}+\phi_{2x}+\hat{n}_1+\hat{n}_2)\mathcal{H}-(f_{11}+f_{21})+(f_{12}-f_{22})_x\\ &+(f_{13}-f_{23})_x+(f_{14}-f_{24})_x\Big]dx\\ \le\;& -\frac{d}{dt}\int_{-\infty}^{+\infty}(\phi_{1x}+\phi_{2x})\mathcal{H}^2dx-\frac{d}{dt}\int_{-\infty}^{+\infty}(p'(n_1)-p'(\bar{n})-\frac{J_1^2}{n_1^2})\mathcal{H}_x^2dx\\ &+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\mathcal{H}_t^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\mathcal{H}^2+\mathcal{H}_x^2+\chi_x^2+\chi_{xx}^2)dx\\ &+C\delta_1e^{-Ct^{\nu_0}}.\end{aligned}\tag{4.26}$$

Moreover, let us show the estimates of the third term on the right hand side of (4.20). First, a simple computation gives

$$(g_{11}-g_{21})+(g_{12}+g_{22})=\frac{w_{1t}+w_{2t}}{w_1}\chi_t-\frac{w_{2t}^2}{w_1w_2}\chi+\left(\frac{n_{1x}}{2w_1}+\frac{n_{2x}}{2w_2}\right)(\mathcal{H}+\hat{E})+\frac{1}{2}(w_1+w_2)^2\chi,$$

then using (4.12), (4.14) and (4.15), we conclude

$$\begin{aligned}&-\int_{-\infty}^{+\infty}2\chi_t((g_{11}-g_{21})+(g_{12}+g_{22}))dx\\ \le\;& -\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}(w_1+w_2)^2\chi^2dx+C(\|w_{1t}(t)\|_{L^\infty(\mathbb{R})}+\|w_{2t}(t)\|_{L^\infty(\mathbb{R})})\int_{-\infty}^{+\infty}(\chi^2+\chi_t^2)dx\\ &+C(\|n_{1x}(t)\|_{L^\infty(\mathbb{R})}+\|n_{2x}(t)\|_{L^\infty(\mathbb{R})})\int_{-\infty}^{+\infty}|\chi_t\mathcal{H}|dx+C\|\hat{E}(t)\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{+\infty}(\chi_t^2+n_{1x}^2+n_{2x}^2)dx\\ \le\;& -\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}(w_1+w_2)^2\chi^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi^2+\chi_t^2)dx\\ &+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\frac{\lambda+1}{2}}|\chi_t\mathcal{H}|dx+C\delta_1e^{-Ct^{\nu_0}}(\|n_{1x}\|^2+\|n_{2x}\|^2)\\ \le\;& -\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}(w_1+w_2)^2\chi^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\chi_t^2dx\\ &+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi^2+\mathcal{H}^2)dx+C\delta_1e^{-Ct^{\nu_0}}.\end{aligned}\tag{4.27}$$

Next, noting that

$$g_{13}-g_{23}\;=\;\left(\frac{J_{1x}^2}{w_1n_1}-\frac{J_{2x}^2}{w_2n_2}\right)+\left(\frac{J_1J_{1xx}}{w_1n_1}-\frac{J_2J_{2xx}}{w_2n_2}\right)-\left(\frac{2J_1J_{1x}n_{1x}}{w_1n_1^2}-\frac{J_2J_{2x}n_{2x}}{w_2n_2^2}\right)$$

$$
-\left(\frac{J_1^2 n_{1xx}}{2w_1 n_1^2}-\frac{J_2^2 n_{2xx}}{2w_2 n_2^2}\right)+\left(\frac{J_1^2 n_{1x}^2}{w_1 n_1^3}-\frac{J_2^2 n_{2x}^2}{w_2 n_2^3}\right)+\left(\frac{p''(n_1)n_{1x}^2}{2w_1}-\frac{p''(n_2)n_{2x}^2}{2w_2}\right)
+\left(\frac{p'(n_1)n_{1xx}}{2w_1}-\frac{p'(n_2)n_{2xx}}{2w_2}\right),
$$

then by a tedious computation but in a similar way as (4.23) and (4.24), and using Lemma 4.1 and (4.13)–(4.17), we can show

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} 2\chi_t((g_{13}-g_{23})dx\\
\le\ &\int_{-\infty}^{+\infty} 2\chi_t\left(-\frac{J_1^2}{n_1^2}\chi_{xx}+p'(n_1)\chi_{xx}\right)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx\\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi^2+\chi_x^2+\mathcal{H}^2+\mathcal{H}_x^2)dx+C\delta_1 e^{-Ct^{\nu_0}}\\
\le\ &-\frac{d}{dt}\int_{-\infty}^{+\infty} p'(n_1)\chi_x^2dx+\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{J_1^2}{n_1^2}\chi_x^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx\\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi^2+\chi_x^2+\mathcal{H}^2+\mathcal{H}_x^2)dx+C\delta_1 e^{-Ct^{\nu_0}}.
\end{aligned}\tag{4.28}
$$

Finally, utilizing the Young inequality, and using (4.16), one gets

$$
\begin{aligned}
\int_{-\infty}^{+\infty} 2\chi_t((g_{14}-g_{24})dx &= \int_{-\infty}^{+\infty} 2\chi_t\left(\frac{w_{1xx}+w_{2xx}}{2w_1}\chi_{xx}-\frac{w_{2xx}^2}{2w_1w_2}\chi\right)dx\\
&\le C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\chi_t^2dx\\
&\quad +C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi^2+\chi_{xx}^2)dx.
\end{aligned}\tag{4.29}
$$

Therefore, combining (4.27)-(4.29), we conclude

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} 2\chi_t\Big[-(g_{11}-g_{21})-(g_{12}+g_{22})+(g_{13}-g_{23})+(g_{14}-g_{24})\Big]dx\\
\le\ &-\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}(w_1+w_2)^2\chi^2dx-\frac{d}{dt}\int_{-\infty}^{+\infty}(p'(n_1)-\frac{J_1^2}{n_1^2})\chi_x^2dx\\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\chi^2+\chi_x^2+\chi_{xx}^2\\
&+\mathcal{H}^2+\mathcal{H}_x^2)dx+C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned}\tag{4.30}
$$

Then, Substituting (4.21), (4.26) and (4.30) into (4.20), we see

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(2\bar{n}+\phi_{1x}+\phi_{2x})\mathcal{H}^2+\mathcal{H}_t^2+\left(p'(\bar{n})-\frac{J_1^2}{n_1^2}\right)\mathcal{H}_x^2+\frac{1}{2}\mathcal{H}_{xx}^2+\frac{1}{2}(w_1+w_2)^2\chi^2\\
&+\chi_t^2+\left(p'(n_1)-\frac{J_1^2}{n_1^2}\right)\chi_x^2+\frac{1}{2}\chi_{xx}^2\Big]dx+\int_{-\infty}^{+\infty}2(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx\\
\le\ &C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_{xx}^2\\
&+\chi^2+\chi_x^2+\chi_{xx}^2)dx+C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned}\tag{4.31}
$$

