APPROXIMATION BY MODIFIED q-GAMMA TYPE OPERATORS IN A POLYNOMIAL WEIGHTED SPACE

Jitendra Kumar Singh[1], Purshottam Narain Agrawal[2] and Arun Kajla[3]

[1,2]Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, India

[3]Department of Mathematics
Central University of Haryana
Haryana-123031, India
[1]jsingh@ma.iitr.ac.in, [2]p.agrawal@ma.iitr.ac.in, [3]arunkajla@cuh.ac.in

Abstract. In this research article, we construct a q-analogue of the operators defined by Betus and Usta (Numer. Methods Partial Differential Eq. 1-12, (2020)) and study approximation properties in a polynomial weighted space. Further, we modify these operators to study the approximation properties of differentiable functions in the same space and show that the modified operators give a better rate of convergence.
Mathematics Subject Classification. (2020) : $41A10, 41A25, 41A36, 41A60$.
Keywords: Gamma type operators; Modulus of continuity; Polynomial weighted space.

1. INTRODUCTION

In 1912, Bernstein [6] gave an elegant and simple constructive proof of Weierstrass first approximation theorem, which deals with the approximation of continuous functions on a compact interval $[a, b]$, by using a sequence of polynomial operators known as Bernstein polynomials [28]. The approximation properties of the Bernstein polynomials and their various modifications and generalizations have been extensively discussed by many researchers in the literature, see for instance [2] and [20]. The essential role of a sequence of positive linear operators was not observed meticulously until the theorem of P .P. Korovkin, which provides a tool to determine the convergence of a sequence of positive linear operators to the identity operator with regard to the supremum norm of the space $\mathcal{C}[a, b]$. Several researchers e.g., Baskakov [5], Szász [37] and Lupaş and Müller [31] etc., introduced the sequences of positive linear operators to approximate continuous functions on the infinite interval $[0, \infty)$. The approximation properties of these most familiar operators have been invesigated by many researchers, for instance please see [12,18,21,35,38] etc. In 1976, the famous mathematician Gadjiev [17] gave the Korovkin type theorem to approximate continuous functions defined on $[0, \infty)$, which has since been applied by many researchers to establish the convergence of various sequences of positive linear operators. The sequence of positive linear operators introduced by Lupaş and Müller [31], widely known in the literature as Gamma operators, is defined as

$$G_n(f; \varkappa) = \int_0^\infty K_n(\varkappa, u) f\left(\frac{n}{u}\right) du, \tag{1.1}$$

where

$$K_n(\varkappa, u) = \frac{\varkappa^{n+1}}{\Gamma(n+1)} e^{-\varkappa u} u^n, \ \varkappa \in (0, \infty).$$

These operators are one of the most extensively discussed operators in approximation theory. Zeng [40] studied some convergence properties of Gamma operators,e.g. the asymptotic rate of convergence for locally bounded functions and the optimal rate of convergence for absolutely continuous functions. Karsli [23] determined the rate of convergence by a new kind of Gamma operators for functions with derivatives of bounded variation. Later, Karsli et al. [26] extended the study of these operators to the pointwise convergence rate of the operators at the Lebesgue points of a function of bounded variation on $[0,\infty)$. Several generalizations and modifications of Gamma operators have been thoroughly investigated by researchers to study the approximation of functions in different function spaces, for instance one may refer to [1,3,4,9–11,13,14,16,19,24,25,29,30,33,34,36,39] etc.

Following King's approach [27], Betus and Usta [7] introduced a modification of the operators 1.1 as

$$\mathcal{G}_n(f;\varkappa)=\int_0^\infty \mathcal{K}_n(\varkappa,u)f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)du, \tag{1.2}$$

where

$$\mathcal{K}_n(\varkappa,u)=\frac{\varkappa^n}{\Gamma(n+1)}e^{-\varkappa u^{1/n}}, \varkappa\in(0,\infty),$$

so as to preserve the function $\varkappa^2$, and studied some approximation properties e.g. Voronovskaya type theorems, pointwise estimates and the rate of convergence of continuous functions in a weighted space. Our goal in this paper is to define a q- analogue of the operators (1.2) and examine some of their approximation properties for functions in a polynomial weighted space. To this end, first we recall some basic notations and definitions of quantum calculus [22], which are used throughout this article.
For any fixed real number q satisfying $0<q<1$ and $r\in\mathbb{N}$, the q-integer of r is defined as

$$[r]_q:=\begin{cases}\frac{1-q^r}{1-q}, & \text{if } q\neq 1,\\ r, & \text{if } q=1,\end{cases}$$

and the q-factorial $[r]_q!$ as

$$[r]_q!:=\begin{cases}[r]_q[r-1]_q...[1]_q, & \text{if } r\in\mathbb{N},\\ 1, & \text{if } r=0.\end{cases}$$

For any $k\in\mathbb{N}\bigcup\{0\}$ such that $0\le k\le r$, the binomial coefficient is defined as

$$\binom{r}{k}_q:=\frac{[r]_q!}{[r-k]_q![k]_q!}.$$

The q-exponential function is defined as

$$E_q(u)=\sum_{\kappa=0}^{\infty}q^{\kappa(\kappa-1)/2}\frac{u^\kappa}{[\kappa]_q!},$$

and for any real valued function h, the q-improper integral is defined as

$$\int_0^{\infty/A}h(u)d_qu=\sum_{\kappa=-\infty}^{\infty}h\left(\frac{q^\kappa}{1-q}\right)q^\kappa.$$

Further, the q-analogue of Gamma function is defined as

$$\Gamma_q(\kappa)=\int_0^{\infty/A}u^{\kappa-1}E_q(-qu)d_qu,\ \kappa>0.$$

For any $\rho\in\mathbb{N}\cup\{0\}$, let us consider the weighted function $w_\rho(u)$ defined as

$$w_0(u)=1,\ w_\rho(u)=1+u^\rho,\ if\ \rho\ge 1,\ u\in[0,\infty).$$

With the above function $w_\rho(u)$, the associated weighted space is given by

$$\mathcal{C}_\rho:=\{f:(0,\infty)\to\mathbb{R}|\frac{f(u)}{w_\rho(u)}\ is\ bounded\ and\ uniformly\ continuous\ in\ (0,\infty)\},$$

with the norm $||f||_\rho = \sup_{u>0} \frac{|f(u)|}{w_\rho(u)}$. For any $\rho_1 < \rho_2$, it is evident that $\mathcal{C}_{\rho_1} \subset \mathcal{C}_{\rho_2}$ and $||f||_{\rho_2} < ||f||_{\rho_1}$. Further, for any $u > 0$ and $\rho,\ k \in \mathbb{N}$, the following inequalities are easily verified:

$$\frac{w_{2\rho}(u)}{(w_\rho(u))^2} \le 1,\ \frac{(w_\rho(u))^2}{w_{2\rho}(u)} \le 4,\ \frac{w_\rho(u) w_k(u)}{w_{\rho+k}(u)} \le 3. \tag{1.3}$$

