On the ground states for the X-ray free electron lasers Schrödinger equation

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Abstract

We consider the following X-ray free electron lasers Schrödinger equation

$$(i\nabla - A)^2u + V(x)u - \frac{\mu}{|x|}u = \left(\frac{1}{|x|} * |u|^2\right)u - K(x)|u|^{q-2}u,\ x \in \mathbb{R}^3,$$

where $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ denotes the magnetic potential such that the magnetic field $B = \operatorname{curl} A$ is $\mathbb{Z}^3$-periodic, $\mu \in \mathbb{R}$, $K \in L^\infty\left(\mathbb{R}^3\right)$ is $\mathbb{Z}^3$ -periodic and non-negative, $q \in (2, 4)$. Using the variational method, based on a profile decomposition of the Cerami sequence in $H^1_A\left(\mathbb{R}^3\right)$, we obtain the existence of the ground state solution for suitable $\mu \geq 0$. When $\mu < 0$ is small, we also obtain the non-existence. Furthermore, we give a description for the asymptotic behaviour of the ground states as $\mu \to 0^+$.

Key words: XFEL Schrödinger equation; Periodic magnetic field; Ground states; Asymptotic behaviour

MSC Classification: 35Q55; 35J20; 46B50; 35Q40;

1 Introduction

In this paper, we consider the following X-ray free electron lasers (XFEL) equation

$$(i\nabla - A)^2u + V(x)u - \frac{\mu}{|x|}u = \left(\frac{1}{|x|} * |u|^2\right)u - K(x)|u|^{q-2}u,\ x \in \mathbb{R}^3, \tag{1.1}$$

where A denotes the magnetic potential, V stands for the electric potential, $K \in L^\infty\left(\mathbb{R}^3\right)$ and $q \in (2, 4)$.

Equation (1.1) comes from the following time-dependent model arises as an effective single particle model in X-ray free electron lasers [27, 17] with an atomic nucleus located at the origin

$$i\partial_t\Phi = (i\nabla - A(t,x))^2\,\Phi + W(x)\Phi + \frac{\lambda_1}{|x|}\Phi + \lambda_2(|\cdot|^{-1} * |\Phi|^2)\Phi + \lambda_3|\Phi|^{2\sigma}\Phi. \tag{1.2}$$

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A solution Φ of (1.2) can be considered as the wavefunction of an electron beam in an electric potential $W(x)$, interacting self-consistently through the Coulomb (Hartree) force with the strength λ_2, the local Fock approximation with the strength λ_3, and interacting with an atomic nucleus, located at the origin, of interaction strength λ_1. Equation (1.1) can be regarded as a generalized stationary equation of (1.2) in the case that A is independent of time, with $\lambda_1 = -\mu$, $\lambda_2 = -1$ and λ_3 being replaced by $K(x)$.

The XFEL has many important applications in physics, recent developments using XFEL include the observation of the motion of atoms [11], measuring the dynamics of atomic vibrations [28], biomolecular imaging [38], etc. Although huge amount of studies have been made in physics, it seems that only very few mathematical results could be found in the literatures. In mathematics, the earliest time that the Cauchy problems for the XFEL equations with $V(x) = 0$ were considered seems to be in [2, 3]. For some recent developments, we refer to [13, 29, 26, 25, 24] and the references therein. Especially, [26, 25, 24] are devoted to the time-dependent problems; in [29], for the case $A = 0$ the authors obtained the existence of ground states and normalized solutions for the XFEL Schrödinger equation with harmonic potential; in [13], the existence and stability properties of standing waves have been discussed under the assumption that the potential A is the vector potential such that the magnetic field $B = \operatorname{curl} A$ is constant. However, for more general magnetic field, the discussion for stationary solutions, especially for ground states, seems still absent.

Equation (1.1) is also a special type of the nonlinear magnetic Schrödinger equation

$$\nabla_A^2 u + V(x)u = g(x, |u|)u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $\nabla_A = (i\nabla - A)$. Equation (1.3) has attracted a great quantity of attentions in recent decades. In the study of magnetic Schrödinger equation, one key problem is to overcome the difficulty brings from the presence of magnetic field. Esteban and Lions [23] obtained firstly the existence result when the magnetic field is constant, and then Arioli and Szulkin [4] generalized the result to periodic magnetic field. Recently, Devillanova and Tintarev [21] obtained the existence for a general bounded external magnetic field. The researches on nonlinear magnetic Schrödinger equations have covered a variety of interesting topics, for more examples, we refer to [12, 5, 22, 39, 31, 32, 33, 1, 18, 19, 20, 16, 36] and the references therein. Inspired by the above works, in this paper, we are interested in the XFEL equation (1.1) with the magnetic field being $\mathbb{Z}^3$-periodic.

We need the following assumptions:

(A) $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ and the magnetic field $B = \operatorname{curl} A$ is $\mathbb{Z}^3$-periodic.

(V1) $V = V_p + V_l$, where $V_p \in L^\infty\left(\mathbb{R}^3\right)$ is $\mathbb{Z}^3$ -periodic, $\operatorname{essinf}_{x\in\mathbb{R}^3} V_p(x) > 0$ and $V_l \in L^\infty\left(\mathbb{R}^3\right) \cap L^3\left(\mathbb{R}^3\right)$ satisfies

$$\lim_{|x|\to+\infty} V_l(x) = 0.$$

(V2) $\operatorname{essinf}_{x\in\mathbb{R}^3} V(x) = V_0 > 0$.

(K) $K \in L^\infty\left(\mathbb{R}^3\right)$ is $\mathbb{Z}^3$ -periodic and non-negative.

In Lemma 2.6 we will show that there exists μ^* such that for all $\mu \in [0, \mu^*)$,

$$\int_{\mathbb{R}^3} |\nabla_A u|^2 dx + \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx - \mu \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx$$

is positive-definite on $H_A^1\left(\mathbb{R}^3\right)$.

Our main results can be stated as follows.

Theorem 1.1 *Suppose that conditions* $(A),(V1),(V2)$ *and* (K) *hold. If* $\mu\in[0,\mu^*)$ *and* V_l *satisfies*

$$V_l(x)\le\frac{\mu}{|x|}\quad \text{for a.e. } x\in\mathbb{R}^3\backslash\{0\},\tag{1.4}$$

then there is a ground state solution $u\in H_A^1\left(\mathbb{R}^3\right)$ *of equation* (1.1).

Theorem 1.2 *Suppose that* $(A),(V1)$ *and* (K) *hold. If* $\mu<0$ *and*

$$V_l(x)>\frac{\mu}{|x|}\quad \text{for a.e. } x\in\mathbb{R}^3\backslash\{0\},\tag{1.5}$$

then equation (1.1) *has no ground state.*

Finally, we give a description for the asymptotic behavior of the ground state when $\mu\to0^+$.

Theorem 1.3 *Suppose that* $(A),(V1),(K)$ *are satisfied and* $V_l\equiv0$. *Let* $\{\mu_n\}\subset[0,\mu^*)$ *be a sequence such that* $\mu_n\to0^+$. *Let* $u_n\in H_A^1\left(\mathbb{R}^3\right)$ *be a ground state of* (1.1) *corresponding to* $\mu=\mu_n$, *then there is a sequence* $\{z_n\}\subset\mathbb{Z}^3$ *such that for* g_{z_n} *defined by* (2.2) *below, up to a subsequence, we have*

$$g_{z_n}u_n\rightharpoonup u_0,\ \text{in } H_A^1\left(\mathbb{R}^3\right),$$

where $u_0\in H_A^1\left(\mathbb{R}^3\right)$ *is a ground state solution of* (1.1) *for* $\mu=0$. *Moreover,* $c_n\to c$, *where* c_n *is the energy of* u_n *and* c *is the energy of* u_0. *The energy of a solution is defined by* (2.1) *below.*

The rest of this paper is organized as follows. In section 2 we give some preliminaries and variational setting. In section 3 we prove the existence and boundedness of a Cerami sequence in $H_A^1\left(\mathbb{R}^3\right)$, and provide a decomposition of bounded minimizing sequences. Section 4 is devoted to the proof of Theorem 1.1 and Theorem 1.2. Finally, we give the proof of Theorem 1.3 in section 6.