On the one hand, when $-1<\lambda\le 0$, performing $\int_{-\infty}^{+\infty}((3.10)\times(1+t)^\lambda\mathcal{H}+(3.21)\times(1+t)^\lambda\chi)dx$, then analysis similar to (4.31), we obtain

$$
\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(1+t)^\lambda\mathcal{H}\mathcal{H}_t+\frac{1}{2}\mathcal{H}^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{H}^2+(1+t)^\lambda\chi\chi_t+\frac{1}{2}\chi^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\chi^2\Big]dx
$$

$$
\begin{aligned}
&-\int_{-\infty}^{+\infty}(1+t)^{\lambda}(\mathcal{H}_t^2+\chi_t^2)dx+\int_{-\infty}^{+\infty}\frac{\lambda(\lambda-1)}{2}(1+t)^{\lambda-2}(\mathcal{H}^2+\chi^2)dx+\int_{-\infty}^{+\infty}2\bar{n}(1+t)^{\lambda}\mathcal{H}^2dx\\
&+\int_{-\infty}^{+\infty}\frac{1}{2}(1+t)^{\lambda}(w_1+w_2)^2\chi^2dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda}p'(\bar{n})(\mathcal{H}_x^2+\chi_x^2)dx+\int_{-\infty}^{+\infty}\frac{1}{2}(1+t)^{\lambda}(\mathcal{H}_{xx}^2+\chi_{xx}^2)dx\\
\leq\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{\lambda}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_t^2+\mathcal{H}_{xx}^2+\chi^2+\chi_x^2+\chi_t^2+\chi_{xx}^2)dx+C\delta_1e^{-Ct^{\nu_0}}. \qquad (4.32)
\end{aligned}
$$

which together with (4.31) implies that

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(2\bar{n}+\phi_{1x}+\phi_{2x})\mathcal{H}^2+\frac{1}{2}\mathcal{H}^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{H}^2+(1+t)^{\lambda}\mathcal{H}\mathcal{H}_t+\mathcal{H}_t^2\\
&+\left(p'(\bar{n})-\frac{J_1^2}{n_1^2}\right)\mathcal{H}_x^2+\frac{1}{2}\mathcal{H}_{xx}^2+\frac{1}{2}(w_1+w_2)^2\chi^2+\frac{1}{2}\chi^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\chi^2+(1+t)^{\lambda}\chi\chi_t+\chi_t^2\\
&+\left(p'(n_1)-\frac{J_1^2}{n_1^2}\right)\chi_x^2+\frac{1}{2}\chi_{xx}^2\Big]dx+\int_{-\infty}^{+\infty}2(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx-\int_{-\infty}^{+\infty}(1+t)^{\lambda}(\mathcal{H}_t^2+\chi_t^2)dx\\
&+\int_{-\infty}^{+\infty}\frac{1}{2}(1+t)^{\lambda}(\mathcal{H}_{xx}^2+\chi_{xx}^2)dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda}p'(\bar{n})(\mathcal{H}_x^2+\chi_x^2)dx+\int_{-\infty}^{+\infty}2\bar{n}(1+t)^{\lambda}\mathcal{H}^2dx\\
&+\int_{-\infty}^{+\infty}\frac{1}{2}(1+t)^{\lambda}(w_1+w_2)^2\chi^2dx+\int_{-\infty}^{+\infty}\frac{\lambda(\lambda-1)}{2}(1+t)^{\lambda-2}(\mathcal{H}^2+\chi^2)dx\\
\leq\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{\lambda}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_t^2+\mathcal{H}_{xx}^2+\chi^2+\chi_x^2+\chi_t^2+\chi_{xx}^2)dx\\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}(\mathcal{H}_t^2+\chi_t^2)dx+C\delta_1e^{-Ct^{\nu_0}}. \qquad (4.33)
\end{aligned}
$$

Further, setting

$$
\begin{aligned}
h_1(t)\ :=\ &\int_{-\infty}^{+\infty}\Big[(2\bar{n}+\phi_{1x}+\phi_{2x})\mathcal{H}^2+\frac{1}{2}\mathcal{H}^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{H}^2+(1+t)^{\lambda}\mathcal{H}\mathcal{H}_t+\mathcal{H}_t^2\\
&+\left(p'(\bar{n})-\frac{J_1^2}{n_1^2}\right)\mathcal{H}_x^2+\frac{1}{2}\mathcal{H}_{xx}^2+\frac{1}{2}(w_1+w_2)^2\chi^2+\frac{1}{2}\chi^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\chi^2\\
&+(1+t)^{\lambda}\chi\chi_t+\chi_t^2+\left(p'(n_1)-\frac{J_1^2}{n_1^2}\right)\chi_x^2+\frac{1}{2}\chi_{xx}^2\Big]dx, \qquad (4.34)
\end{aligned}
$$

and using the smallness of $N(T)+\delta_1$, there exists a positive constant θ_1 such that

$$\frac{d}{dt}h_1+\theta_1(1+t)^{\lambda}h_1\leq C\delta_1e^{-Ct^{\nu_0}}.$$

Then applying Gronwall's inequality, we have

$$h_1(t)\leq C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}},$$

which together with (4.34) leads to

$$\int_{-\infty}^{+\infty}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_t^2+\mathcal{H}_{xx}^2+\chi^2+\chi_x^2+\chi_t^2+\chi_{xx}^2)dx\leq C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}. \qquad (4.35)$$

On the other hand, when $0<\lambda<1$, by performing $\int_{-\infty}^{+\infty}((3.10)\times(1+t)^{-\lambda}\mathcal{H}+(3.21)\times(1+t)^{-\lambda}\chi)dx$, and analysis similar to (4.35), we also have

$$\int_{-\infty}^{+\infty}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_t^2+\mathcal{H}_{xx}^2+\chi^2+\chi_x^2+\chi_t^2+\chi_{xx}^2)dx\leq C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.$$

Thus, for $-1<\lambda<1$, we have (4.19). The proof is completed.

Lemma 4.4 *There exists a positive constant $\varepsilon_2 < \varepsilon_1$ such that if $N(T)+\delta_1 \le \varepsilon_2$, then the following estimate holds for $-1<\lambda<1$ and $t\in[0,T]$*

$$\|(\chi_{xt},\chi_{xxx})(t)\|^2 \le C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}. \tag{4.36}$$

Proof. By calculating $\int_{-\infty}^{+\infty}\partial_x(3.21)\times 2\chi_{xt}dx$, then employing Lemma 4.3, we can prove that

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{+\infty}\left[\chi_{xt}^2+\frac{1}{2}\chi_{xxx}^2+\left(p'(n_1)-\frac{J_1^2}{n_1}^2\right)\chi_{xx}^2\right]dx+\int_{-\infty}^{+\infty}2(1+t)^{-\lambda}\chi_{xt}^2dx\\ \le\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-\lambda}\chi_{xt}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{-1}\chi_{xxx}^2dx+C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.\end{aligned} \tag{4.37}$$

Firstly, when $-1<\lambda\le 0$, by taking $\int_{-\infty}^{+\infty}\partial_x(3.21)\times(1+t)^{\lambda}\chi_x dx$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{+\infty}\left[(1+t)^{\lambda}\chi_x\chi_{xt}-\frac{\lambda}{2}(1+t)^{\lambda-1}\chi_x^2+\frac{1}{2}\chi_x^2\right]dx-\int_{-\infty}^{+\infty}(1+t)^{\lambda}(\chi_{xt}^2-\frac{1}{2}\chi_{xxx}^2)dx\\ \le\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(1+t)^{\lambda}\chi_{xt}^2dx+C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.\end{aligned} \tag{4.38}$$

Adding (4.38) to (4.37), using the smallness of $N(T)+\delta_1$ and Gronwall's inequality, one gets

$$\int_{-\infty}^{+\infty}(\chi_{xt}^2+\chi_{xxx}^2)dx \le C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.$$

On the other hand, when $0<\lambda<1$, by taking $\int_{-\infty}^{+\infty}\partial_x(3.21)\times(1+t)^{-\lambda}\chi_x dx$, then analysis similar to above, we also have

$$\int_{-\infty}^{+\infty}(\chi_{xt}^2+\chi_{xxx}^2)dx \le C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.$$

Thus, for $-1<\lambda<1$, it holds

$$\int_{-\infty}^{+\infty}(\chi_{xt}^2+\chi_{xxx}^2)dx \le C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}. \tag{4.39}$$

We complete the proof.