For $f \in \mathcal{C}_\rho$, and $q \in (0,1)$, we present a q- analogue of the modified Gamma operators (1.2) as follows:

$$\mathcal{G}_{n,q}(f;\varkappa) = \int_0^{\infty/A} \mathcal{K}_{n,q}(\varkappa,u) f\left(\frac{\sqrt{[n-1]_q[n-2]_q}}{u^{1/n}}\right) d_q u, \tag{1.4}$$

where

$$\mathcal{K}_{n,q}(\varkappa,u) = \frac{\varkappa^n}{\Gamma[n+1]_q} E_q(-q\varkappa u^{1/n}).$$

It is clear that as $q \to 1^-$, the operator $\mathcal{G}_{n,q}(f;\varkappa)$ tends to the modified Gamma operator (1.2). We also consider a modification of the operators (1.4) to approximate functions in the space

$$\mathcal{C}^\rho := \{f \in \mathcal{C}_\rho : f^{(m)} \in \mathcal{C}_{\rho-m}\ for\ all\ \ 0 \le m \le \rho\},\ \rho \in \mathbb{N} \cup \{0\}$$

.

For $f \in \mathcal{C}^\rho,\ \rho \in \mathbb{N} \cup \{0\}$, the modified operator is defined as

$$\mathcal{G}_{n,q,\rho}(f;\varkappa) = \int_0^{\infty/A} \mathcal{K}_{n,q}(\varkappa,u) \mathcal{J}_\rho\left(\varkappa, \frac{\sqrt{[n-1]_q[n-2]_q}}{u^{1/n}}\right) d_q u, \tag{1.5}$$

$\varkappa \in (0,\infty)$ and $n(\in \mathbb{N}) > 2\rho$ and

$$\mathcal{J}_\rho(\varkappa,u) = \sum_{k=0}^{\rho} \frac{f^{(k)}(u)}{k!} (\varkappa-u)^k,\ \varkappa, u > 0. \tag{1.6}$$

Note that the operator $\mathcal{G}_{n,q,\rho}$ reduces to the operator $\mathcal{G}_{n,q}$ defined by (1.4), if we take $\rho = 0$ and $f \in \mathcal{C}^0$. In Section 2, we shall show that $\mathcal{G}_{n,q,\rho}$ is a positive linear operator from $\mathcal{C}^\rho$ to $\mathcal{C}_\rho$ for all $n > 2\rho$. In section 3, we show that the operators $\mathcal{G}_{n,q,\rho}$ have better rate of convergence than the operators $\mathcal{G}_{n,q}$.

The main purpose of this paper is to study the approximation properties of the operators (1.4) in a polynomial weighted space $f \in \mathcal{C}_\rho$ of continuous functions on $(0,\infty)$ and the approximation degree of the modified operators for the differentiable functions in the above weighted space. Following the ideas developed in the paper [34], we determine the convergence rate of the operators (1.4) in terms of the moduli of continuity of first and second orders through the approach of Steklov means. We also establish the Voronovskaya type asymptotic theorems for the q- operators (1.4) and for their modification defined for a polynomial weighted space of differentiable functions.

2. Auxiliary results

This section is devoted to preliminary results, which will be useful to prove our main results.

Lemma 1. *For $q \in (0,1), \varkappa \in (0,\infty)$ and $k \in \mathbb{N} \cup \{0\}$ we have*

$$\mathcal{G}_{n,q}(u^k;\varkappa) = \frac{\Gamma[n-k]_q \sqrt{([n-1]_q[n-2]_q)^k}}{\Gamma[n]_q} \varkappa^k.$$

Proof. Putting $\varkappa u^{1/n} = t$, we have

$$\begin{aligned}
\mathcal{G}_{n,q}(u^k;\varkappa) &= \frac{\varkappa^n}{\Gamma[n+1]_q}\int_0^{\infty/A} E(-q\varkappa u^{1/n})\frac{\sqrt{([n-1]_q[n-2]_q)^k}}{u^{k/n}}d_qu \\
&= \frac{[n]_q\sqrt{([n-1]_q[n-2]_q)^k}}{\Gamma[n+1]_q}\varkappa^k\int_0^{\infty/A} E(-qt)t^{n-k-1}d_qt \\
&= \frac{\Gamma[n-k]_q\sqrt{([n-1]_q[n-2]_q)^k}}{\Gamma[n]_q}\varkappa^k.
\end{aligned}$$

Consequently, a simple calculation yields:

Lemma 2. *The operators $\mathcal{G}_{n,q}(f;\varkappa)$ satisfy*

(i) $\mathcal{G}_{n,q}(1;\varkappa) = 1$;

(ii) $\mathcal{G}_{n,q}(u;\varkappa) = \sqrt{\dfrac{[n-2]_q}{[n-1]_q}}\ \varkappa$;

(iii) $\mathcal{G}_{n,q}(u^2;\varkappa) = \varkappa^2$;

(iv) $\mathcal{G}_{n,q}(u^3;\varkappa) = \dfrac{\sqrt{[n-1]_q[n-2]_q}}{[n-3]_q}\varkappa^3$;

(v) $\mathcal{G}_{n,q}(u^4;\varkappa) = \dfrac{[n-1]_q[n-2]_q}{[n-3]_q[n-4]_q}\varkappa^4$.