2 Preliminaries and variational setting

In this section, we collect some preliminary results that we will use later.

Suppose that $A\in L^2_{loc}(\mathbb{R}^3,\mathbb{R}^3)$. For $\nabla_A=(i\nabla-A)$, let

$$H_A^1(\mathbb{R}^3)=\left\{u\in L^2(\mathbb{R}^3):\nabla_Au\in L^2(\mathbb{R}^3)\right\},$$

then $H_A^1(\mathbb{R}^3)$ is a Hilbert space with inner product

$$\langle u,v\rangle=\Re\int_{\mathbb{R}^3}(\nabla_Au\cdot\overline{\nabla_Av}+u\overline{v})dx,$$

where $\overline{u}$ denotes the conjugation of u and $\Re a$ denotes the real part of $a\in\mathbb{C}$. The corresponding norm is $\|u\|_{H_A^1(\mathbb{R}^3)}=\sqrt{\langle u,u\rangle}$. We have the continuous embedding $H_A^1(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ for $2\le p\le6$ (see [23]). If $u\in H_A^1(\mathbb{R}^3,\mathbb{C})$, then $|u|\in H^1(\mathbb{R}^3)$ and the following diamagnetic inequality holds

$$|\nabla|u|(x)|\le|(i\nabla-A(x))u(x)|,\ \text{for a.e. } x\in\mathbb{R}^3.$$

The energy functional $\mathcal{E}: H_A^1(\mathbb{R}^3) \to \mathbb{R}$ of a solution u of (1.1) is defined by

$$\begin{aligned}\mathcal{E}(u) =& \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_A u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(V(x) - \frac{\mu}{|x|}\right) |u|^2 dx \\ & - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy + \frac{1}{q}\int_{\mathbb{R}^3} K(x)|u|^q dx, \end{aligned} \tag{2.1}$$

$\mathcal{E}$ is of C^1-class on $H_A^1(\mathbb{R}^3)$. The corresponding Nehari manifold is defined by

$$\mathcal{N} := \left\{u \in H_A^1(\mathbb{R}^3)\backslash\{0\} \mid \mathcal{E}'(u)u = 0\right\}.$$

Note that, if the magnetic field $B = \operatorname{curl} A$ is $\mathbb{Z}^3$-periodic, i.e.

$$B(\cdot - y) = B(\cdot), \quad \forall\, y \in \mathbb{Z}^3,$$

then, in terms of a fixed magnetic potential A,

$$\operatorname{curl}\left(A(\cdot - y) - A(\cdot)\right) = 0, \quad \forall\, y \in \mathbb{Z}^3.$$

Therefore, from [34, Lemma 1.1], we have

$$\forall y \in \mathbb{Z}^3, \;\; \exists\, \phi_y \in H^1_{loc}(\mathbb{R}^3), \;\; \text{such that} \;\; A(\cdot - y) = A(\cdot) + \nabla\phi_y(\cdot).$$

Set

$$g_y := u \to e^{i\phi_y(\cdot)}u(\cdot - y), \tag{2.2}$$

$$g_y^{-1} := v \to e^{-i\phi_y(\cdot + y)}v(\cdot + y), \tag{2.3}$$

we have

Lemma 2.1 ([4]) *Let $u \in H_A^1\left(\mathbb{R}^3\right), z \in \mathbb{Z}^3$ and $v := g_z u$. Then $v \in H_A^1\left(\mathbb{R}^3\right)$, $\int_{\mathbb{R}^3} |\nabla_A v|^2 = \int_{\mathbb{R}^3} |\nabla_A u|^2$ and $\|v\|_{H_A^1(\mathbb{R}^3)} = \|u\|_{H_A^1(\mathbb{R}^3)}$. In particular, for each $z \in \mathbb{Z}^3$ the operator g_z is an isometry on $H_A^1\left(\mathbb{R}^3\right)$.*

Define

$$\mathcal{E}_{\text{per}}(u) := \mathcal{E}(u) - \frac{1}{2}\int_{\mathbb{R}^3} V_l(x)|u|^2 dx + \frac{\mu}{2}\int_{\mathbb{R}^3} \frac{|u|^2}{|x|} dx, \tag{2.4}$$

Obviously, $\mathcal{E}_{\text{per}}$ is differential on $H_A^1\left(\mathbb{R}^3\right)$.

Lemma 2.2 ([34]) *For all $y \in \mathbb{Z}^3$, there hold $\mathcal{E}_{per}\left(g_y u\right) = \mathcal{E}_{per}(u)$ and $\mathcal{E}'_{per}\left(g_y u\right) = g_y \mathcal{E}'_{per}(u)$.*

We also need the following lemmas.

Lemma 2.3 *(Hardy-Littlewood-Sobolev inequality [30]) Let $p, r > 1$, $0 < \lambda < n$ and $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{r} = 2$. If $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$, then there exists a constant C independent of g and h, such that*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}h(y)dxdy\right| \leq C\|f\|_p\|h\|_r.$$

Lemma 2.4 ([15]) *Let $1 \leq p < \infty$. If $q < N$ is such that $0 \leq q \leq p$, then $\frac{|u(\cdot)|^p}{|\cdot|^q} \in L^1(\mathbb{R}^N)$ for every $u \in W^{1,p}(\mathbb{R}^N)$. Furthermore,*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^q} dx \leq \left(\frac{p}{N-q}\right)^q \|u\|_{L^p}^{p-q}\|\nabla u\|_{L^p}^q.$$

Remark 2.5 *Let $q = 1$, $p = 2$, by the diamagnetic inequality and Lemma 2.4, we know that for any $u \in H_A^1(\mathbb{R}^3)$ there holds*

$$\frac{1}{2}(\|\nabla_A u\|_2^2 + \|u\|_2^2) \geq \|\nabla |u|\|_2 \|u\|_2 \geq \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx.$$

Lemma 2.6 *There exists μ^* such that for any $0 \leq \mu < \mu^*$ the quadratic form*

$$Q_\mu : u \mapsto \int_{\mathbb{R}^3} |\nabla_A u|^2 dx + \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx - \mu \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx$$

is positive-definite and gives a norm on $H_A^1(\mathbb{R}^3)$ which is equivalent to the standard one.

Proof. By the assumption on $V(x)$, for any $u \in H_A^1(\mathbb{R}^3)$, we have

$$Q_\mu(u) \leq \int_{\mathbb{R}^3} |\nabla_A u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2 dx \leq \max\{1, \|V(x)\|_{L^\infty}\} \left(\|\nabla_A u\|_2^2 + \|u\|_2^2\right).$$

On the other hand, by Lemma 2.4, let $\mu^* = 2\min\{1, V_0\}$, then for any $0 < \mu < \mu^*$,

$$\begin{aligned} Q_\mu(u) &= \int_{\mathbb{R}^3} |\nabla_A u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2 dx - \mu \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx \\ &\geq \int_{\mathbb{R}^3} |\nabla_A u|^2 dx + V_0 \|u\|_2^2 - \mu \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx \\ &\geq \frac{1}{2}(\mu^* - \mu) \left(\|\nabla_A u\|_2^2 + \|u\|_2^2\right), \end{aligned}$$

which implies the conclusion. □

Lemma 2.6 yields that for $0 \leq \mu < \mu^*$, $\|u\|_\mu := \sqrt{Q_\mu(u)}$ is an equivalent norm on $H_A^1(\mathbb{R}^3)$. In the rest of the paper we will always assume $\mu \in [0, \mu^*)$ if we use the symbol $\langle \cdot, \cdot \rangle_\mu$ to denote the scalar product. Moreover, we write

$$\mathcal{D}(u) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} dx dy.$$

It is known that $\mathcal{D}(u)$ is well-defined on $H_A^1(\mathbb{R}^3)$. By the Hardy-Littlewood-Sobolev inequality and Sobolev inequality, there is a constant $C > 0$ such that

$$\mathcal{D}(u) \leq C\|u\|_\mu^4.$$

Now we can rewrite the energy functional as

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{1}{4}\mathcal{D}(u) + \frac{1}{q}\int_{\mathbb{R}^3} K(x)|u(x)|^q dx,$$

it is standard to check that $\mathcal{E}$ is of $\mathcal{C}^1$-class and its critical points are weak solutions of equation (1.1).