Lemma 4.5 *There exists a positive constant $\varepsilon_3 < \varepsilon_2$ such that if $N(T)+\delta_1 \le \varepsilon_3$, then the following estimates hold for $t\in[0,T]$ and i=1,2*

$$\begin{aligned}&\|(\phi_i,z_i)(t)\|^2+(1+t)^{\lambda+1}\|(\phi_{ix},\phi_{it},\phi_{ixx},z_{ix},z_{it},z_{ixx})(t)\|^2\\ &+\int_0^t(1+\tau)^{\lambda}\|(\phi_{ix},\phi_{ixx},z_{ix},z_{ixx})(\tau)\|^2d\tau+\int_0^t(1+\tau)\|(\phi_{it},z_{it})(\tau)\|^2d\tau\\ \le\ & C(\Phi_0^2+\delta_1),\quad -1<\lambda<\frac{1}{7},\end{aligned}$$

$$\begin{aligned}&\|(\phi_i,z_i)(t)\|^2+(1+t)^{\lambda+1}\|(\phi_{ix},\phi_{it},\phi_{ixx},z_{ix},z_{it},z_{ixx})(t)\|^2\\ &+\int_0^t(1+\tau)^{\lambda}\|(\phi_{ix},\phi_{ixx},z_{ix},z_{ixx})(\tau)\|^2d\tau+\int_0^t(1+\tau)\|(\phi_{it},z_{it})(\tau)\|^2d\tau\\ \le\ & C(\Phi_0^2+\delta_1)\ln^2(2+t),\quad \lambda=\frac{1}{7}\end{aligned}$$

and

$$\|(\phi_i,z_i)(t)\|^2+(1+t)^{\lambda+1}\|(\phi_{ix},\phi_{it},\phi_{ixx},z_{ix},z_{it},z_{ixx})(t)\|^2$$

$$
\begin{aligned}
&+\int_0^t (1+\tau)^\lambda \|(\phi_{ix},\phi_{ixx},z_{ix},z_{ixx})(\tau)\|^2 d\tau + \int_0^t (1+\tau)\|(\phi_{it},z_{it})(\tau)\|^2 d\tau \\
\leq\ & C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}}, \quad \frac{1}{7}<\lambda<1.
\end{aligned}
$$

Proof. Let $\alpha\geq 0, \beta>0, \kappa\geq 0$ be constants which will be determined later. Multiplying $(3.8)_1$ by $\big(2(\alpha+t)^\kappa\phi_{1t}+\beta(\alpha+t)^{\kappa-1}\phi_1\big)$ and $(3.18)_1$ by $\big(2(\alpha+t)^\kappa z_{1t}+\beta(\alpha+t)^{\kappa-1}z_1\big)$, adding them together and then integrating the resultant equation with respect to x over $\mathbb{R}$, one has

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(\alpha+t)^\kappa(\phi_{1t}^2+z_{1t}^2)+\frac{1}{2}(\alpha+t)^\kappa(\phi_{1xx}^2+z_{1xx}^2)+(\alpha+t)^\kappa p'(\bar{n})\phi_{1x}^2 \\
&+\beta(\alpha+t)^{\kappa-1}(\phi_1\phi_{1t}+z_1z_{1t})-\frac{\beta(\kappa-1)}{2}(\alpha+t)^{\kappa-2}(\phi_1^2+z_1^2)+\frac{\beta(\alpha+t)^{\kappa-1}}{2(1+t)^\lambda}(\phi_1^2+z_1^2)\Big]dx \\
&+\int_{-\infty}^{+\infty}\frac{\beta-\kappa}{2}(\alpha+t)^{\kappa-1}(\phi_{1xx}^2+z_{1xx}^2)dx+\int_{-\infty}^{+\infty}(\beta-\kappa)(\alpha+t)^{\kappa-1}p'(\bar{n})\phi_{1x}^2dx \\
&+\int_{-\infty}^{+\infty}\Big[\frac{2(\alpha+t)^\kappa}{(1+t)^\lambda}-(\kappa+\beta)(\alpha+t)^{\kappa-1}\Big](\phi_{1t}^2+z_{1t}^2)dx \\
&+\int_{-\infty}^{+\infty}\Big[\frac{\beta}{2}(\kappa-1)(\kappa-2)(\alpha+t)^{\kappa-3}-\frac{\beta(\kappa-1)(\alpha+t)^{\kappa-2}}{2(1+t)^\lambda}+\frac{\beta\lambda(\alpha+t)^{\kappa-1}}{2(1+t)^{\lambda-1}}\Big](\phi_1^2+z_1^2)dx \\
=\ &\int_{-\infty}^{+\infty}(\alpha+t)^\kappa p''(\bar{n})\bar{n}_t\phi_{1x}^2dx-\int_{-\infty}^{+\infty}\Big(2(\alpha+t)^\kappa\phi_{1t}+\beta(\alpha+t)^{\kappa-1}\phi_1\Big)(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}dx \\
&+\int_{-\infty}^{+\infty}\Big(2(\alpha+t)^\kappa\phi_{1t}+\beta(\alpha+t)^{\kappa-1}\phi_1\Big)(-f_0-f_{11}+f_{12x}+f_{13x}+f_{14x})dx \\
&+\int_{-\infty}^{+\infty}\Big(2(\alpha+t)^\kappa z_{1t}+\beta(\alpha+t)^{\kappa-1}z_1\Big)(-g_0-g_{11}-g_{12}+g_{13}+g_{14})dx.
\end{aligned}
\tag{4.40}
$$

Now let us estimate the terms on the right-hand side of (4.40) one by one. First, from Lemma 2.2, 4.1 and 4.3, and using the Young and Hölder inequalities and (4.13)-(4.15), we have

$$
\int_{-\infty}^{+\infty}(\alpha+t)^\kappa p''(\bar{n})\bar{n}_t\phi_{1x}^2dx\leq C\delta_1\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}\phi_{1x}^2dx, \tag{4.41}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\left(2(\alpha+t)^\kappa\phi_{1t}+\beta(\alpha+t)^{\kappa-1}\phi_1\right)\left(-(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}-f_0-f_{11}+f_{12x}\right)dx \\
\leq\ &\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{J_1^2}{n_1^2}(\alpha+t)^\kappa\phi_{1x}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}\phi_{1t}^2dx \\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}\phi_{1x}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-3}\phi_1^2dx+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-5}{2}} \\
&+C\delta_1(1+t)^{\frac{2\kappa-3\lambda-5}{2}}+C\delta_1(1+t)^{\frac{4\kappa+3\lambda-9}{4}}\|\phi_1(t)\|+C\delta_1(1+t)^{\frac{4\kappa-5\lambda-9}{4}}\|\phi_1(t)\|+C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned}
\tag{4.42}
$$

Next, applying Taylor's formula, and using Lemma 2.2, 4.1, (4.14) and (4.15), one gets

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\left(2(\alpha+t)^\kappa\phi_{1t}+\beta(\alpha+t)^{\kappa-1}\phi_1\right)f_{13x}dx \\
\leq\ &-\frac{d}{dt}\int_{-\infty}^{+\infty}(\alpha+t)^\kappa(p'(n_1)-p'(\bar{n}))\phi_{1x}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}\phi_{1x}^2dx \\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}\phi_{1t}^2dx+C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned}
\tag{4.43}
$$

Moreover, noting that

$$f_{14x} = \left(\frac{n_{1x}^2}{2n_1}\right)_x = 4w_{1x}w_{1xx},$$

then with the help of integration by parts, and using Young's inequality, Lemma 4.1 and (4.14), we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left(2(\alpha+t)^\kappa \phi_{1t} + \beta(\alpha+t)^{\kappa-1}\phi_1\right) f_{14x}dx \\
=\ & \int_{-\infty}^{+\infty} 8(\alpha+t)^\kappa \phi_{1t} w_{1x} w_{1xx} dx - \int_{-\infty}^{+\infty} \beta(\alpha+t)^{\kappa-1}\phi_{1x}\left(\frac{n_{1x}^2}{2n_1}\right)dx \\
\leq\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty} (\alpha+t)^{\kappa-\frac{\lambda+1}{2}}(|\phi_{1t}z_{1xx}| + |\phi_{1t}\bar{w}_{xx}|)dx \\
& +C(N(t)+\delta_1)\int_{-\infty}^{+\infty} (\alpha+t)^{\kappa-\frac{\lambda+3}{2}}|\phi_{1x}|(|\phi_{1xx}| + |\hat{n}_{1x}| + |\bar{n}_x|)dx \\
\leq\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty} [(\alpha+t)^{\kappa-\lambda}\phi_{1t}^2 + (\alpha+t)^{\kappa-1}(\phi_{1x} + \phi_{1xx}^2 + z_{1xx}^2)]dx \\
& +C\delta_1(1+t)^{\frac{2\kappa-3\lambda-5}{2}} + C\delta_1 e^{-Ct^{\nu_0}}.
\end{aligned} \tag{4.44}$$