□

Using the linearity of the operators (1.4) and applying Lemma 2, we easily obtain the following result:

Lemma 3. *The operators $\mathcal{G}_{n,q}(f;\varkappa)$ satisfy*

(a) $\mathcal{G}_{n,q}(u-\varkappa;\varkappa) = \left(\sqrt{\frac{[n-2]_q}{[n-1]_q}} - 1\right)\varkappa$;

(b) $\mathcal{G}_{n,q}((u-\varkappa)^2;\varkappa) = 2\left(1 - \sqrt{\frac{[n-2]_q}{[n-1]_q}}\right)\varkappa^2$;

(c) $\mathcal{G}_{n,q}((u-\varkappa)^3;\varkappa) = \left(\frac{\sqrt{[n-1]_q[n-2]_q}}{[n-3]_q} + 3\sqrt{\frac{[n-2]_q}{[n-1]_q}} - 4\right)\varkappa^3$;

(d) $\mathcal{G}_{n,q}((u-\varkappa)^4;\varkappa) = \left(\frac{[n-1]_q[n-2]_q}{[n-3]_q[n-4]_q} - 4\frac{\sqrt{[n-1]_q[n-2]_q}}{[n-3]_q} - 4\sqrt{\frac{[n-2]_q}{[n-1]_q}} + 7\right)\varkappa^4$.

Remark 1. *Throughout this paper, we assume that $\{q_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} q_n^n = \lambda, (0 \le \lambda < 1)$, then using Lemma 3, we easily derive:*

$$\begin{aligned}
\lim_{n\to\infty}[n]_{q_n}\ \mathcal{G}_{n,q_n}(u-\varkappa;\varkappa) &= -\frac{1}{2}\lambda\varkappa, \\
\lim_{n\to\infty}[n]_{q_n}\ \mathcal{G}_{n,q_n}((u-\varkappa)^2;\varkappa) &= \lambda\varkappa^2 \\
\lim_{n\to\infty}[n]_{q_n}^2\ \mathcal{G}_{n,q_n}((u-\varkappa)^3;\varkappa) &= \left(\frac{3\lambda^2+2\lambda}{2}\right)\varkappa^3 \\
\lim_{n\to\infty}[n]_{q_n}^2\ \mathcal{G}_{n,q_n}((u-\varkappa)^4;\varkappa) &= 3\lambda^2\varkappa^4.
\end{aligned}$$

Remark 2. *Continuing the process further in an analogous manner in Lemma 3, it can be observed that for $k \in \mathbb{N}$ and $n > 2k$, there exists a positive constant $M_0(k)$, depending on k such that*

$$\mathcal{G}_{n,q}((u-\varkappa)^{2k};\varkappa) \le M_0(k)\frac{\varkappa^{2k}}{[n]_q^k},\ \varkappa > 0.$$

Lemma 4. *Let $k \in \mathbb{N}$. Then there exists $M_1(k)$, a positive constant depending on k such that*

$$\mathcal{G}_{n,q}(|u-\varkappa|^k;\varkappa) \leq M_1(k)\frac{\varkappa^k}{[n]_q^{k/2}},\ \varkappa > 0.$$

The proof of the lemma easily follows from Cauchy-Schwarz inequality and Remark 2.

The following result shows that the operators $\mathcal{G}_{n,q}$ are positive linear operators from the space $\mathcal{C}_\rho$ into $\mathcal{C}_\rho$, for all $n > \rho$.

Lemma 5. *Let $\rho \in \mathbb{N}\cup\{0\}$, then for every $f \in \mathcal{C}_\rho$, there exists $M_2(\rho)$ a positive constant depending on ρ such that*

$$||\mathcal{G}_{n,q}(f)||_\rho \leq M_2(\rho)||f||_\rho,\ \forall n > \rho.$$

Proof. From equation (1.4) and Lemma (1), we have

$$\begin{aligned}
\mathcal{G}_{n,q}(w_\rho(u);\varkappa) &= \mathcal{G}_{n,q}(1+u^\rho;\varkappa)\\
&= 1+\frac{\Gamma[n-\rho]_q\sqrt{([n-1]_q[n-2]_q)^\rho}}{\Gamma[n]_q}\varkappa^\rho\\
&= 1+\frac{\sqrt{([n-1]_q[n-2]_q)^\rho}}{[n-1]_q[n-2]_q...[n-\rho]_q}\varkappa^\rho,\ \forall n > \rho.
\end{aligned}$$

This shows that

$$\frac{1}{w_\rho(\varkappa)}\mathcal{G}_{n,q}(w_\rho(u);\varkappa) \leq M_2(\rho),\ \forall \varkappa > 0 \text{ and } n > \rho, \tag{2.1}$$

where $M_2(\rho) > 0$ is a constant depending upon ρ.
Now, from equation (1.4) and (2.1), we get

$$||\mathcal{G}_{n,q}(f)||_\rho \leq ||f||_\rho||\mathcal{G}_{n,q}(w_\rho)||_\rho \leq M_2(\rho)||f||_\rho,\ n > \rho.$$

□

Lemma 6. *Let $k \in \mathbb{N}$ then for all $n > 2(\rho+k)$, there exists a constant $M_3(k,\rho) > 0$, such that*

$$\frac{1}{w_{\rho+k}(\varkappa)}\mathcal{G}_{n,q}\bigg(|u-\varkappa|^k w_\rho(u);\varkappa\bigg) \leq M_3(k,\rho)\frac{1}{[n]_q^{k/2}},\ \varkappa \geq 0.$$

Proof. Since

$$\frac{1}{w_{\rho+k}(\varkappa)}\mathcal{G}_{n,q}\bigg(|u-\varkappa|^k w_\rho(u);\varkappa\bigg) = \mathcal{G}_{n,q}\bigg(\frac{|u-\varkappa|^k}{w_{\rho+k}(\varkappa)}w_\rho(u);\varkappa\bigg).$$

By using Cauchy-Schwarz inequality and equation (1.3) , we obtain

$$\begin{aligned}
\mathcal{G}_{n,q}\bigg(\frac{|u-\varkappa|^k}{w_{\rho+k}(\varkappa)}w_\rho(u);\varkappa\bigg) &\leq \bigg\{\mathcal{G}_{n,q}((u-\varkappa)^{2k};\varkappa)\bigg\}^{1/2}\bigg\{\mathcal{G}_{n,q}\bigg(\frac{(w_\rho(u))^2}{(w_{(\rho+k)}(\varkappa))^2};\varkappa\bigg)\bigg\}^{1/2}\\
&\leq 4\bigg\{\mathcal{G}_{n,q}((u-\varkappa)^{2k};\varkappa)\bigg\}^{1/2}\bigg\{\frac{1}{w_{2(\rho+k)}(\varkappa)}\mathcal{G}_{n,q}(w_{2(\rho+k)}(u);\varkappa)\bigg\}^{1/2}.
\end{aligned}$$

From Remark (2) and equation (2.1), it is evident that for all $n > 2(k+\rho)$, the proof of the lemma follows. □