Let $(E, \|\cdot\|_E)$ be a Hilbert space. Suppose that $\mathcal{H} : E \to \mathbb{R}$ is a nonlinear functional of the form

$$\mathcal{H}(u) = \frac{1}{2}\|u\|_E^2 - \mathcal{I}(u), \tag{2.5}$$

where $\mathcal{I}$ is of $\mathcal{C}^1$ class and $\mathcal{I}(0) = 0$. Let

$$\mathcal{N} := \{u \in E \backslash \{0\} \mid \mathcal{H}'(u)u = 0\}$$

be the Nehari manifold, then it is obvious that any nontrivial critical point of $\mathcal{H}$ belongs to $\mathcal{N}$. Recall that a Cerami sequence for $\mathcal{H}$ at level c is a sequence $\{u_n\}_n \subset E$ such that

$$\mathcal{H}(u_n) \to c, \quad (1 + \|u_n\|_E)\mathcal{H}'(u_n) \to 0.$$

The next result establishes the existence of Cerami sequence, which can be found in [37, 8].

Lemma 2.7 *Suppose that the functional $\mathcal{H}(u)$ defined by (2.5) satisfies the following conditions:*

(J1) *there exists $r > 0$ such that*

$$\inf_{\|u\|_E = r} \mathcal{H}(u) > 0.$$

(J2) $\frac{\mathcal{I}(t_n u_n)}{t_n^2} \to +\infty$ *as $t_n \to +\infty$, and $u_n \to u$, with $u \in E \backslash \{0\}$.*

(J3) *for all $t > 0$ and $u \in \mathcal{N}$,*

$$\frac{t^2 - 1}{2} \mathcal{I}'(u)(u) - \mathcal{I}(tu) + \mathcal{I}(u) \leq 0.$$

Then $\mathcal{N} \neq \emptyset$, $\Gamma \neq \emptyset$ and

$$c := \inf_{\mathcal{N}} \mathcal{H} = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{H}(\gamma(t)) = \inf_{u \in E \backslash \{0\}} \sup_{t \geq 0} \mathcal{H}(tu) > 0,$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0,\ \|\gamma(1)\|_E > r,\ \mathcal{H}(\gamma(1)) < 0\}.$$

Moreover, there is a Cerami sequence for $\mathcal{H}$ at level c.

3 Profile decomposition of minimizing sequences

In this section, we will give a profile decomposition for the minimizing sequences. We firstly prove the existence and boundedness of Cerami sequence via Lemma 2.7.

Lemma 3.1 *Suppose (A), (V1), (V2) and (K) are satisfied. Let $\mathcal{H} = \mathcal{E}$, $E = H_A^1(\mathbb{R}^3)$ and*

$$\mathcal{I}(u) := \frac{1}{4} \mathcal{D}(u) - \frac{1}{q} \int_{\mathbb{R}^3} K(x) |u(x)|^q dx.$$

Then (J1)-(J3) hold.

Proof. The idea is similar to [6, Lemma 3.3]. It is easy to check that (J1) and (J2) hold. To prove (J3), let $u \in \mathcal{N}$, set

$$\varphi(t) = \frac{t^2 - 1}{2} \mathcal{I}'(u) u - \mathcal{I}(tu) + \mathcal{I}(u), \text{ for } t \geq 0,$$

we see that $\varphi(1) = 0$. As $u \in \mathcal{N}$, we have $\|u\|_\mu^2 = \mathcal{I}'(u) u > 0$, which implies

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} dx dy > \int_{\mathbb{R}^3} K(x) |u(x)|^q dx. \tag{3.1}$$

Furthermore, we have

$$\frac{d\varphi(t)}{dt} = (t - t^3) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} dx dy + (t^{q-1} - t) \int_{\mathbb{R}^3} K(x) |u(x)|^q dx.$$

For any fixed $x, y \in \mathbb{R}^3$, define $\psi : (0, +\infty) \to \mathbb{R}$ by

$$\psi(t) := \psi_{(x,y)}(t) := \frac{|u(x)|^2 |u(y)|^2}{t^{q-4}}. \tag{3.2}$$

If $t \in (0, 1]$, we have

$$\frac{d\varphi(t)}{dt} \geq (t^{q-1} - t^3) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} dx dy = t^{q-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\psi_{(x,y)}(1) - \psi_{(x,y)}(t)}{|x - y|} dx dy.$$

As ψ is non-decreasing on $(0,1]$, we see that $\frac{d\varphi(t)}{dt} \geq 0$ for $t \in (0,1]$ and then $\varphi(t) \leq \varphi(1) = 0$, so (J3) holds. Similarly, if $t \in (1,+\infty)$, because $q \in (2,4)$, we have

$$\frac{d\varphi(t)}{dt} \leq (t^{q-1} - t^3) \int_{\mathbb{R}^3} K(x)|u(x)|^q dx \leq 0.$$

Therefore, $\varphi(t) \leq \varphi(1) = 0$ for $t \in (1,+\infty)$. This completes the proof for (J3). □

Lemma 3.2 *Any Cerami sequence $\{u_n\}$ of $\mathcal{E}$ is bounded.*

Proof. It can be deduced from the properties of Cerami sequences that

$$\begin{aligned}
\lim_{n\to+\infty} \mathcal{E}(u_n) &= \lim_{n\to+\infty} \left(\mathcal{E}(u_n) - \frac{1}{q}\mathcal{E}'(u_n) u_n \right) \\
&= \lim_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{q} \right) \|u_n\|_\mu^2 + (\frac{1}{q} - \frac{1}{4}) \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} dxdy \right] \\
&\geq \lim_{n\to+\infty} \left(\frac{1}{2} - \frac{1}{q} \right) \|u_n\|_\mu^2,
\end{aligned}$$

now the boundedness of $\{u_n\}$ follows from $\limsup_{n\to+\infty} \mathcal{E}(u_n) < +\infty$. □

In the next we will give a profile decomposition for the minimizing sequences.

Lemma 3.3 *Suppose that $\{u_n\}_n \subset H_A^1(\mathbb{R}^3)$ is a bounded sequence such that $u_n \rightharpoonup u_0$ in $H_A^1(\mathbb{R}^3)$. Then*

$$\mathcal{D}(u_n - u_0) - \mathcal{D}(u_n) + \mathcal{D}(u_0) \to 0 \text{ as } n \to +\infty.$$

Lemma 3.3 can be proved by the similar arguments as in [14, Lemma 2.2], we omit it here.

Lemma 3.4 *Suppose that $\{u_n\}_n \subset H_A^1(\mathbb{R}^3)$, $\ell \geq 1$, and for $k = 1, \ldots, \ell$, there exists a sequence $\{z_n^k\}_n \subset \mathbb{Z}^3$ satisfying $|z_n^k| \to +\infty$ and $\left|z_n^k - z_n^{k'}\right| \to +\infty$ for $k \neq k'$. Let $g_{z_n^k}$ and $g_{z_n^k}^{-1}$ be defined by (2.2) and (2.3). Assume that $u_n \rightharpoonup u_0$ and $w^k \in H_A^1(\mathbb{R}^3)$ satisfying $g_{z_n^k}^{-1} u_n \rightharpoonup w^k$ in $H_A^1(\mathbb{R}^3)$. If*

$$\left\| u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k \right\|_\mu \to 0, \tag{3.3}$$

then we have

$$\mathcal{D}(u_n) \to \mathcal{D}(u_0) + \sum_{k=1}^{\ell} \mathcal{D}(w^k).$$

Proof. Set $a_0^n := u_n - u_0$, and

$$a_m^n := u_n - u_0 - \sum_{k=1}^{m} g_{z_n^k} w^k, \quad m \in \{1, \ldots, \ell\}.$$