Next, similar to (4.41)-(4.42), by a simple computation, we also have

$$\begin{aligned}
& -\int_{-\infty}^{+\infty} \left(2(\alpha+t)^\kappa z_{1t} + \beta(\alpha+t)^{\kappa-1}z_1\right)(g_0+g_{11}+g_{12})dx. \\
\leq\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty} [(\alpha+t)^{\kappa-\lambda}z_{1t} + (\alpha+t)^{\kappa-3}z_1^2]dx + C\delta_1((1+t)^{\frac{4\kappa+3\lambda-9}{4}} \\
& +(1+t)^{\frac{4\kappa-5\lambda-9}{4}})\|z_1(t)\| + C\delta_1((1+t)^{\frac{2\kappa+5\lambda-5}{2}} + (1+t)^{\frac{2\kappa+\lambda-5}{2}} + (1+t)^{\frac{2\kappa-3\lambda-5}{2}}) \\
& +C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.
\end{aligned} \tag{4.45}$$

Moreover, taking integration by parts, we have

$$\begin{aligned}
& \int_{-\infty}^{+\infty} 2(\alpha+t)^\kappa z_{1t} g_{13} dx \\
=\ & \int_{-\infty}^{+\infty} \Big[-(\alpha+t)^\kappa(2p'(w_1^2)w_{1x} - 2p'(\bar{w}^2)\bar{w}_x)z_{1xt} + (\alpha+t)^\kappa\Big(\frac{w_{1x}n_{1x}}{n_1}p'(n_1) - \frac{\bar{w}_x\bar{n}_x}{\bar{n}}p'(\bar{n})\Big)z_{1t} \\
& -(\alpha+t)^\kappa \frac{1}{w_1}\left(\frac{2J_1J_{1x}}{w_1^2} - \frac{2J_1^2 w_{1x}}{w_1^3}\right)z_{1xt} + (\alpha+t)^\kappa\frac{w_{1x}}{w_1^2}\left(\frac{2J_1J_{1x}}{w_1^2} - \frac{2J_1^2 w_{1x}}{w_1^3}\right)z_{1t}dx \\
=:\ & I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.46}$$

First, going through a tedious computation, employing Young's inequality and Taylor's formula, and using Lemma 4.1 and (4.14)-(4.15), we obtain

$$\begin{aligned}
I_1 =\ & -\int_{-\infty}^{+\infty} (\alpha+t)^\kappa(2p'(w_1^2)w_{1x} - 2p'(\bar{w}^2)\bar{w}_x)z_{1xt}dx \\
=\ & -\int_{-\infty}^{+\infty} 2(\alpha+t)^\kappa p'(n_1)z_{1x}z_{1xt}dx - \int_{-\infty}^{+\infty} 2(\alpha+t)^\kappa(p'(n_1)-p'(\bar{n}))\bar{w}_x z_{1xt}dx \\
=\ & -\frac{d}{dt}\int_{-\infty}^{+\infty} (\alpha+t)^\kappa p'(n_1)z_{1x}^2dx + \int_{-\infty}^{+\infty} \kappa(\alpha+t)^{\kappa-1}p'(n_1)z_{1x}^2dx + \int_{-\infty}^{+\infty} (\alpha+t)^\kappa p''(n_1)n_{1t}z_{1x}^2dx \\
& + \int_{-\infty}^{+\infty} 2(\alpha+t)^\kappa(p'(n_1)-p'(\bar{n}))\bar{w}_{xx}z_{1t}dx + \int_{-\infty}^{+\infty} 2(\alpha+t)^\kappa(p''(n_1)n_{1x} - p''(\bar{n})\bar{n}_x)\bar{w}_x z_{1t}dx \\
\leq\ & -\frac{d}{dt}\int_{-\infty}^{+\infty} (\alpha+t)^\kappa p'(n_1)z_{1x}^2dx + \int_{-\infty}^{+\infty} \kappa(\alpha+t)^{\kappa-1}p'(n_1)z_{1x}^2dx + C\delta_1 e^{-Ct^{\nu_0}}
\end{aligned}$$

$$+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(z_{1x}^2+\phi_{1x}^2+\phi_{1xx}^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}z_{1t}^2dx$$
$$+C\delta_1 e^{-Ct^{\nu_0}}, \tag{4.47}$$

and

$$\begin{aligned}
I_2 &= \int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\left(\frac{w_{1x}n_{1x}}{n_1}p'(n_1)-\frac{\bar{w}_x\bar{n}_x}{\bar{n}}p'(\bar{n})\right)z_{1t}dx \\
&= \int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{z_{1t}}{n_1}(z_{1x}n_{1x}p'(n_1)+\bar{w}_xp'(n_1)(\phi_{1xx}+\hat{n}_{1x})+\bar{w}_x\bar{n}_x(p'(n_1)-p'(\bar{n})))dx \\
&\quad -\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{\bar{w}_x\bar{n}_xp'(\bar{n})}{n_1\bar{n}}(\phi_{1x}+\hat{n}_1)z_{1t}dx \\
&\leq C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(z_{1x}^2+\phi_{1x}^2+\phi_{1xx}^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}z_{1t}^2dx \\
&\quad +C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned} \tag{4.48}$$

Next, using the fact that $J_{1x}=-2w_1w_{1t}$, and utilizing (4.13)-(4.15), one gets

$$\begin{aligned}
I_3 &= -\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{1}{w_1}\left(\frac{2J_1J_{1x}}{w_1^2}-\frac{2J_1^2w_{1x}}{w_1^3}\right)z_{1xt}dx \\
&= \int_{-\infty}^{+\infty}4(\alpha+t)^{\kappa}\frac{J_1}{w_1}^2(z_{1t}+\bar{w}_t)z_{1xt}dx+\int_{-\infty}^{+\infty}2(\alpha+t)^{\kappa}\frac{J_1^2}{w_1^4}(z_{1x}+\bar{w}_x)z_{1xt}dx \\
&\leq \frac{d}{dt}\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{J_1^2}{n_1^2}z_{1x}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}z_{1t}^2dx \\
&\quad +C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}z_{1x}^2dx+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-5}{2}},
\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}
I_4 &= \int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{w_{1x}}{w_1^2}\left(\frac{2J_1J_{1x}}{w_1^2}-\frac{2J_1^2w_{1x}}{w_1^3}\right)z_{1t}dx \\
&= -\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{4J_1w_{1x}}{w_1^3}z_{1t}(z_{1t}+\bar{w}_t)dx-\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{2J_1^2w_{1x}}{w_1^5}(z_{1x}+\bar{w}_x)z_{1t}dx \\
&\leq C(N(t)+\delta_1)\int_{-\infty}^{+\infty}\left[(\alpha+t)^{\kappa-\lambda}z_{1t}^2+(\alpha+t)^{\kappa-1}z_{1x}^2\right]dx+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-5}{2}}.
\end{aligned} \tag{4.50}$$

Therefore, combining (4.46)-(4.50), we see

$$\begin{aligned}
&\int_{-\infty}^{+\infty}2(\alpha+t)^{\kappa}z_{1t}g_{13}dx \\
\leq\ & -\frac{d}{dt}\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}p'(n_1)z_{1x}^2dx+\frac{d}{dt}\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa}\frac{J_1^2}{n_1^2}z_{1x}^2dx \\
&+\int_{-\infty}^{+\infty}\kappa(\alpha+t)^{\kappa-1}p'(n_1)z_{1x}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(z_{1x}^2+\phi_{1x}^2+\phi_{1xx}^2)dx \\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}z_{1t}^2dx+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-5}{2}}+C\delta_1e^{-Ct^{\nu_0}}.
\end{aligned} \tag{4.51}$$