Lemma 7. *For every $f \in \mathcal{C}^\rho$ and $n > 2\rho$, there exists a positive constant $M_4(\rho)$ such that*

$$||\mathcal{G}_{n,q,\rho}(f)||_\rho \leq M_4(\rho)\sum_{k=0}^{\rho}||f^{(k)}||_{\rho-k}.$$

Proof. From equations (1.5) and (1.6), we may write

$$|\mathcal{G}_{n,q,\rho}(f;\varkappa)| \leq \sum_{k=0}^{\rho} \mathcal{G}_{n,q}(|f^{(k)}(u)||u-\varkappa|^k;\varkappa)$$

Since $f^{(k)} \in \mathcal{C}_{\rho-k}$, from Lemma (6), we have

$$\begin{aligned} |\mathcal{G}_{n,q,\rho}(f;\varkappa)| &\leq \sum_{k=0}^{\rho} ||f^{(k)}||_{\rho-k}\mathcal{G}_{n,q}(w_{\rho-k}(u)|u-\varkappa|^k;\varkappa) \\ &\leq w_\rho(\varkappa)M_4(\rho)\sum_{k=0}^{\rho}||f^{(k)}||_{\rho-k}{[n]_q}^{-k/2} \end{aligned}$$

holds for all $\varkappa > 0$, hence

$$||\mathcal{G}_{n,q,\rho}(f)||_\rho \leq M_4(\rho)\sum_{k=0}^{\rho}||f^{(k)}||_{\rho-k}.$$

□

3. Main Results

3.1. Approximation properties of the operator $\mathcal{G}_{n,q}$.

Theorem 3.1. *Suppose that for a fixed $\rho \in \mathbb{N}\cup\{0\}$, $f \in \mathcal{C}_\rho$ such that $f', f'' \in \mathcal{C}_\rho$. Then, there exists a positive constant $M_5(\rho)$ such that*

$$\frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q}(f;\varkappa)-f(\varkappa)| \leq \varkappa||f'||_\rho\left(\frac{q^{n-2}}{[n-2]_q}\right)^{1/2} + M_5(\rho)||f''||_\rho\frac{\varkappa^2}{[n]_q},$$

for all $\varkappa > 0$ and $n \geq 2\rho+4$.

Proof. Since $f \in \mathcal{C}_\rho$ and $f',\ f'' \in \mathcal{C}_\rho$, by the Taylor formula, for each $\varkappa > 0$, we may write

$$f(u) = f(\varkappa) + f'(\varkappa)(u-\varkappa) + \int_\varkappa^u (u-s)f''(s)ds.$$

Therefore, we have

$$\mathcal{G}_{n,q}(f;\varkappa) = f(\varkappa) + f'(\varkappa)\mathcal{G}_{n,q}((u-\varkappa);\varkappa) + \mathcal{G}_{n,q}\left(\int_\varkappa^u (u-s)f''(s)ds;\varkappa\right). \tag{3.1}$$

Now, from [34], we get

$$\left|\int_\varkappa^u (u-s)f''(s)ds\right| \leq ||f''||_\rho(w_\rho(u)+w_\rho(\varkappa))(u-\varkappa)^2. \tag{3.2}$$

By using equations (3.1), (3.2), Lemma 3 and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q}(f;\varkappa)-f(\varkappa)| &\leq ||f'||_\rho\sqrt{\mathcal{G}_{n,q}((u-\varkappa)^2;\varkappa)} + \mathcal{G}_{n,q}\left(\left|\int_\varkappa^u (u-s)f''(s)ds\right|;\varkappa\right) \\ &\leq \sqrt{2}\varkappa||f'||_\rho\left(1-\sqrt{\frac{[n-2]_q}{[n-1]_q}}\right)^{1/2} + ||f''||_\rho\left[\frac{1}{w_\rho(\varkappa)}\mathcal{G}_{n,q}\left((u-\varkappa)^2 w_\rho(u);\varkappa\right)\right. \\ &\quad \left. + \mathcal{G}_{n,q}((u-\varkappa)^2;\varkappa)\right]. \end{aligned}$$

By using Hölder's inequality, equations (1.3), (2.1) and Remark 2, we get

$$\begin{aligned}\frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q}(f;\varkappa)-f(\varkappa)| &\leq \varkappa||f'||_\rho\left(\frac{q^{n-2}}{[n-2]_q}\right)^{1/2}+||f''||_\rho\Bigg[\sqrt{\mathcal{G}_{n,q}((u-\varkappa)^4;\varkappa)}\left\{4\mathcal{G}_{n,q}\left(\frac{w_{2\rho}(u)}{w_{2\rho}(\varkappa)};\varkappa\right)\right\}^{1/2}\\ &\quad+\mathcal{G}_{n,q}((u-\varkappa)^2;\varkappa)\Bigg]\\ &\leq \varkappa||f'||_\rho\left(\frac{q^{n-2}}{[n-2]_q}\right)^{1/2}+M_5(\rho)||f''||_\rho\frac{\varkappa^2}{[n]_q}.\end{aligned}$$

□

For $f\in\mathcal{C}_\rho$, $\rho\in\mathbb{N}\cup\{0\}$, the k-th ($k=1,2$) order modulus of continuity ω_k is defined as

$$\omega_k(f,\mathcal{C}_\rho;\wp)=\sup_{0\leq h\leq\wp}||\triangle_h^k f(.)||_\rho,\ \wp\geq 0,$$

where $\triangle_h^1 f(u)\equiv f(u+h)-f(u)$ and $\triangle_h^2 f(u)\equiv f(u+2h)-2f(u+h)+f(u)$. From [8,15], we have

$$\lim_{\wp\to 0+}\omega_k(f,\mathcal{C}_\rho;\wp)=0,\ k=1,2,\ \text{for all } f\in\mathcal{C}_\rho.$$

In the following theorem, we derive an estimate of the rate of convergence by the operators (1.4) with the aid of the modulus of continuity of order 2.

Theorem 3.2. *For every $f\in\mathcal{C}_\rho$, $\rho\in\mathbb{N}\cup\{0\}$ and $\varkappa>0$, there exists a positive constant $M_8(\rho)$ such that*

$$\frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q}(f;\varkappa)-f(\varkappa)|\leq M_8(\rho)\left\{\varkappa\left(\frac{q^{n-2}}{[n-2]_q}\right)^{1/2}+\omega_2\left(f;\frac{\varkappa}{\sqrt{[n]_q}}\right)\right\},$$

for all $n>2\rho+4$.