As $u_n \rightharpoonup u_0$, by Lemma 3.3 we get

$$\mathcal{D}(a_0^n) - \mathcal{D}(u_n) + \mathcal{D}(u_0) \to 0.$$

Since $g_{z_n^1}^{-1} u_n \rightharpoonup w^1$, $g_{z_n^1}^{-1} u_0 \rightharpoonup 0$ and $g_{z_n^1}^{-1} a_0^n = g_{z_n^1}^{-1}(u_n - u_0)$, we have $g_{z_n^1}^{-1} a_0^n \rightharpoonup w^1$. Applying Lemma 3.3 again, we have

$$\mathcal{D}\left(g_{z_n^1}^{-1} a_0^n - w^1\right) - \mathcal{D}\left(g_{z_n^1}^{-1} a_0^n\right) + \mathcal{D}(w^1) \to 0,$$

or equivalently

$$\mathcal{D}(a_1^n) - \mathcal{D}(a_0^n) + \mathcal{D}(w^1) \to 0. \tag{3.4}$$

Similarly,

$$\mathcal{D}\left(a_2^n\right)-\mathcal{D}\left(a_1^n\right)+\mathcal{D}\left(w^2\right)\to 0. \tag{3.5}$$

It yields from (3.4) and (3.5) that

$$\mathcal{D}\left(a_2^n\right)+\mathcal{D}\left(w^2\right)+\mathcal{D}\left(w^1\right)-\mathcal{D}\left(a_0^n\right)\to 0.$$

Repeating the above process, we obtain

$$\mathcal{D}\left(a_\ell^n\right)+\sum_{k=1}^{\ell}\mathcal{D}\left(w^k\right)-\mathcal{D}\left(u_n-u_0\right)\to 0. \tag{3.6}$$

In view of (3.3), $a_\ell^n\to 0$ holds. Therefore

$$\mathcal{D}\left(a_\ell^n\right)\to 0,$$

then from (3.6) we get

$$\mathcal{D}\left(u_n-u_0\right)\to\sum_{k=1}^{\ell}\mathcal{D}\left(w^k\right).$$

Using Lemma 3.3 again, we obtain

$$\mathcal{D}\left(u_n\right)\to\mathcal{D}\left(u_0\right)+\sum_{k=1}^{\ell}\mathcal{D}\left(w^k\right),$$

this completes the proof. □

Similar to [6, Lemma 4.3, Corollary 4.4], we can get the following lemma and corollary.

Lemma 3.5 *$\mathcal{D}':H_A^1\left(\mathbb{R}^3\right)\to\left(H_A^1\left(\mathbb{R}^3\right)\right)^*$ is weak-to-weak * continuous, i.e. if $\{u_n\}_n$ is bounded and $u_n\rightharpoonup u_0$ in $H_A^1\left(\mathbb{R}^3\right)$, then for $\varphi\in H_A^1\left(\mathbb{R}^3\right)$,*

$$\mathcal{D}'\left(u_n\right)(\varphi)\to\mathcal{D}'\left(u_0\right)(\varphi).$$

Corollary 3.1 *$\mathcal{E}':H_A^1\left(\mathbb{R}^3\right)\to\left(H_A^1\left(\mathbb{R}^3\right)\right)^*$ is weak-to-weak * continuous.*

Theorem 3.2 *Under the conditions of Theorem 1.1, let $\{u_n\}$ be a bounded Cerami sequence in $H_A^1\left(\mathbb{R}^3\right)$. Then there exists $u_0\in H_A^1\left(\mathbb{R}^3\right)$ such that $u_n\rightharpoonup u_0$ and $\mathcal{E}'\left(u_0\right)=0$. Furthermore, up to a subsequence, either $u_n\to u_0$ in $H_A^1\left(\mathbb{R}^3\right)$, or there exist an integer $\ell\geq 1$ and sequences $\{w^k\}_{k=1}^l\subset H_A^1\left(\mathbb{R}^3\right)$, $\left\{z_n^k\right\}\subset\mathbb{Z}^3$ and $\{\phi_{z_n^k}\}_{k=1}^l\subset H_{loc}^1(\mathbb{R}^3)$, such that*

1) $\left|z_n^k\right|\to+\infty$ *and* $\left|z_n^k-z_n^{k'}\right|\to+\infty$ *for* $k\neq k'$;
2) $w^k\neq 0$ *and* $\mathcal{E}_{per}'\left(w^k\right)=0$ *for* $1\leq k\leq\ell$;
3) $u_n-u_0-\sum_{k=1}^{\ell}g_{z_n^k}w^k\to 0$;
4) $\mathcal{E}\left(u_n\right)\to\mathcal{E}\left(u_0\right)+\sum_{k=1}^{\ell}\mathcal{E}_{per}\left(w^k\right)$.

Proof. We divide the proof into six steps.

Step 1. Since $\{u_n\}_n$ is bounded, up to a subsequence, there exists u_0 such that $u_n\rightharpoonup u_0$. As $\mathcal{E}'\left(u_n\right)\to 0$, by Corollary 3.1 we get $\mathcal{E}'\left(u_0\right)=0$.

Step 2. Set $v_n^0:=u_n-u_0$.

Case i). If

$$\lim_{n\to+\infty}\sup_{z\in\mathbb{R}^3}\int_{B(z,1)}\left|v_n^0(x)\right|^2dx=0, \tag{3.7}$$

we confirm that $u_n \to u_0$ in $H_A^1(\mathbb{R}^3)$ and the theorem is true in this situation. In fact, from

$$\mathcal{E}'(u_n)(v_n^0) = \|v_n^0\|_\mu^2 + \langle u_0, v_n^0\rangle_\mu - \frac{1}{4}\mathcal{D}'(u_n)(v_n^0) + \Re\int_{\mathbb{R}^3} K(x)|u_n|^{q-2} u_n\overline{v_n^0}dx,$$

we see that

$$\|v_n^0\|_\mu^2 = \mathcal{E}'(u_n)(v_n^0) - \langle u_0, v_n^0\rangle_\mu + \frac{1}{4}\mathcal{D}'(u_n)(v_n^0) - \Re\int_{\mathbb{R}^3} K(x)|u_n|^{q-2} u_n\overline{v_n^0}dx. \tag{3.8}$$

As $\mathcal{E}'(u_0) = 0$, i.e.,

$$\langle u_0, v_n^0\rangle_\mu - \frac{1}{4}\mathcal{D}'(u_0)(v_n^0) + \Re\int_{\mathbb{R}^3} K(x)|u_0|^{q-2} u_0\overline{v_n^0}dx = 0,$$

and then from (3.8) we have

$$\begin{aligned}\|v_n^0\|_\mu^2 =& \mathcal{E}'(u_n)(v_n^0) - \frac{1}{4}\mathcal{D}'(u_0)(v_n^0) + \Re\int_{\mathbb{R}^3} K(x)|u_0|^{q-2} u_0\overline{v_n^0}dx \\ &+ \frac{1}{4}\mathcal{D}'(u_n)(v_n^0) - \Re\int_{\mathbb{R}^3} K(x)|u_n|^{q-2} u_n\overline{v_n^0}dx.\end{aligned}$$

Since $\{v_n^0\}_n$ is bounded, it follows from $\mathcal{E}'(u_n) \to 0$ that

$$\mathcal{E}'(u_n)(v_n^0) \to 0.$$

By Hölder's inequality and the lemma of Lions [35], we have

$$\left|\int_{\mathbb{R}^3} K(x)|u_0|^{q-2}u_0\overline{v_n^0}dx\right| \le \|K\|_\infty \|u_0\|_q^{q-1}\|v_n^0\|_q \to 0,$$
$$\left|\int_{\mathbb{R}^3} K(x)|u_n|^{q-2}u_n\overline{v_n^0}dx\right| \le \|K\|_\infty \|u_n\|_q^{q-1}\|v_n^0\|_q \to 0.$$

Moreover, by the Hardy-Littlewood-Sobolev inequality, we have

$$\mathcal{D}'(u_n)(v_n^0) \to 0 \quad \text{and} \quad \mathcal{D}'(u_0)(v_n^0) \to 0.$$

Thus $\|v_n^0\|_\mu^2 \to 0$ and $u_n \to u_0$, which yields $\mathcal{E}(u_n) \to \mathcal{E}(u_0)$, then in the first case, the theorem is true.

Case ii). If (3.7) does not hold, we can find a sequence $\{z_n^1\}_n \subset \mathbb{Z}^3$ such that

$$\liminf_{n\to+\infty}\int_{B(z_n^1, 1+\sqrt{3})} |v_n^0|^2 dx > 0.$$

Obviously we have $|z_n^1| \to +\infty$, and there are $w^1 \in H_A^1(\mathbb{R}^3)$ and $\phi_{z_n^1} \in H_{loc}^1(\mathbb{R}^3)$ such that (up to a subsequence) $g_{z_n^1}^{-1}u_n(x) \rightharpoonup w^1 \neq 0$. We confirm that $\mathcal{E}'_{\mathrm{per}}(w^1) = 0$.