Next, in a similar way, we also have

$$\int_{-\infty}^{+\infty}\beta(\alpha+t)^{\kappa-1}z_1g_{13}dx \quad\leq\quad -\int_{-\infty}^{+\infty}\beta(\alpha+t)^{\kappa-1}p'(n_1)z_{1x}^2dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}z_{1t}^2dx$$

$$
\begin{aligned}
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(z_{1x}^2+\phi_{1x}^2+\phi_{1xx}^2)dx\\
&+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-5}{2}}+C\delta_1 e^{-Ct^{\nu_0}}. \qquad (4.52)
\end{aligned}
$$

Finally, utilizing Lemma 4.1, (4.6) and (4.9), we have

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\left(2(\alpha+t)^{\kappa}z_{1t}+\beta(\alpha+t)^{\kappa-1}z_1\right)f_{24}dx\\
\leq\ & CN(t)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(z_{1xx}^2+\bar{w}_{xx}^2)dx\\
\leq\ & CN(t)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}z_{1xx}^2dx+C\delta_1(1+t)^{\frac{2\kappa-3\lambda-5}{2}}. \qquad (4.53)
\end{aligned}
$$

Therefore, substituting (4.41)–(4.45), (4.51)–(4.53) into (4.40) yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(\alpha+t)^{\kappa}(\phi_{1t}^2+z_{1t}^2)+\beta(\alpha+t)^{\kappa-1}(\phi_1\phi_{1t}+z_1z_{1t})+\frac{1}{2}(\alpha+t)^{\kappa}(\phi_{1xx}^2+z_{1xx}^2)\\
&+(\alpha+t)^{\kappa}\left(p'(n_1)-\frac{J_1^2}{n_1^2}\right)(\phi_{1x}^2+z_{1x}^2)-\frac{\beta(\kappa-1)}{2}(\alpha+t)^{\kappa-2}(\phi_1^2+z_1^2)\\
&+\frac{\beta(\alpha+t)^{\kappa-1}}{2(1+t)^{\lambda}}(\phi_1^2+z_1^2)\Big]dx+\int_{-\infty}^{+\infty}\frac{\beta-\kappa}{2}(\alpha+t)^{\kappa-1}(\phi_{1xx}^2+z_{1xx}^2)dx\\
&+\int_{-\infty}^{+\infty}\left[\frac{2(\alpha+t)^{\kappa}}{(1+t)^{\lambda}}-(\kappa+\beta)(\alpha+t)^{\kappa-1}\right](\phi_{1t}^2+z_{1t}^2)dx\\
&+\int_{-\infty}^{+\infty}(\beta-\kappa)(\alpha+t)^{\kappa-1}p'(\bar{n})\phi_{1x}^2dx+\int_{-\infty}^{+\infty}(\beta-\kappa)(\alpha+t)^{\kappa-1}p'(n_1)z_{1x}^2dx\\
&+\int_{-\infty}^{+\infty}\left[\frac{\beta}{2}(\kappa-1)(\kappa-2)(\alpha+t)^{\kappa-3}-\frac{\beta(\kappa-1)(\alpha+t)^{\kappa-2}}{2(1+t)^{\lambda}}+\frac{\beta\lambda(\alpha+t)^{\kappa-1}}{2(1+t)^{\lambda-1}}\right](\phi_1^2+z_1^2)dx\\
\leq\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-3}(\phi_1^2+z_1^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}(\phi_{1t}^2+z_{1t}^2)dx\\
&+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dx+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-5}{2}}\\
&+C\delta_1(1+t)^{\frac{2\kappa+\lambda-5}{2}}+C\delta_1(1+t)^{\frac{2\kappa-3\lambda-5}{2}}+C\delta_1(1+t)^{\frac{4\kappa+3\lambda-9}{4}}(\|\phi_1(t)\|+\|z_1(t)\|)\\
&+C\delta_1(1+t)^{\frac{4\kappa-5\lambda-9}{4}}(\|\phi_1(t)\|+\|z_1(t)\|)+C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}. \qquad (4.54)
\end{aligned}
$$

In the following, to obtain the desired estimates, we need to choose proper constant α, β, κ. Indeed, when $-1<\lambda<0$, taking $\alpha=1,\beta=1,\kappa=\lambda+1$ in (4.54), integrating the resultant equation over $(0,t)$, then using the (4.4) and $N(T)+\delta_1\ll 1$, we have

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}(\phi_1^2+z_1^2)dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda+1}(\phi_{1x}^2+\phi_{1t}^2+\phi_{1xx}^2+z_{1x}^2+z_{1t}^2+z_{1xx}^2)dx\\
&+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\lambda}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dxd\tau+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau\\
\leq\ & C(\Phi_0^2+\delta_1).
\end{aligned}
$$

Nest, when $\lambda=0$, we take $\alpha=\beta=1,\kappa=0$ in (4.54), then integrating it over $(0,t)$, one has

$$
\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{-2}(\phi_1^2+z_1^2)dxd\tau\leq C(\Phi_0^2+\delta_1). \qquad (4.55)
$$

Moreover, taking $\alpha = \beta = 2, \kappa = 1$ in (4.54), then integrating the resultant equation over $(0,t)$, by using (4.55), we can prove

$$
\begin{aligned}
& \int_{-\infty}^{+\infty}(\phi_1^2+z_1^2)dx + \int_{-\infty}^{+\infty}(1+t)(\phi_{1x}^2+\phi_{1t}^2+\phi_{1xx}^2+z_{1x}^2+z_{1t}^2+z_{1xx}^2)dx \\
+ & \int_0^t\int_{-\infty}^{+\infty}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dxd\tau + \int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau \le C(\Phi_0^2+\delta_1).
\end{aligned}
$$

Finally, when $0<\lambda<1$, taking $\beta = 2(\lambda+1), \kappa = \lambda+1$ in (4.54), we get

$$
\begin{aligned}
& \frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(\alpha+t)^{\lambda+1}(\phi_{1t}^2+z_{1t}^2)+(2\lambda+2)(\alpha+t)^{\lambda}(\phi_1\phi_{1t}+z_1z_{1t})+\frac{1}{2}(\alpha+t)^{\lambda+1}(\phi_{1xx}^2+z_{1xx}^2) \\
& +(\alpha+t)^{\lambda+1}\left(p'(n_1)-\frac{J_1^2}{n_1^2}\right)(\phi_{1x}^2+z_{1x}^2)-\lambda(\lambda+1)(\alpha+t)^{\lambda-1}(\phi_1^2+z_1^2) \\
& +\frac{(\lambda+1)(\alpha+t)^{\lambda}}{(1+t)^{\lambda}}(\phi_1^2+z_1^2)\Big]dx+\int_{-\infty}^{+\infty}\frac{\lambda+1}{2}(\alpha+t)^{\lambda}(\phi_{1xx}^2+z_{1xx}^2)dx \\
& +\int_{-\infty}^{+\infty}\left[\frac{2(\alpha+t)^{\lambda+1}}{(1+t)^{\lambda}}-3(\lambda+1)(\alpha+t)^{\lambda}\right](\phi_{1t}^2+z_{1t}^2)dx \\
& +\int_{-\infty}^{+\infty}(\lambda+1)(\alpha+t)^{\lambda}p'(\bar n)\phi_{1x}^2dx+\int_{-\infty}^{+\infty}(\lambda+1)(\alpha+t)^{\lambda}p'(n_1)z_{1x}^2dx \\
& +\int_{-\infty}^{+\infty}\left[\lambda(\lambda-1)(\lambda+1)(\alpha+t)^{\lambda-2}-\frac{\lambda(\lambda+1)(\alpha+t)^{\lambda-1}}{(1+t)^{\lambda}}+\frac{\lambda(\lambda+1)(\alpha+t)^{\lambda}}{(1+t)^{\lambda+1}}\right](\phi_1^2+z_1^2)dx \\
\le\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\lambda-2}(\phi_1^2+z_1^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)(\phi_{1t}^2+z_{1t}^2)dx \\
& +C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\lambda}(z_{1x}^2+\phi_{1x}^2+\phi_{1xx}^2+z_{1xx}^2)dx+C\delta_1(1+t)^{\frac{7\lambda-3}{2}} \\
& +C\delta_1(1+t)^{\frac{7\lambda-5}{4}}(\|\phi_1(t)\|+\|z_1(t)\|)+C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}.
\end{aligned}
\tag{4.56}
$$