Proof. Following [8], for $f\in\mathcal{C}_\rho$ the Steklov function f_h is defined as

$$f_h(u)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\{2f(u+s+t)-f(u+2(s+t))\}dsdt$$

for $u>0,$ *and* $h>0$. If $f\in\mathcal{C}_\rho$ then from [8], for all $h>0$, it is known that $f_h^{(k)}\in\mathcal{C}_\rho$ for $k=0,1,2$ and

$$||f-f_h||_\rho\leq\omega_2(f;h), \tag{3.3}$$

$$||f_h''||_\rho\leq\frac{9}{h^2}\omega_2(f;h). \tag{3.4}$$

For any $f\in\mathcal{C}_\rho$, we may write

$$\frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q}(f;\varkappa)-f(\varkappa)|\leq\frac{1}{w_\rho(\varkappa)}(|\mathcal{G}_{n,q}(f(u)-f_h(u);\varkappa)|+|\mathcal{G}_{n,q}(f_h;\varkappa)-f_h(\varkappa)|+|f_h(\varkappa)-f(\varkappa)|).$$

By using Lemma (5), Theorem (3.1), equations (3.3) and (3.4), we obtain

$$\begin{aligned}\frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q}(f;\varkappa)-f(\varkappa)| &\leq M_6(\rho)||f-f_h||_\rho+\varkappa||f_h'||_\rho\left(\frac{q^{n-2}}{[n-2]_q}\right)^{1/2}\\ &\quad+M_7(\rho)||f_h''||_\rho\frac{\varkappa^2}{[n]_q}+||f-f_h||_\rho\\ &\leq \varkappa||f_h'||_\rho\left(\frac{q^{n-2}}{[n-2]_q}\right)^{1/2}+\{M_6(\rho)+1+9M_7(\rho)\frac{\varkappa^2}{h^2[n]_q}\}\omega_2(f;h)\end{aligned}$$

Taking $h=\frac{\varkappa}{\sqrt{[n]_q}}$, we reach the desired result. □

As a consequence of the above theorem, we have an immediate corollary as follows:

Corollary 1. *For any* $f \in \mathcal{C}_\rho$, $\rho \in \mathbb{N} \cup \{0\}$ *and* $\varkappa > 0$, *we have*

$$\lim_{n\to\infty} \mathcal{G}_{n,q_n}(f;\varkappa) = f(\varkappa).$$

Note that this convergence is uniform on any compact interval $[a,b]$ *of* $(0,\infty)$.

In the next result, we establish the Voronovskaja type asymptotic theorem for the operators $\mathcal{G}_{n,q_n}$.

Theorem 3.3. *Let* $\rho \in \mathbb{N} \cup \{0\}$ *and* $f \in \mathcal{C}_\rho$ *such that* $f', f'' \in \mathcal{C}_\rho$, *then for any* $\varkappa > 0$, *there holds*

$$\lim_{n\to\infty} [n]_{q_n}(\mathcal{G}_{n,q_n}(f;\varkappa) - f(\varkappa)) = \frac{\lambda}{2}\{\varkappa^2 f''(\varkappa) - \varkappa f'(\varkappa)\}.$$

Proof. By our hypothesis, from Taylor's formula for f about $\varkappa > 0$, we have

$$f(u) = f(\varkappa) + (u-\varkappa)f'(\varkappa) + \frac{1}{2}(u-\varkappa)^2 f''(\varkappa) + \xi_\varkappa(u)(u-\varkappa)^2,\ u > 0, \tag{3.5}$$

where $\xi_\varkappa(u)$ is a function satisfying $\lim_{u\to\varkappa} \xi_\varkappa(u) = \xi_\varkappa(\varkappa) = 0$.
Operating $\mathcal{G}_{n,q_n}(.;\varkappa)$ on both sides of the above equation (3.5) and using Lemma 3, we get

$$\begin{aligned} [n]_{q_n}(\mathcal{G}_{n,q_n}(f;\varkappa) - f(\varkappa)) &= [n]_{q_n}\left(\sqrt{\frac{[n-2]_{q_n}}{[n-1]_{q_n}}} - 1\right)\{\varkappa f'(\varkappa) - \varkappa^2 f''(\varkappa)\} \\ &\quad + [n]_{q_n}\mathcal{G}_{n,q_n}((u-\varkappa)^2\xi_\varkappa(u);\varkappa). \end{aligned} \tag{3.6}$$

Now, we show that $[n]_{q_n}\mathcal{G}_{n,q_n}((u-\varkappa)^2\xi_\varkappa(u);\varkappa) \to 0$, as $n \to \infty$, for any $\varkappa > 0$.
By using Cauchy-Schwarz inequality, we arrive at

$$|\mathcal{G}_{n,q_n}((u-\varkappa)^2\xi_\varkappa(u);\varkappa)| \le \sqrt{\mathcal{G}_{n,q_n}((u-\varkappa)^4;\varkappa)}\sqrt{\mathcal{G}_{n,q_n}((\xi_\varkappa(u))^2;\varkappa)}. \tag{3.7}$$

From the properties of $\xi_\varkappa(u)$ and Corollary (1), it follows that

$$\lim_{n\to\infty} \mathcal{G}_{n,q_n}((\xi_\varkappa(u))^2;\varkappa) = (\xi_\varkappa(\varkappa))^2 = 0, \tag{3.8}$$

for any $\varkappa > 0$.
Further from Remark 1, for all $\varkappa > 0$, we have

$$[n]_{q_n}\sqrt{\mathcal{G}_{n,q_n}((u-\varkappa)^4;\varkappa)} = O(1), as\ n \to \infty. \tag{3.9}$$

Therefore combining (3.7)-(3.9), we obtain

$$\lim_{n\to\infty} [n]_{q_n}\mathcal{G}_{n,q_n}((u-\varkappa)^2\xi_\varkappa(u);\varkappa) = 0.$$

Consequently, applying Remark 1 to the equation (3.6), we reach the required result. $\square$

3.2. **Approximation properties of the operator $\mathcal{G}_{n,q,\rho}$.**

Our following result is an analogue of Theorem 3.2 for the operators (1.5).