Indeed, let $w_n^1 := g_{z_n^1}^{-1}u_n(x)$, it follows from $w_n^1 \rightharpoonup w^1$ that

$$\mathcal{E}'_{\mathrm{per}}(w_n^1)(\varphi) \to \mathcal{E}'_{\mathrm{per}}(w^1)(\varphi),\ \forall\ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3).$$

Moreover,

$$\begin{aligned} o(1) = \mathcal{E}'(u_n)g_{z_n^1}\varphi &= \mathcal{E}'_{\mathrm{per}}(w_n^1)\varphi + \Re\int_{\mathbb{R}^3} V_l(x+z_n^1)w_n^1\overline{\varphi}dx - \mu\Re\int_{\mathbb{R}^3}\frac{u_n\overline{g_{z_n^1}\varphi}}{|x|}dx \\ &= \mathcal{E}'_{\mathrm{per}}(w^1)\varphi + \Re\int_{\mathrm{supp}\,\varphi} V_l(x+z_n^1)w_n^1\overline{\varphi}dx - \mu\Re\int_{\mathbb{R}^3}\frac{u_n\overline{g_{z_n^1}\varphi}}{|x|}dx + o(1),\end{aligned}$$

and

$$\mathcal{E}'_{\mathrm{per}}(w^1)(\varphi) = -\Re\int_{\mathrm{supp}\,\varphi} V_l(x+z_n^1)w_n^1\overline{\varphi}dx + \mu\Re\int_{\mathbb{R}^3}\frac{u_n\overline{g_{z_n^1}\varphi}}{|x|}dx + o(1).$$

On the other hand, Lemma 2.4 implies that $\{u_n\}_n$ is bounded in $L^2\left(\mathbb{R}^3;|x|^{-1}dx\right)$, and from [9, Lemma 2.5], we obtain that

$$\left|\int_{\mathbb{R}^3}\frac{u_n\overline{g_{z_n^1}\varphi}}{|x|}dx\right|\leq\left(\int_{\mathbb{R}^3}\frac{|u_n|^2}{|x|}dx\right)^{1/2}\left(\int_{\mathbb{R}^3}\frac{\left|\varphi\left(x-z_n^1\right)\right|^2}{|x|}dx\right)^{1/2}\to 0.$$

Hence it is sufficient to show that $\int_{\operatorname{supp}\varphi}V_l\left(x+z_n^1\right)w_n^1\overline{\varphi}dx\to 0$. Fix any measurable set $E\subset\operatorname{supp}\varphi$, by Hölder's inequality,

$$\int_E\left|V_l\left(x+z_n^1\right)w_n^1\overline{\varphi}\right|dx\leq\|V_l\|_\infty\left\|w_n^1\right\|_2\|\varphi\chi_E\|_2.$$

Thus the boundedness of $\left\{w_n^1\right\}_n$ in $L^2\left(\mathbb{R}^3\right)$ implies that the family $\left\{V_l\left(\cdot+z_n^1\right)w_n^1\overline{\varphi}\right\}_n$ is uniformly integrable on $\operatorname{supp}\varphi$. The Vitali's convergence theorem yields

$$\Re\int_{\operatorname{supp}\varphi}V_l\left(x+z_n^1\right)w_n^1\overline{\varphi}dx\to 0,$$

which implies $\mathcal{E}_{\mathrm{per}}'(w^1)=0$.

Step 3. Set $v_n^1:=u_n-u_0-g_{z_n^1}w^1=v_n^0-g_{z_n^1}w^1$.

Case i). If

$$\sup_{z\in\mathbb{R}^3}\int_{B(z,1)}\left|v_n^1\right|^2dx\to 0\quad\text{ as }\quad n\to+\infty,\tag{3.9}$$

let us prove $v_n^1\to 0$ in $H_A^1\left(\mathbb{R}^3\right)$. Firstly, from (3.9), by virtue of the Lemma of Lions, we have $v_n^1\to 0$ in $L^t\left(\mathbb{R}^3\right)$ for any $2<t<6$. Repeating the argument of [7, Lemma 4.3, Step 4], from $\mathcal{E}'\left(u_0\right)\left(v_n^1\right)=0$ we conclude that

$$\begin{aligned}\left\|v_n^1\right\|_\mu^2=&-\left\langle u_0,v_n^1\right\rangle_\mu-\left\langle g_{z_n^1}w^1(x),v_n^1\right\rangle_\mu\\&+\frac{1}{4}\mathcal{D}'\left(u_n\right)\left(v_n^1\right)-\Re\int_{\mathbb{R}^3}K(x)\left|u_n\right|^{q-2}u_n\overline{v_n^1}dx+o(1)\\=&-\frac{1}{4}\mathcal{D}'\left(u_0\right)\left(v_n^1\right)+\Re\int_{\mathbb{R}^3}K(x)\left|u_0\right|^{q-2}u_0\overline{v_n^1}dx-\left\langle g_{z_n^1}w^1(x),v_n^1\right\rangle_\mu\\&+\frac{1}{4}\mathcal{D}'\left(u_n\right)\left(v_n^1\right)-\Re\int_{\mathbb{R}^3}K(x)\left|u_n\right|^{q-2}u_n\overline{v_n^1}dx+o(1).\end{aligned}$$

Since w^1 is a critical point of $\mathcal{E}_{\mathrm{per}}$, we have

$$\begin{aligned}\left\|v_n^1\right\|_\mu^2=&\frac{1}{4}\mathcal{D}'\left(u_n\right)\left(v_n^1\right)-\frac{1}{4}\mathcal{D}'\left(u_0\right)\left(v_n^1\right)-\sum_{k=1}^m\frac{1}{4}\mathcal{D}'(g_{z_n^1}w^1(x))\left(v_n^1\right)\\&-\Re\int_{\mathbb{R}^3}K(x)\left(\left|u_n\right|^{q-2}u_n-\left|u_0\right|^{q-2}u_0-\left|g_{z_n^1}w^1(x)\right|^{q-2}g_{z_n^1}w^1(x)\right)\overline{v_n^1}dx\\&-\Re\int_{\mathbb{R}^3}V_l(x)g_{z_n^1}w^1(x)\overline{v_n^1}dx+\mu\Re\int_{\mathbb{R}^3}\frac{g_{z_n^1}w^1(x)\overline{v_n^1}}{|x|}dx+o(1).\end{aligned}$$

From [9, Lemma 2.5], we obtain

$$\Re\int_{\mathbb{R}^3}\frac{g_{z_n^1}w^1(x)\overline{v_n^1}}{|x|}dx\to 0.$$

Similarly to [7, Lemma 4.3, Step 4], we get

$$\Re\int_{\mathbb{R}^3}K(x)\left(\left|u_n\right|^{q-2}u_n-\left|u_0\right|^{q-2}u_0-\left|g_{z_n^1}w^1(x)\right|^{q-2}g_{z_n^1}w^1(x)\right)\overline{v_n^1}dx\to 0,$$

and

$$\Re\int_{\mathbb{R}^3}V_l(x)g_{z_n^1}w^1(x)\overline{v_n^1}dx\to 0.$$

Hence

$$\left\|v_n^1\right\|_\mu^2=\frac{1}{4}\mathcal{D}'\left(u_n\right)\left(v_n^1\right)-\frac{1}{4}\mathcal{D}'\left(u_0\right)\left(v_n^1\right)-\frac{1}{4}\mathcal{D}'\left(g_{z_n^1}w^1(x)\right)\left(v_n^1\right)+o(1).$$

To prove $\left|\mathcal{D}'\left(u_n\right)\left(v_n^1\right)\right|\to 0$, it follows from Hardy-Littlewood-Sobolev and Hölder inequalities that

$$\left|\mathcal{D}'\left(u_n\right)\left(v_n^1\right)\right|\leq\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u_n(x)|^2|u_n(y)||v_n^1(y)|}{|x-y|}dxdy\leq C\left\|u_n\right\|_{\frac{12}{5}}^3\left\|v_n^1\right\|_{\frac{12}{5}}\to 0. \tag{3.10}$$

Similar argument yields that $\mathcal{D}'\left(u_0\right)\left(v_n^1\right)\to 0$ and $\mathcal{D}'\left(g_{z_n^1}w^1(x)\right)\left(v_n^1\right)\to 0$. Then we have $v_n^1\to 0$ in $H_A^1\left(\mathbb{R}^3\right)$.