Moreover, when α is sufficiently large, we can prove that

$$
\begin{aligned}
& (\alpha+t)^{\lambda+1}(\phi_{1t}^2+z_{1t}^2)+(2\lambda+2)(\alpha+t)^{\lambda}(\phi_1\phi_{1t}+z_1z_{1t}) \\
& +\Big(\frac{(\lambda+1)(\alpha+t)^{\lambda}}{(1+t)^{\lambda}}-\lambda(\lambda+1)(\alpha+t)^{\lambda-1}\Big)(\phi_1^2+z_1^2) \\
\ge\ & \frac{1}{2}(\alpha+t)^{\lambda+1}(\phi_{1t}^2+z_1^2)+\frac{\lambda+1}{2}(\phi_1^2+z_1^2),
\end{aligned}
$$

$$
\frac{2(\alpha+t)^{\lambda+1}}{(1+t)^{\lambda}}-3(\lambda+1)(\alpha+t)^{\lambda}\ge\alpha+t,
$$

and

$$
\frac{\lambda(\lambda+1)(\alpha+t)^{\lambda}}{(1+t)^{\lambda+1}}-\frac{\lambda(\lambda+1)(\alpha+t)^{\lambda-1}}{(1+t)^{\lambda}}+\lambda(\lambda-1)(\lambda+1)(\alpha+t)^{\lambda-2}\ge\frac{\lambda(\lambda+1)}{2}(\alpha+t)^{-1}.
$$

Then integrating (4.56) over $(0,t)$ and using the smallness of $N(T)+\delta_1$, one gets

$$
\begin{aligned}
& \int_{-\infty}^{+\infty}(\phi_1^2+z_1^2)dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda+1}(\phi_{1x}^2+\phi_{1t}^2+\phi_{1xx}^2+z_{1x}^2+z_{1t}^2+z_{1xx}^2)dx \\
& +\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\lambda}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dxd\tau+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau \\
\le\ & C\delta_1\int_0^t(1+\tau)^{\frac{7\lambda-5}{4}}(\|\phi_1(\tau)\|+\|z_1(\tau)\|d\tau+C\delta_1\int_0^t(1+\tau)^{\frac{7\lambda-3}{2}}+C(\Phi_0^2+\delta_1).
\end{aligned}
\tag{4.57}
$$

Firstly, when $0 < \lambda < \frac{1}{7}$, using (4.4), we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(\phi_1^2+z_1^2)dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda+1}(\phi_{1x}^2+\phi_{1t}^2+\phi_{1xx}^2+z_{1x}^2+z_{1t}^2+z_{1xx}^2)dx\\
&+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\lambda}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dxd\tau+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau\\
\leq\ & C(\Phi_0^2+\delta_1).
\end{aligned}$$

Secondly, when $\lambda = \frac{1}{7}$, from (4.4), it follows that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(\phi_1^2+z_1^2)dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda+1}(\phi_{1x}^2+\phi_{1t}^2+\phi_{1xx}^2+z_{1x}^2+z_{1t}^2+z_{1xx}^2)dx\\
&+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\lambda}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dxd\tau+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau\\
\leq\ & C(\Phi_0^2+\delta_1)\ln^2(2+t).
\end{aligned}$$

Finally, when $\frac{1}{7} < \lambda < 1$, by (4.4), we can show

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(\phi_1^2+z_1^2)dx+\int_{-\infty}^{+\infty}(1+t)^{\lambda+1}(\phi_{1x}^2+\phi_{1t}^2+\phi_{1xx}^2+z_{1x}^2+z_{1t}^2+z_{1xx}^2)dx\\
&+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\lambda}(\phi_{1x}^2+\phi_{1xx}^2+z_{1x}^2+z_{1xx}^2)dxd\tau+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau\\
\leq\ & C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}}.
\end{aligned}$$

Moreover, similar arguments apply to ϕ_2, z_2, we also can obtain the desired estimates. This completes the proof.

Lemma 4.6 *There exists a positive constant $\varepsilon_4 < \varepsilon_3$ such that if $N(T) + \delta_1 \leq \varepsilon_4$, then the following estimates hold for $t \in [0, T]$ and $i = 1, 2$*

$$\begin{aligned}
&(1+t)^{\lambda+1}\|z_i(t)\|^2+(1+t)^{2\lambda+2}\|(z_{ix},z_{ixx},z_{it})(t)\|^2+\int_0^t(1+\tau)^{2\lambda+1}\|(z_{ix},z_{ixx})(\tau)\|^2d\tau\\
+\ &\int_0^t(1+\tau)^{\lambda+2}\|(z_{it}(\tau)\|^2d\tau\leq C(\Phi_0^2+\delta_1),\quad -1<\lambda<\frac{1}{7},
\end{aligned}$$

$$\begin{aligned}
&(1+t)^{\frac{8}{7}}\|z_i(t)\|^2+(1+t)^{\frac{16}{7}}\|(z_{ix},z_{ixx},z_{it})(t)\|^2+\int_0^t(1+\tau)^{\frac{9}{7}}\|(z_{ix},z_{ixx})(\tau)\|^2d\tau\\
+\ &\int_0^t(1+\tau)^{\frac{15}{7}}\|(z_{it}(\tau)\|^2d\tau\leq C(\Phi_0^2+\delta_1)\ln^2(2+t),\quad \lambda=\frac{1}{7},
\end{aligned}$$

and

$$\begin{aligned}
&(1+t)^{\lambda+1}\|z_i(t)\|^2+(1+t)^{2\lambda+2}\|(z_{ix},z_{ixx},z_{it})(t)\|^2+\int_0^t(1+\tau)^{2\lambda+1}\|(z_{ix},z_{ixx})(\tau)\|^2d\tau\\
+\ &\int_0^t(1+\tau)^{\lambda+2}\|(z_{it}(\tau)\|^2d\tau\leq C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}},\quad \frac{1}{7}<\lambda<1.
\end{aligned}$$

Proof. Let α be a proper large constant. Taking $\int_{-\infty}^{+\infty}(3.18)_1 \times (2(\alpha+t)^{2\lambda+2}z_{1t} + 4(\alpha+t)^{2\lambda+1}z_1)dx$, then by the same method as in the proof of Lemma 4.5, we can prove that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(\alpha+t)^{2\lambda+2}z_{it}^2+4(\alpha+t)^{2\lambda+1}z_1z_{it}+\frac{1}{2}(\alpha+t)^{2\lambda+2}z_{1xx}^2+(\alpha+t)^{2\lambda+2}\big(p'(w_1^2)-\frac{J_1^2}{w_1^4}\big)z_{1x}^2$$

$$
\begin{aligned}
& -2(2\lambda+1)(\alpha+t)^{2\lambda} z_1^2 + \frac{2(\alpha+t)^{2\lambda+1}}{(1+t)^\lambda} z_1^2 \Big] dx + \int_{-\infty}^{+\infty} (2-2\lambda)(\alpha+t)^{2\lambda+1} p'(w_1^2) z_{1x}^2 dx \\
& + \int_{-\infty}^{+\infty} (1-\lambda)(\alpha+t)^{2\lambda+1} z_{1xx}^2 dx + \int_{-\infty}^{+\infty} \Big[\frac{2(\alpha+t)^{2\lambda+2}}{(1+t)^\lambda} - (2\lambda+6)(\alpha+t)^{2\lambda+1} \Big] z_{1t}^2 dx \\
\leq\ & C \int_{-\infty}^{+\infty} (\alpha+t)^\lambda z_1^2 dx + C(N(t)+\delta_1) \int_{-\infty}^{+\infty} (\alpha+t)^{2\lambda+1} (z_{1x}^2 + z_{1xx}^2) dx \\
& + C(N(t)+\delta_1) \int_{-\infty}^{+\infty} (\alpha+t)^{\lambda+2} z_{1t}^2 dx + C\delta_1 (1+t)^{\frac{7\lambda-3}{2}} + C\delta_1 (1+t)^{\frac{-\lambda-3}{2}} + C(\Phi_0^2+\delta_1) e^{-Ct^{\nu_0}}.
\end{aligned}
$$

Then integrating it over $(0,t)$, and using relation $z_1 = \frac{\phi_{1x}+\hat{n}_1}{\sqrt{n_1}+\sqrt{\bar{n}}}$ and Lemma 4.5, we obtain the desired estimates for $i=1$. The estimates for $i=2$ can be obtained in the same way. This completes the proof.