Theorem 3.4. *For every* $f \in \mathcal{C}^\rho$ *and* $\rho \in \mathbb{N}$, *there exists a constant* $M_9(\rho)$ *such that the following inequality holds*

$$\frac{1}{w_\rho(\varkappa)}|\mathcal{G}_{n,q,\rho}(f;\varkappa) - f(\varkappa)| \le \frac{M_9(\rho)}{[n]_q^{\rho/2}}\omega_1\left(f^{(\rho)}, \mathcal{C}_0; \frac{\varkappa}{\sqrt{[n]_q}}\right),\ \varkappa > 0$$

for $n > 2\rho + 2$. *Furthermore,*

$$||\mathcal{G}_{n,q,\rho}(f) - f||_{\rho+1} \le \frac{3M_9(\rho)}{[n]_q^{\rho/2}}\omega_1\left(f^{(\rho)}, \mathcal{C}_0; \frac{1}{\sqrt{[n]_q}}\right)$$

for $n > 2\rho + 2$.

Proof. For every $f \in \mathcal{C}^\rho$ and $\varkappa > 0$, from equations (1.5) and (1.6), we may write

$$\mathcal{G}_{n,q}(f;\varkappa) - f(\varkappa) = \mathcal{G}_{n,q}((\mathcal{J}_{\rho,q}(\varkappa,u) - f(\varkappa));\varkappa)$$

. For any $f \in \mathcal{C}^\rho$, the Taylor's formula about $u > 0$, is given by

$$f(\varkappa) = \sum_{k=0}^{\rho} \frac{f^{(k)}(u)}{k!}(\varkappa - u)^k + \frac{(\varkappa - u)^\rho}{(\rho-1)!} h_\rho(\varkappa,u),\ \varkappa > 0$$

where

$$h_\rho(\varkappa,u) = \int_0^1 (1-v)^{\rho-1}\{f^{(\rho)}(u + v(\varkappa - u)) - f^{(\rho)}(u)\}dv. \tag{3.10}$$

From equation (1.6) we have

$$f(\varkappa) - \mathcal{J}_{\rho,q}(\varkappa,u) = \frac{(\varkappa-u)^\rho}{(\rho-1)!} h_\rho(\varkappa,u)$$

and therefore, we obtain

$$\begin{aligned} |\mathcal{G}_{n,q,\rho}(f;\varkappa) - f(\varkappa)| &\le \mathcal{G}_{n,q}(|\mathcal{J}_{\rho,q}(\varkappa,u) - f(\varkappa)|;\varkappa) \\ &= \frac{1}{(\rho-1)!}\mathcal{G}_{n,q}(|\varkappa - u|^\rho |h_\rho(\varkappa,u)|;\varkappa). \end{aligned} \tag{3.11}$$

Now, we estimate $|h_\rho(\varkappa,u)|$ in terms of the first order modulus of continuity ω_1. Since $f^{(\rho)} \in \mathcal{C}_0$, from equation (3.10), we get

$$\begin{aligned} |h_\rho(\varkappa,u)| &\le \int_0^1 (1-v)^{\rho-1}\omega_1(f^{(\rho)},\mathcal{C}_0; v|\varkappa - u|)dv \\ &\le \omega_1(f^{(\rho)},\mathcal{C}_0;|\varkappa - u|)\int_0^1 (1-v)^{\rho-1}dv \\ &\le \frac{1}{\rho}\omega_1\left(f^{(\rho)},\mathcal{C}_0;\frac{\varkappa}{\sqrt{[n]_q}}\right)\left(\frac{\sqrt{[n]_q}}{\varkappa}|\varkappa - u| + 1\right),\ \varkappa > 0. \end{aligned} \tag{3.12}$$

From equations (3.11) and (3.12), we have

$$|\mathcal{G}_{n,q,\rho}(f;\varkappa) - f(\varkappa)| \le \frac{1}{\rho!}\omega_1\left(f^{(\rho)},\mathcal{C}_0;\frac{\varkappa}{\sqrt{[n]_q}}\right)\left\{\frac{\sqrt{[n]_q}}{\varkappa}\mathcal{G}_{n,q,\rho}(|\varkappa - u|^{\rho+1};\varkappa) + \mathcal{G}_{n,q,\rho}(|\varkappa - u|^\rho;\varkappa)\right\}. \tag{3.13}$$

The first assertion of the theorem follows from Lemma (3). Further from [32], for any $\varkappa > 0$ and $n \in \mathbb{N}$, we have

$$\omega_1\left(f^{(\rho)};\frac{\varkappa}{\sqrt{[n]_q}}\right) \le (\varkappa + 1)\omega_1\left(f^{(\rho)};\frac{1}{\sqrt{[n]_q}}\right).$$

Thus, from equations (1.3) and (3.13), the second assertion of the theorem follows. □

Remark 3. *If we take $\rho = 2$ in the above Theorem 3.4 and comparing with the Theorem 3.3, then it can be observe that the operators $\mathcal{G}_{n,q,\rho}$ have better rate of convergence than the operators $\mathcal{G}_{n,q}$*

Corollary 2. *Let $\rho \in \mathbb{N}$, then for every $f \in \mathcal{C}^\rho$, we have*

$$\lim_{n\to\infty}[n]_{q_n}^{\rho/2}(\mathcal{G}_{n,q_n,\rho}(f;\varkappa) - f(\varkappa)) = 0,\ \varkappa > 0.$$

Further

$$\lim_{n\to\infty}[n]_{q_n}^{\rho/2}||\mathcal{G}_{n,q_n,\rho}(f) - f||_{\rho+1} = 0.$$

In the next corollary, we investigate the convergence rate of the operators (1.5) for those functions $f \in \mathcal{C}^\rho$, where $f^{(\rho)}$ belongs to the Lipschitz class $Lip_M\mu$, $0 < \mu \le 1$, defined as

$$Lip_M\mu = \{f \in \mathcal{C}_0 : |f(\varkappa) - f(u)| \le M|\varkappa - u|^\mu,\ 0 < \mu \le 1,\ \varkappa, u > 0\},$$

$M > 0$ is a constant depending on f.

Corollary 3. *For any $f \in \mathcal{C}^\rho$ such that $f^{(\rho)} \in Lip_M^\mu$, $0 < \mu \leq 1$, and $\rho \in \mathbb{N}$, for all $\varkappa > 0$, we have*

$$\mathcal{G}_{n,q_n,\rho}(f;\varkappa) - f(\varkappa) = O\left(\frac{1}{[n]_{q_n}^{(\rho+\mu)/2}}\right), \text{ as } n \to \infty.$$

Also,

$$||\mathcal{G}_{n,q_n,\rho}(f) - f||_{\rho+1} = O\left(\frac{1}{[n]_{q_n}^{(\rho+\mu)/2}}\right), \text{ as } n \to \infty.$$

Lastly, we establish a Voronovskaya type result for the operators (1.5).