Case ii). If (3.9) does not hold, then there is a sequence $\left\{z_n^2\right\}_n\subset\mathbb{Z}^3$ satisfying

$$\liminf_{n\to+\infty}\int_{B\left(z_n^2,1+\sqrt{3}\right)}\left|v_n^1\right|^2dx>0, \tag{3.11}$$

which yields that there exist $w^2\in H_A^1\left(\mathbb{R}^3\right)$ and $\phi_{z_n^2}\in H_{loc}^1(\mathbb{R}^3)$ such that (up to subsequences)

$$\begin{aligned}&\left|z_n^2\right|\to+\infty,\quad\left|z_n^2-z_n^1\right|\to+\infty,\\&g_{z_n^2}^{-1}u_n(x)\rightharpoonup w^2\neq 0.\end{aligned}$$

Furthermore, we confirm that

$$\mathcal{E}_{\text{per}}'\left(w^2\right)=0.$$

Indeed, let $w_n^2:=g_{z_n^2}^{-1}u_n(x)$, it follows from the discussion in Step 3 that

$$\mathcal{E}_{\text{per}}'\left(w_n^2\right)(\varphi)-\mathcal{E}_{\text{per}}'\left(w^2\right)(\varphi)\to 0$$

and $\mathcal{E}_{\text{per}}'\left(w_n^2\right)(\varphi)\to 0$ for any $\varphi\in\mathcal{C}_0^\infty\left(\mathbb{R}^3\right)$, so we have $\mathcal{E}_{\text{per}}'\left(w^2\right)=0$, which gives the desired result.

Step 4. Repeating the procedure in Step 2 and 3, we can suppose that for an integer $k\geq 1$, there exist $\left\{z_n^k\right\}_n\subset\mathbb{Z}^3$, $\{\phi_{z_n^k}\}_n\subset H_{loc}^1(\mathbb{R}^3)$ and $w^k\in H_A^1\left(\mathbb{R}^3\right)$ such

$$\begin{aligned}&\left|z_n^k\right|\to+\infty,\left|z_n^k-z_n^{k'}\right|\to+\infty\text{ for }k'>k\geq 1,\\&w_n^k:=g_{z_n^k}^{-1}u_n(x)\rightharpoonup w^k\neq 0,\\&\mathcal{E}_{\text{per}}'\left(w^k\right)=0.\end{aligned}$$

Set

$$v_n^k=v_n^{k-1}-g_{z_n^k}w^k,\quad k\geq 1.$$

Similar to Step 3, we can prove that if

$$\sup_{z\in\mathbb{R}^3}\int_{B(z,1)}\left|v_n^k\right|^2dx\to 0\ \text{ as }n\to+\infty, \tag{3.12}$$

then

$$v_n^k\to 0;$$

or if

$$\sup_{z\in\mathbb{R}^3}\int_{B(z,1)}\left|v_n^k\right|^2dx>0\ \text{ as }n\to+\infty, \tag{3.13}$$

then there exist two sequences $\left\{z_n^{k+1}\right\}_n\subset\mathbb{Z}^3$, $\{\phi_{z_n^{k+1}}\}_n\subset H_{loc}^1(\mathbb{R}^3)$, and $w^{k+1}\in H_A^1\left(\mathbb{R}^3\right)$ such that (up to subsequences)

$$\left|z_n^{k+1}\right|\to+\infty,\quad\left|z_n^{k+1}-z_n^{k'}\right|\to+\infty,\quad\text{for}\quad k'\leq k,$$

$$g_{z_n^{k+1}}^{-1} u_n(x) \rightharpoonup w^{k+1} \neq 0, \quad \mathcal{E}_{\text{per}}' \left(w^{k+1}\right) = 0.$$

Step 5. We prove that the above procedure will finish after finite number of steps. Since for any fixed $\ell \geq 1$ and $k \in \{1, 2, \cdots, \ell\}$, we have $\mathcal{E}_{\text{per}}'(w^k) = 0$, and there is some $\rho > 0$ such that $\left\|w^k\right\|_\mu \geq \rho$. Indeed, from $\mathcal{E}_{\text{per}}'(w^k)w^k = 0$ we have

$$\|\nabla_A w^k\|_2^2 + \int_{\mathbb{R}^3} V_p(x) \left|w^k(x)\right|^2 dx = \mathcal{D}(w^k) - \int_{\mathbb{R}^3} K(x) \left|w^k(x)\right|^q dx.$$

Then by $\|\nabla_A w^k\|_2^2 + \int_{\mathbb{R}^3} V_p(x) \left|w^k(x)\right|^2 dx \geq c_1 \|w^k\|_\mu^2$ we obtain $\|w^k\|_\mu^2 \leq c_2 \|w^k\|_\mu^4$, where c_1, c_2 are constants. Hence there is some $\rho > 0$ such that $\left\|w^k\right\|_\mu \geq \rho$. In addition it results from the properties of a weak convergence sequence that

$$\begin{aligned}
0 \leq & \lim_{n \to +\infty} \left\| u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k(x) \right\|_\mu^2 \\
= & \lim_{n \to +\infty} \left(\|u_n\|_\mu^2 - \|u_0\|_\mu^2 - \sum_{k=1}^{\ell} \left\|w^k\right\|_\mu^2 \right) \leq \limsup_{n \to +\infty} \|u_n\|_\mu^2 - \|u_0\|_\mu^2 - \ell \rho^2.
\end{aligned}$$

Hence

$$\rho^2 \ell \leq \limsup_{n \to +\infty} \|u_n\|_\mu^2 - \|u_0\|_\mu^2,$$

which concludes that ℓ is bounded. This completes the proof of 3). Combining step 1, step 3 and step 4, we conclude 1) and 2).

Step 6. In this step, we will prove 4). Indeed,

$$\begin{aligned}
\mathcal{E}(u_n) = & \frac{1}{2} \langle u_n, u_n \rangle_\mu - \frac{1}{4} \mathcal{D}(u_n) + \frac{1}{q} \int_{\mathbb{R}^3} K(x) |u_n(x)|^q dx \\
= & \frac{1}{2} \langle u_0, u_0 \rangle_\mu + \frac{1}{2} \langle u_n - u_0, u_n - u_0 \rangle_\mu + \langle u_0, u_n - u_0 \rangle_\mu \\
& - \frac{1}{4} \mathcal{D}(u_n) + \frac{1}{q} \int_{\mathbb{R}^3} K(x) |u_n(x)|^q dx \\
= & \mathcal{E}(u_0) + \mathcal{E}_{\text{per}}(u_n - u_0) + \langle u_0, u_n - u_0 \rangle_\mu + \frac{1}{4} \mathcal{D}(u_n - u_0) \\
& - \frac{1}{4} \mathcal{D}(u_n) + \frac{1}{4} \mathcal{D}(u_0) - \frac{1}{q} \int_{\mathbb{R}^3} K(x) \left[|u_n - u_0|^q + |u_0|^q - |u_n|^q\right] dx \\
& + \frac{1}{2} \int_{\mathbb{R}^3} V_l(x) |u_n - u_0|^2 dx - \frac{\mu}{2} \int_{\mathbb{R}^3} \frac{|u_n - u_0|^2}{|x|} dx.
\end{aligned}$$

As $u_n \rightharpoonup u_0$, we have

$$\langle u_0, u_n - u_0 \rangle_\mu \to 0.$$

Using Lemma 3.3 we deduce that

$$\mathcal{D}(u_n - u_0) - \mathcal{D}(u_n) + \mathcal{D}(u_0) \to 0.$$

Similar to the proof for the classical Brezis-Lieb lemma [10, Proposition 4.7.30], we can get

$$\int_{\mathbb{R}^3} K(x) \left[|u_n - u_0|^q + |u_0|^q - |u_n|^q\right] dx \to 0.$$