By performing $\int_{-\infty}^{+\infty} \partial_x (3.18)_i \times (2(\alpha+t)^{3\lambda+3} z_{ixt} + 6(\alpha+t)^{3\lambda+2} z_{ix})]dx$ and $\int_{-\infty}^{+\infty} \partial_x^2 (3.18)_i \times (2(\alpha+t)^{4\lambda+4} z_{ixxt} + 8(\alpha+t)^{4\lambda+3} z_{ixx})]dx$ for some large number α and $i=1,2$, and analysis similar to the proof of Lemma 4.6, we can prove the following two lemmas.

Lemma 4.7 *There exists a positive constant $\varepsilon_5 < \varepsilon_4$ such that if $N(T)+\delta_1 \leq \varepsilon_5$, then the following estimates hold for $t \in [0,T]$ and $i=1,2$*

$$
\begin{aligned}
& (1+t)^{3\lambda+3} \|(z_{ixx}, z_{ixt}, z_{ixxx})(t)\|^2 + \int_0^t (1+\tau)^{3\lambda+2} \|(z_{ixx}, z_{ixxx})(\tau)\|^2 d\tau \\
+ & \int_0^t (1+\tau)^{2\lambda+3} \|z_{ixt}(\tau)\|^2 d\tau \leq C(\Phi_0^2+\delta_1), \quad -1 < \lambda < \frac{1}{7},
\end{aligned}
$$

$$
\begin{aligned}
& (1+t)^{\frac{24}{7}} \|(z_{ixx}, z_{ixt}, z_{ixxx})(t)\|^2 + \int_0^t (1+\tau)^{\frac{17}{7}} \|(z_{ixx}, z_{ixxx})(\tau)\|^2 d\tau \\
+ & \int_0^t (1+\tau)^{\frac{23}{7}} \|z_{ixt}(\tau)\|^2 d\tau \leq C(\Phi_0^2+\delta_1) \ln^2(2+t), \quad \lambda = \frac{1}{7},
\end{aligned}
$$

and

$$
\begin{aligned}
& (1+t)^{3\lambda+3} \|(z_{ixx}, z_{ixt}, z_{ixxx})(t)\|^2 + \int_0^t (1+\tau)^{3\lambda+2} \|(z_{ixx}, z_{ixxx})(\tau)\|^2 d\tau \\
+ & \int_0^t (1+\tau)^{2\lambda+3} \|z_{ixt}(\tau)\|^2 d\tau \leq C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}}, \quad \frac{1}{7} < \lambda < 1.
\end{aligned}
$$

Lemma 4.8 *There exists a positive constant $\varepsilon_6 < \varepsilon_5$ such that if $N(T)+\delta_1 \leq \varepsilon_6$, then the following estimates hold for $t \in [0,T]$ and $i=1,2$*

$$
\begin{aligned}
& (1+t)^{4\lambda+4} \|(z_{ixxx}, z_{ixxt}, z_{ixxxx})(t)\|^2 + \int_0^t (1+\tau)^{4\lambda+3} \|(z_{ixxx}, z_{ixxxx})(\tau)\|^2 d\tau \\
+ & \int_0^t (1+\tau)^{3\lambda+4} \|z_{ixxt}(\tau)\|^2 d\tau \leq C(\Phi_0^2+\delta_1), \quad -1 < \lambda < \frac{1}{7},
\end{aligned}
$$

$$
\begin{aligned}
& (1+t)^{\frac{32}{7}} \|(z_{ixxx}, z_{ixxt}, z_{ixxxx})(t)\|^2 + \int_0^t (1+\tau)^{\frac{25}{7}} \|(z_{ixxx}, z_{ixxxx})(\tau)\|^2 d\tau \\
+ & \int_0^t (1+\tau)^{\frac{31}{7}} \|z_{ixxt}(\tau)\|^2 d\tau \leq C(\Phi_0^2+\delta_1) \ln^2(2+t), \quad \lambda = \frac{1}{7},
\end{aligned}
$$

and

$$
(1+t)^{4\lambda+4} \|(z_{ixxx}, z_{ixxt}, z_{ixxxx})(t)\|^2 + \int_0^t (1+\tau)^{4\lambda+3} \|(z_{ixxx}, z_{ixxxx})(\tau)\|^2 d\tau
$$

$$
+ \quad \int_0^t (1+\tau)^{3\lambda+4}\|z_{ixxt}(\tau)\|^2 d\tau \le C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}}, \quad \frac{1}{7}<\lambda<1.
$$

Lemma 4.9 *There exists a positive constant $\varepsilon_7<\varepsilon_6$ such that if $N(T)+\delta_1\le\varepsilon_7$, then the following estimates hold for $t\in[0,T]$ and $i=1,2$*

$$
\begin{aligned}
& (1+t)^2\|(\phi_{it},z_{it})(t)\|^2+(1+t)^{\lambda+3}\|(\phi_{ixt},\phi_{itt},\phi_{ixxt},z_{ixt},z_{itt},z_{ixxt})(t)\|^2 \\
& +\int_0^t (1+\tau)^{\lambda+2}\|(\phi_{ixt},\phi_{ixxt},z_{ixt},z_{ixxt})(\tau)\|^2 d\tau+\int_0^t (1+\tau)^3\|(\phi_{itt},z_{itt})(\tau)\|^2 d\tau \\
\le\ & C(\Phi_0^2+\delta_1), \quad 0<\lambda<\frac{1}{7},
\end{aligned}
$$

$$
\begin{aligned}
& (1+t)^2\|(\phi_{it},z_{it})(t)\|^2+(1+t)^{\frac{22}{7}}\|(\phi_{ixt},\phi_{itt},\phi_{ixxt},z_{ixt},z_{itt},z_{ixxt})(t)\|^2 \\
& +\int_0^t (1+\tau)^{\frac{15}{7}}\|(\phi_{ixt},\phi_{ixxt},z_{ixt},z_{ixxt})(\tau)\|^2 d\tau+\int_0^t (1+\tau)^3\|(\phi_{itt},z_{itt})(\tau)\|^2 d\tau \\
\le\ & C(\Phi_0^2+\delta_1)\ln^2(2+t), \quad \lambda=\frac{1}{7},
\end{aligned}
$$

and

$$
\begin{aligned}
& (1+t)^2\|(\phi_{it},z_{it})(t)\|^2+(1+t)^{\lambda+3}\|(\phi_{ixt},\phi_{itt},\phi_{ixxt},z_{ixt},z_{itt},z_{ixxt})(t)\|^2 \\
& +\int_0^t (1+\tau)^{\lambda+2}\|(\phi_{ixt},\phi_{ixxt},z_{ixt},z_{ixxt})(\tau)\|^2 d\tau+\int_0^t (1+\tau)^3\|(\phi_{itt},z_{itt})(\tau)\|^2 d\tau \\
\le\ & C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}}, \quad \frac{1}{7}<\lambda<1.
\end{aligned}
$$