Theorem 3.5. *Let $f \in \mathcal{C}^\rho$ and $\rho \in \mathbb{N} \cup \{0\}$ such that the derivatives $f^{(\rho+1)}$ and $f^{(\rho+2)}$ are bounded and continuous on $(0, \infty)$, then for all $\varkappa > 0$, we have*

$$\begin{aligned}\mathcal{G}_{n,q_n,\rho}(f;\varkappa) - f(\varkappa) \quad &= \quad \frac{(-1)^\rho f^{(\rho+1)}(\varkappa)}{(\rho+1)!}\mathcal{G}_{n,q_n}((u-\varkappa)^{\rho+1};\varkappa) \\ &\quad + \frac{(-1)^\rho(\rho+1) f^{(\rho+2)}(\varkappa)}{(\rho+2)!}\mathcal{G}_{n,q_n}((u-\varkappa)^{\rho+2};\varkappa) + o\left(\frac{1}{[n]_q{}^{(\rho+2)/2}}\right), \text{ as } n \to \infty.\end{aligned}$$

Proof. To prove this theorem, we use the Taylor's expansion of $f^{(k)}$, $1 \leq k \leq \rho$ at $\varkappa > 0$, which is given as

$$f^{(k)}(u) = \sum_{j=0}^{\rho+2-k} \frac{f^{(k+j)}(\varkappa)}{j!}(u-\varkappa)^j + \eta_k(u,\varkappa)(u-\varkappa)^{\rho+2-k}, \text{ for } u \geq 0,$$

where $\eta_k(u,\varkappa)$ is a function such that for a fixed $\varkappa > 0$, $\eta_k(u,\varkappa)u^2 \in \mathcal{C}_0$ and $\lim_{u\to\varkappa} \eta_k(u,\varkappa) = 0$ for all k, $0 \leq k \leq \rho$.
Then, equation (1.6) can be written as

$$\begin{aligned}\mathcal{J}_\rho(\varkappa,u) \quad &= \quad \sum_{k=0}^{\rho} \frac{1}{k!}\left\{\sum_{j=0}^{\rho+2-k} \frac{f^{(k+j)}(\varkappa)}{j!}(u-\varkappa)^j + \eta_k(u,\varkappa)(u-\varkappa)^{\rho+2-k}\right\}(\varkappa-u)^k, \ \varkappa, u > 0 \\ &= \quad \sum_{k=0}^{\rho} \frac{(-1)^k}{k!} \sum_{j=0}^{\rho+2-k} \frac{f^{(k+j)}(\varkappa)}{j!}(u-\varkappa)^{k+j} + \left\{\sum_{k=0}^{\rho}\frac{(-1)^k}{k!}\eta_k(u,\varkappa)\right\}(u-\varkappa)^{\rho+2}\end{aligned}$$

On simplification, the above expression becomes

$$\begin{aligned}\mathcal{J}_\rho(\varkappa,u) \quad &= \quad \sum_{k=0}^{\rho}(-1)^k \sum_{i=k}^{\rho+2}\binom{i}{k}\frac{f^{(i)}(\varkappa)}{i!}(u-\varkappa)^i + \eta_\rho(u,\varkappa)(u-\varkappa)^{\rho+2} \\ &= \quad \sum_{i=0}^{\rho}\frac{f^{(i)}(\varkappa)}{i!}(u-\varkappa)^i \sum_{k=0}^{i}\binom{i}{k}(-1)^k + \frac{f^{(\rho+1)}(\varkappa)(u-\varkappa)^{\rho+1}}{(\rho+1)!}\sum_{k=0}^{\rho}\binom{\rho+1}{k}(-1)^k \\ &\quad + \frac{f^{(\rho+2)}(\varkappa)(u-\varkappa)^{\rho+2}}{(\rho+2)!}\sum_{k=0}^{\rho}\binom{\rho+2}{k}(-1)^k + \eta_\rho(u,\varkappa)(u-\varkappa)^{\rho+2} \qquad (3.14)\end{aligned}$$

where $\eta_\rho(u,\varkappa)=\sum_{k=0}^{\rho}\frac{(-1)^k}{k!}\eta_k(u,\varkappa)$.
It can be easily seen that for a non-negative integer i there hold:

$$\begin{aligned}\sum_{k=0}^{i}(-1)^k\binom{i}{k} &= \begin{cases}1 & \text{if } i=0\\ 0 & \text{if } i\geq 1,\end{cases}\\ \sum_{k=0}^{i}(-1)^k\binom{i+1}{k} &= (-1)^i,\\ \sum_{k=0}^{i}(-1)^k\binom{i+2}{k} &= (i+1)(-1)^i.\end{aligned}$$

By virtue of these identities and equations (1.5) and (3.14), we have

$$\begin{aligned}\mathcal{G}_{n,q_n,\rho}(f;\varkappa)-f(\varkappa) &= \frac{(-1)^\rho f^{(\rho+1)}(\varkappa)}{(\rho+1)!}\mathcal{G}_{n,q_n,\rho}((u-\varkappa)^{\rho+1};\varkappa)+\frac{(-1)^\rho(\rho+1)f^{(\rho+2)}(\varkappa)}{(\rho+2)!}\mathcal{G}_{n,q_n,\rho}((u-\varkappa)^{\rho+2};\varkappa)\\ &\quad +\mathcal{G}_{n,q_n,\rho}(\eta_\rho(u,\varkappa)(u-\varkappa)^{\rho+2};\varkappa). \qquad (3.15)\end{aligned}$$

Since, $\lim_{u\to\varkappa}\eta_\rho(u,\varkappa)=\eta_\rho(\varkappa,\varkappa)=0$, hence from Corollary (1), we have

$$\lim_{n\to\infty}\mathcal{G}_{n,q_n,\rho}((\eta_\rho(u,\varkappa))^2;\varkappa)=(\eta_\rho(\varkappa,\varkappa))^2=0.$$

Thus, from Hölder's inequality and remark (2), we get

$$\mathcal{G}_{n,q_n,\rho}(\eta_\rho(u,\varkappa)(u-\varkappa)^{\rho+2};\varkappa)=o\left(\frac{1}{[n]_q^{(\rho+2)/2}}\right),\ as\ n\to\infty. \qquad (3.16)$$

The proof of the theorem follows from equations (3.15) and (3.16).