Let $E \subset \mathbb{R}^3$ be a measurable set, it follows from (V1) and Hölder's inequality that

$$\int_E |V_l(x)| \, |u_n - u_0|^2 dx \leq \|V_l \chi_E\|_3 \|u_n - u_0\|_3^2.$$

Furthermore, since $\{u_n - u_0\}_n$ is bounded in $H_A^1(\mathbb{R}^3)$, by Vitali's convergence theorem,

$$\int_{\mathbb{R}^3} V_l(x) |u_n - u_0|^2 \, dx \to 0.$$

Notice that

$$\int_{\mathbb{R}^3} \frac{|u_n - u_0|^2}{|x|} dx = \int_{\mathbb{R}^3} \frac{(u_n - u_0)\, \overline{u_n}}{|x|} dx - \int_{\mathbb{R}^3} \frac{(u_n - u_0)\, \overline{u_0}}{|x|} dx,$$

repeating the similar arguments as above, we have $\int_{\mathbb{R}^3} \frac{(u_n - u_0)\overline{u_0}}{|x|} dx \to 0$ and

$$\int_{\mathbb{R}^3} \frac{(u_n - u_0)\, \overline{u_n}}{|x|} dx = \int_{\mathbb{R}^3} \frac{\left(u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k(x)\right) \overline{u_n}}{|x|} + \sum_{k=1}^{\ell} \frac{g_{z_n^k} w^k(x) \overline{u_n}}{|x|} dx.$$

We now deduce that

$$\left| \int_{\mathbb{R}^3} \frac{g_{z_n^k} w^k(x) \overline{u_n}}{|x|} dx \right| \leq \left(\int_{\mathbb{R}^3} \frac{\left| w^k\left(\cdot - z_n^k\right)\right|^2}{|x|} dx \right)^{1/2} \left(\int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx \right)^{1/2} \to 0,$$

and

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \frac{\left(u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k(x)\right) \overline{u_n}}{|x|} dx \right| \\
\leq & \left(\int_{\mathbb{R}^3} \frac{\left| u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k(x)\right|^2}{|x|} dx \right)^2 \left(\int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx \right)^{1/2} \\
\leq & C \left\| u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k(x) \right\|_{\mu} \left(\int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx \right)^{1/2} \to 0,
\end{aligned}$$

where the Hölder's inequality has been used, thus we get $\int_{\mathbb{R}^3} \frac{|u_n - u_0|^2}{|x|} dx \to 0$. It follows from the above conclusions that

$$\mathcal{E}(u_n) = \mathcal{E}(u_0) + \mathcal{E}_{\text{per}}(u_n - u_0) + o(1). \tag{3.14}$$

In the next let us prove

$$\mathcal{E}_{\text{per}}(u_n - u_0) \to \sum_{k=1}^{\ell} \mathcal{E}_{\text{per}}\left(w^k\right). \tag{3.15}$$

Firstly we have

$$\begin{aligned}
\mathcal{E}_{\text{per}}(u_n - u_0) = & \frac{1}{2} \|u_n - u_0\|_{\mu}^2 - \frac{1}{4} \mathcal{D}(u_n - u_0) + \frac{1}{q} \int_{\mathbb{R}^3} K(x) |u_n - u_0|^q \, dx \\
& - \frac{1}{2} \int_{\mathbb{R}^3} V_l(x) |u_n - u_0|^2 \, dx + \frac{\mu}{2} \int_{\mathbb{R}^3} \frac{|u_n - u_0|^2}{|x|} dx \\
= & \frac{1}{2} \left\| u_n - u_0 - \sum_{k=1}^{\ell} g_{z_n^k} w^k(x) \right\|_{\mu}^2 - \frac{1}{4} \mathcal{D}(u_n - u_0) \\
& + \frac{1}{q} \int_{\mathbb{R}^3} K(x) |u_n - u_0|^q \, dx + \frac{1}{2} \sum_{k=1}^{\ell} \left\| g_{z_n^k} w^k(x) \right\|_{\mu}^2 + o(1) \\
= & \sum_{k=1}^{\ell} \mathcal{E}_{\text{per}}\left(w^k\right) + \frac{1}{4} \sum_{k=1}^{\ell} \mathcal{D}\left(g_{z_n^k} w^k(x)\right) - \frac{1}{q} \sum_{k=1}^{\ell} \int_{\mathbb{R}^3} K(x) \left| g_{z_n^k} w^k(x) \right|^q dx \\
& - \frac{1}{4} \mathcal{D}(u_n - u_0) + \frac{1}{q} \int_{\mathbb{R}^3} K(x) |u_n - u_0|^q \, dx + o(1).
\end{aligned}$$

From Lemma 3.3, by iterating and Lemma 3.4, we have

$$\mathcal{D}\left(u_n-u_0\right)-\sum_{k=1}^{\ell} \mathcal{D}\left(g_{z_n^k} w^k(x)\right) \rightarrow 0, \text { as } n \rightarrow+\infty .$$

Using the similar arguments as above, we also have

$$\int_{\mathbb{R}^3} K(x)\left|u_n-u_0\right|^q d x-\sum_{k=1}^{\ell} \int_{\mathbb{R}^3} K(x)\left|g_{z_n^k} w^k(x)\right|^q d x \rightarrow 0, \text { as } n \rightarrow+\infty .$$

Hence, (3.15) holds, and 4) follows from (3.14) and (3.15). So we complete the proof. □

4 Existence and nonexistence of ground states

Proof of Theorem 1.1. Write $c_{\text{per}} := \inf_{\mathcal{N}_{\text{per}}} \mathcal{E}_{\text{per}}$, where $\mathcal{E}_{\text{per}}$ is defined by (2.4) and $\mathcal{N}_{\text{per}}$ is the corresponding Nehari manifold. From 2) and 4) of Theorem 3.2, either

$$c=\lim_{n \rightarrow+\infty} \mathcal{E}\left(u_n\right)=\mathcal{E}\left(u_0\right) \tag{4.1}$$

or

$$c=\lim_{n \rightarrow+\infty} \mathcal{E}\left(u_n\right)=\mathcal{E}\left(u_0\right)+\sum_{k=1}^{\ell} \mathcal{E}_{\text{per}}\left(w^k\right) \geq \mathcal{E}\left(u_0\right)+\ell c_{\text{per}}, \tag{4.2}$$

where u_0 and w^k are the critical point of $\mathcal{E}_{\text{per}}$.

We assert that there is a $u_{\text{per}} \in \mathcal{N}_{\text{per}}$ such that

$$\mathrm{c}_{\text{per}}=\mathcal{E}_{\text{per}}(u_{\text{per}})>0.$$

In the case $V_l(x)=\frac{\mu}{|x|}$, we have $\mathcal{E}=\mathcal{E}_{\text{per}}$ and $c=c_{\text{per}}$. If $u_0 \neq 0$, then (4.1) holds, $c_{\text{per}}=\mathcal{E}_{\text{per}}\left(u_0\right)>0$, and $u_0 \in \mathcal{N}_{\text{per}}$, the assertion is true. If $u_0=0$, due to $c>0$, which is guaranteed by Lemma 2.7 and Lemma 3.1, we know that (4.1) does not hold. So

$$c_{\text{per}}=\sum_{k=1}^{\ell} \mathcal{E}_{\text{per}}\left(w^k\right) \geq \ell c_{\text{per}},$$

we obtain $\ell=1$, $c_{\text{per}}=\mathcal{E}_{\text{per}}\left(w^1\right)>0$ and $w^1 \in \mathcal{N}_{\text{per}}$, then the assertion follows.

In the case $V_l(x)<\frac{\mu}{|x|}$, let $t_{\text{per}}>0$ be the number such that $t_{\text{per}} u_{\text{per}} \in \mathcal{N}$, then

$$V(x)-\frac{\mu}{|x|}=V_p(x)+V_l(x)-\frac{\mu}{|x|}<V_p(x) \text{ for } a.e.\ x \in \mathbb{R}^3,$$

which implies that

$$c_{\text{per}}=\mathcal{E}_{\text{per}}\left(u_{\text{per}}\right) \geq \mathcal{E}_{\text{per}}\left(t_{\text{per}} u_{\text{per}}\right)>\mathcal{E}\left(t_{\text{per}} u_{\text{per}}\right) \geq \inf_{\mathcal{N}} \mathcal{E}=c>0.$$

Hence $c_{\text{per}}>c>0$.