Proof. Let $\alpha\ge 0,\beta>0,\kappa\ge 0$ be constants which will be determined later. By performing $\int_{-\infty}^{+\infty}[\partial_t(3.8)_1\times(2(\alpha+t)^\kappa\phi_{1tt}+\beta(\alpha+t)^{\kappa-1}\phi_{1t})+\partial_t(3.18)_1\times(2(\alpha+t)^\kappa z_{1tt}+\beta(\alpha+t)^{\kappa-1}z_{1t})]dx$, we have

$$
\begin{aligned}
& \frac{d}{dt}\int_{-\infty}^{+\infty}\Big[(\alpha+t)^\kappa(\phi_{1tt}^2+z_{1tt}^2)+(\alpha+t)^\kappa\left(p'(\bar n)-\frac{J_1^2}{n_1^2}\right)(\phi_{1xt}^2+z_{1xt}^2) \\
& +\frac{1}{2}(\alpha+t)^\kappa(\phi_{1xxt}^2+z_{1xxt}^2)+\beta(\alpha+t)^{\kappa-1}(\phi_{1t}\phi_{1tt}+z_{1t}z_{1tt}) \\
& +\Big(\frac{\beta(\alpha+t)^{\kappa-1}}{2(1+t)^\lambda}-\frac{\lambda(\alpha+t)^\kappa}{(1+t)^{\lambda+1}}-\frac{\beta}{2}(\kappa-1)(\alpha+t)^{\kappa-2}\Big)(\phi_{1t}^2+z_{1t}^2)\Big]dx \\
& +\int_{-\infty}^{+\infty}\left(\frac{2(\alpha+t)^\kappa}{(1+t)^\lambda}-(\kappa+\beta)(\alpha+t)^{\kappa-1}\right)(\phi_{1tt}^2+z_{1tt}^2)dx \\
& +\int_{-\infty}^{+\infty}\frac{\beta-\kappa}{2}(\alpha+t)^{\kappa-1}(\phi_{1xxt}^2+z_{1xxt}^2)dx+\int_{-\infty}^{+\infty}(\beta-\kappa)(\alpha+t)^{\kappa-1}p'(\bar n)\phi_{1xt}^2dx \\
& +\int_{-\infty}^{+\infty}(\beta-\kappa)(\alpha+t)^{\kappa-1}p'(n_1)z_{1xt}^2dx+\int_{-\infty}^{+\infty}\Big[\frac{\beta}{2}(\kappa-1)(\kappa-2)(\alpha+t)^{\kappa-3} \\
& +\frac{(\kappa-\beta)\lambda(\alpha+t)^{\kappa-1}}{2(1+t)^{\lambda+1}}-\frac{\lambda(\lambda+1)(\alpha+t)^\kappa}{(1+t)^{\lambda+2}}-\frac{\beta(\kappa-1)(\alpha+t)^{\kappa-2}}{2(1+t)^\lambda}\Big](\phi_{1t}^2+z_{1t}^2)dx \\
\le\ & C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda}(\phi_{1tt}^2+z_{1tt}^2)dx+C(N(t)+\delta_1)\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-1}(\phi_{1xt}^2+\phi_{1xxt}^2 \\
& +z_{1xt}^2+z_{1xxt}^2)dx+C\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-3}(\phi_{1x}^2+z_{1x}^2)dx+C\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa-\lambda-2}(\phi_{1t}^2+z_{1t}^2)dx \\
& +C\int_{-\infty}^{+\infty}(\alpha+t)^{\kappa+\lambda-2}(\phi_{1xx}^2+z_{1xx}^2)dx+C\delta_1(1+t)^{\frac{2\kappa+5\lambda-9}{2}}
\end{aligned}
$$

$$+C\delta_1(1+t)^{\frac{2\kappa-3\lambda-9}{2}}+C(\Phi_0^2+\delta_1)e^{-Ct^{\nu_0}}. \tag{4.58}$$

To obtain the desired estimates, first, we taking α be a large number, $\beta=4, \kappa=2\lambda+2$ in (4.58), then analysis similar to the proof of Lemma 4.5, we can prove that

$$\begin{aligned}
&\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^2(\phi_{1tt}^2+z_{1tt}^2)dxd\tau \le C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^\lambda(\phi_{1x}^2+z_{1x}^2)dxd\tau\\
+\quad & C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)(\phi_{1t}^2+z_{1t}^2)dxd\tau+C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{2\lambda+1}(\phi_{1xx}^2+z_{1xx}^2)dxd\tau\\
+\quad & C\delta_1\int_0^t(1+\tau)^{\frac{7\lambda-3}{2}}d\tau+C\delta_1\int_0^t(1+\tau)^{\frac{-\lambda-3}{2}}d\tau+C(\Phi_0^2+\delta_1)
\end{aligned} \tag{4.59}$$

Next, taking $\alpha=1, \beta=4, \kappa=\lambda+3$ in (4.58), and using Lemma 4.5, 4.6, 4.7 and (4.59), we obtain the desired estimates for $i=1$. Applying the same argument for $i=2$, we then complete the proof.

Lemma 4.10 *There exists a positive constant $\varepsilon_8<\varepsilon_7$ such that if $N(T)+\delta_1\le\varepsilon_8$, then the following estimates hold for $t\in[0,T]$ and $i=1,2$*

$$\begin{aligned}
&(1+t)^{\lambda+3}\|z_{it}(t)\|^2+(1+t)^{2\lambda+4}\|(z_{ixt},z_{itt},z_{ixxt})(t)\|^2\\
&+\int_0^t(1+\tau)^{2\lambda+3}\|(z_{ixt},z_{ixxt})(\tau)\|^2d\tau+\int_0^t(1+\tau)^{\lambda+4}\|z_{itt}(\tau)\|^2d\tau\\
\le\ & C(\Phi_0^2+\delta_1),\quad 0<\lambda<\frac{1}{7},
\end{aligned}$$

$$\begin{aligned}
&(1+t)^{\frac{22}{7}}\|z_{it}(t)\|^2+(1+t)^{\frac{30}{7}}\|(z_{ixt},z_{itt},z_{ixxt}(t)\|^2\\
&+\int_0^t(1+\tau)^{\frac{23}{7}}\|(z_{ixt},z_{ixxt})(\tau)\|^2d\tau+\int_0^t(1+\tau)^{\frac{29}{7}}\|(z_{itt}(\tau)\|^2d\tau\\
\le\ & C(\Phi_0^2+\delta_1)\ln^2(2+t),\quad \lambda=\frac{1}{7},
\end{aligned}$$

and

$$\begin{aligned}
&(1+t)^{\lambda+3}\|z_{it}(t)\|^2+(1+t)^{2\lambda+4}\|(z_{ixt},z_{itt},z_{ixxt})(t)\|^2\\
&+\int_0^t(1+\tau)^{2\lambda+3}\|(z_{ixt},z_{ixxt})(\tau)\|^2d\tau+\int_0^t(1+\tau)^{\lambda+4}\|z_{itt}(\tau)\|^2d\tau\\
\le\ & C(\Phi_0^2+\delta_1)(1+t)^{\frac{7\lambda-1}{2}},\quad \frac{1}{7}<\lambda<1.
\end{aligned}$$

Proof. By performing $\int_{-\infty}^{+\infty}\partial_t(3.18)_i\times(2(\alpha+t)^{2\lambda+4}z_{1tt}+6(\alpha+t)^{2\lambda+3}z_{1t})dx$ for some large number α and $i=1,2$, then analysis similar to Lemma 4.9 and applying Lemma 4.3–4.9, we can obtain the desired estimates. We omit the details here.

Recalling Lemma 4.3–4.10, we complete the proof Proposition 4.2. Then by the definition of $(\phi_1,\phi_2,\mathcal{H},z_1,z_2,\chi)$ shown in (3.4)–(3.5), (3.17), (3.20), we complete the proofs of Theorem 1.1–1.3.

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