□

4. Acknowledgement

The first author is thankful to the Council of Scientific and Industrial Research New Delhi, India, [Grant No.-09/143(0914)/2018-EMR-I] for providing the financial assistance to carry out the above research work.

References

[1] Acar T., Mursaleen M. and Deveci Ş. N., Gamma operators reproducing exponential functions. Adv. Difference Equ. doi:10.1186/s13662-020-02880-x, (2020).
[2] Aral A., Gupta V. and Agarwal R. P., Applications of q- Calculus in Operator Theory, Vol. XII. Springer: New York, 2013.
[3] Arpaguş S. and Olgun A. Approximation properties of modified Baskakov Gamma operators. Facta Univ. Ser. Math. Inform. 36(1), 125-141, (2020).
[4] Artee, Approximation by modified Gamma type operators. Int. J. Adv. Appl. Math. Mech. 5(4), 12-19, (2018).
[5] Baskakov V. A., An instance of a sequence of the linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR, 113, 249-251, (1957).
[6] Bernstein S. N., Démonstration du théorém de Weierstrass fondée sur la calcul des probabilitiés, Comm. Soc. Math. Charkow Sér, 13(1), 1–2, (1912).
[7] Betus Ö. and Usta F., Approximation of functions by a new type of Gamma operators. Numer Methods Partial Differential Eq., 1–12, (2020).
[8] Becker, M., Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces, Indiana University Mathematics Journal, 27(1), 127–142, (1978).
[9] Chen W. and Guo S., On the rate of convergence of the Gamma operators for functions of bounded variation. Approx. Theory Appl. 1(5), 8596, (1985).
[10] Cheng W. T. and Tang X. J., Approximation properties of λ-Gamma operators based on q -integers", J. Funct. Spaces, Article ID 5710510, https://doi.org/10.1155/2020/5710510, (2020).
[11] Cheng W. T., Zhang W. H. and Zhang J., Approximation properties of modified q-Gamma operators preserving linear functions, Filomat, 34(5), 16011609, (2020).
[12] Ciupa A., Approximation by a generalized Szász type operator. J. Comput. Anal. Appl., 5(4), 413-424, (2003).

[13] Deveci S. N., Acar T. and Alagoz O., AlagozApproximation by Gamma type operators, Math. Meth. Appl. Sci. 43(5), 2772-2782, (2020).
[14] Devore R. A. and Lorentz G. G., Constructive Approximation. Springer, Berlin 1993.
[15] Ditzian, Z. and Totik, V., Moduli of Smoothness, Springer-Verlag, New York, 1987.
[16] Erdogan S. and Olgun A., Approximation properties of modified Jain-Gamma operators, Carpathian Math. Publ. 13(3), 651-665, (2021).
[17] Gadjiev A. D., On P. P. Korovkin type theorems. Math. Zametki 20(5), 781-786, (1976).
[18] Gupta P. and Gupta V., Rate of convergence on Baskakov-Szász type operators. Fasc. Math., 31, 37-44, (2001).
[19] Gupta V., Convergence Estimates for Gamma Operator. Bull. Malays. Math. Sci. Soc. 43, 2065-2075, (2020).
[20] Gupta V. and Agarwal R. P., Convergence Estimates in Approximation Theory, Springer (Berlin), 2014.
[21] Lesniewicz M. and Rempulska L., Approximation by some operators of the Szász-Mirakyan type in exponential weight spaces. Glas. Mat. Ser. III, 32(52)(1), 57-69, (1997).
[22] Kac V. and Cheung P., Quantum Calculus. Universitext. Springer, New York, 2002.
[23] Karsli H., Rate of convergence of a new Gamma Type Operators for functions with derivatives of bounded variation, Math. Comput. Modelling, 45(56), 617-624, (2007).
[24] Karsli, H., On convergence of general Gamma type operators. Anal. Theory Appl. 27, 288300 (2011).
[25] Karsli H., Agrawal P. N. and Goyal M., General Gamma type operators based on q-integers. Appl. Math. Comput. 251, 564-575, (2015).
[26] Karsli H., Gupta V. and Izgi A., Rate of pointwise convergence of a new kind of Gamma operators for functions of bounded variation. Appl. Math. Lett. 22(4), 505510, (2009).
[27] King J. P., Positive linear operators which preserve $\varkappa^2$, Acta Math. Hungar. 99(3), 203-208, (2003).
[28] Korovkin P. P., Linear Operators and Approximation Theory, Hindustan Publ. Corp., India, 1960.
[29] Krech G., On the rate of convergence for modified Gamma operators, Rev. Un. Mat. Argentina, 55(2), 123-131, (2014).
[30] Leviatan D., On Gamma-Type Approximation Operators, Math. Z. 124, 208-212, (1972).
[31] Lupaş A. and Müller M., Approximations eigenschaften der Gamma operatoren. Math Zeitschr., 98, 208–226, (1967).
[32] Natanson I. P., Constructive Function Theory. Frederick Ungar Publishing Co. New York, 1964.
[33] Özçelik R., Kara E. E., Usta, F., Approximation properties of a new family of Gamma operators and their applications. Adv Differ Equ 2021, 508 (2021). https://doi.org/10.1186/s13662-021-03666-5
[34] Rempulska L. and Skorupka M., Approximation properties of modified Gamma operators, Integral Transforms and Special Functions, 18:9, 653–662, (2007).
[35] Rempulska L. and Walczak Z., Modified Szász-Mirakyan operators. Math. Balcanica, 18,, 53-63, (2004).
[36] Singh J. K., Agrawal P. N. and Kajla A., Approximation by modified q-Gamma type operators via A-statistical convergence and power series method, Linear Multilinear Algebra, DOI: 10.1080/03081087.2021.1960260, (2021).
[37] Sz*á*sz O., Generalization of S. Bernstein's polynomials to infinite interval. J. Res. Nat. Bur. Standards, 45, 239-245 , (1959).
[38] Walczak Z., On the convergence of the modified Szász-Mirakyan operators. Yokohama Math. J., 51(1), 11-18, (2004).
[39] Xu X. W. and Wang J. Y., Approximation properties of modified Gamma operator. J. Math. Anal. Appl. 332, 798-813, (2007).
[40] Zeng X. M., Approximation properties of Gamma operator. J. Math. Anal. Appl. 311(2),389-401, (2005).