By the same process of Lemma 3.2, we conclude that $\mathcal{E}(u_0) \geq 0$. If $\ell \geq 1$, from (4.2) we obtain

$$c \geq \ell c_{\text{per}}$$

which contradicts to $c_{\text{per}}>c$. Hence, (4.1) holds, and by Theorem 3.2 we have $u_n \rightarrow u_0$ in $H_A^1\left(\mathbb{R}^3\right)$, $0<c=\mathcal{E}(u_0)$, and $u_0 \neq 0$ is a ground state solution of (1.1). □

Proof of Theorem 1.2. We assume by contradiction that u_0 is a ground state for $\mathcal{E}$. In particular

$$c=\inf_{\mathcal{N}} \mathcal{E}=\mathcal{E}\left(u_0\right)>0.$$

The inequality (1.5) implies that

$$V(x)-\frac{\mu}{|x|}=V_p(x)+V_l(x)-\frac{\mu}{|x|}>V_p(x) \quad \text{for a.e. } \quad x \in \mathbb{R}^3,$$

by the similar argument as before, we have $c>c_{\mathrm{per}}$. On the other hand, fix $u \in \mathcal{N}_{\mathrm{per}}$, where $\mathcal{E}_{\mathrm{per}}$ is given by (2.4) with the corresponding Nehari manifold $\mathcal{N}_{\mathrm{per}}$, we may choose $t_z>0$ such that $t_z e^{i \phi_z} u(\cdot-z) \in \mathcal{N}$ for any $z \in \mathbb{Z}^3$. Then

$$\begin{aligned}
\mathcal{E}_{\mathrm{per}}(u) & =\mathcal{E}_{\mathrm{per}}\left(e^{i \phi_z} u(\cdot-z)\right) \geq \mathcal{E}_{\mathrm{per}}\left(t_z e^{i \phi_z} u(\cdot-z)\right) \\
& =\mathcal{E}\left(t_z e^{i \phi_z} u(\cdot-z)\right)-\frac{1}{2} \int_{\mathbb{R}^3} V_l(x)\left|t_z u(\cdot-z)\right|^2 d x+\frac{\mu}{2} \int_{\mathbb{R}^3} \frac{\left|t_z u(\cdot-z)\right|^2}{|x|} d x \\
& \geq c-\frac{1}{2} \int_{\mathbb{R}^3} V_l(x)\left|t_z u(\cdot-z)\right|^2 d x+\frac{\mu}{2} \int_{\mathbb{R}^3} \frac{\left|t_z u(\cdot-z)\right|^2}{|x|} d x .
\end{aligned}$$

Note that $\mathcal{E}_{\mathrm{per}}\left(t_z e^{i \phi_z} u(\cdot-z)\right)=\mathcal{E}_{\mathrm{per}}\left(t_z u\right) \leq c_{\mathrm{per}}$, by the coercivity of $\mathcal{E}_{\mathrm{per}}$ on $\mathcal{N}_{\mathrm{per}}$ we confirm that $\sup_{z \in \mathbb{Z}^3} t_z<+\infty$. Hence we have

$$\int_{\mathbb{R}^3} V_l(x)\left|t_z u(\cdot-z)\right|^2 d x=t_z^2 \int_{\mathbb{R}^3} V_l(x+z) u^2 d x \rightarrow 0 \text{ as } |z| \rightarrow+\infty .$$

It follows from [9, Lemma 2.5] that

$$\int_{\mathbb{R}^3} \frac{\left|t_z u(\cdot-z)\right|^2}{|x|} d x=t_z^2 \int_{\mathbb{R}^3} \frac{|u(\cdot-z)|^2}{|x|} d x \rightarrow 0 \text{ as } |z| \rightarrow+\infty,$$

and therefore

$$\mathcal{E}_{\mathrm{per}}(u) \geq c+o(1) .$$

Taking infimum over $u \in \mathcal{N}_{\mathrm{per}}$ we see that

$$c_{\mathrm{per}}=\inf_{\mathcal{N}_{\mathrm{per}}} \mathcal{E}_{\mathrm{per}} \geq c,$$

which is a contradiction. This ends the proof of Theorem 1.2. □

5 Compactness of Ground states sequence

Suppose that $\{\mu_n\}_n \subset (0, \mu^*)$ is a sequence such that $\mu_n \rightarrow 0^+$ as $n \rightarrow+\infty$. Let $\mathcal{E}_n$ be the energy functional corresponding to $\mu=\mu_n$, $\mathcal{E}_0$ and $\mathcal{N}_0$ be the energy functional and the Nehari manifold for $\mu=0$, respectively. Define

$$c_n:=\mathcal{E}_n\left(u_n\right)=\inf_{u \in \mathcal{N}_n} \mathcal{E}_n(u), \quad c_0:=\mathcal{E}_0\left(u_0\right)=\inf_{u \in \mathcal{N}_0} \mathcal{E}_0(u),$$

where $u_n \in \mathcal{N}_n$ is the ground state solution for $\mathcal{E}_n$, in particular $\mathcal{E}_n^{\prime}\left(u_n\right)=0$.

Lemma 5.1 *The sequence $\{u_n\}_n$ is bounded in the norm $\|\cdot\|_{H_A^1}$.*

Proof. By Lemma 2.6, we know that $\frac{\mu^*}{4}\|u_n\|_{H_A^1} \leq \|u_n\|_{\mu_n}$ for $\mu_n \in [0, \frac{\mu^*}{2})$. If $\|u_n\|_{\mu_n}$ is bounded, it follows that $\|u_n\|_{H_A^1}$ is bounded. Suppose by contradiction that $\|u_n\|_{\mu_n} \rightarrow+\infty$. Consider $s_n>0$ such that

$s_n u_0 \in \mathcal{N}_n$, we have

$$c_0 = \mathcal{E}_0(u_0) \geq \mathcal{E}_0(s_n u_0) = \mathcal{E}_n(s_n u_0) + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx \geq c_n + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx.$$

Then we obtain $c_0 \geq c_n$. Thus

$$\begin{aligned} c_0 &\geq \lim_{n\to+\infty} \mathcal{E}_n(u_n) = \lim_{n\to+\infty} \left(\mathcal{E}_n(u_n) - \frac{1}{q}\mathcal{E}'(u_n)(u_n) \right) \\ &= \lim_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{q} \right) \|u_n\|_{\mu_n}^2 + \frac{1}{q} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} dxdy \right. \\ &\quad \left. - \frac{1}{4} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} dxdy \right] \\ &= \lim_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{q} \right) \|u_n\|_{\mu_n}^2 + \left(\frac{1}{q} - \frac{1}{4} \right) \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} dxdy \right] \\ &\geq \lim_{n\to+\infty} \left(\frac{1}{2} - \frac{1}{q} \right) \|u_n\|_{\mu_n}^2 = +\infty, \end{aligned}$$

since $2 < q < 4$. This yields a contradiction. □

Lemma 5.2 $\lim_{n\to+\infty} c_n = c_0$.

Proof. Consider $t_n > 0$ and $s_n > 0$ such that $t_n u_n \in \mathcal{N}_0$ and $s_n u_0 \in \mathcal{N}_n$, we have

$$c_n = \mathcal{E}_n(u_n) \geq \mathcal{E}_n(t_n u_n) = \mathcal{E}_0(t_n u_n) - \frac{\mu_n t_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx \geq c_0 - \frac{\mu_n t_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx, \tag{5.1}$$

and

$$c_0 = \mathcal{E}_0(u_0) \geq \mathcal{E}_0(s_n u_0) = \mathcal{E}_n(s_n u_0) + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx \geq c_n + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx. \tag{5.2}$$

By (5.1) and (5.2) we obtain

$$c_0 \geq c_n + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx \geq c_n \geq c_0 - \frac{\mu_n t_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx,$$

that is

$$c_0 - \frac{\mu_n t_n^2}{2} \int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|} dx \leq c_n \leq c_0. \tag{5.3}$$

Since $\{u_n\}_n$ is bounded, then from Lemma 2.4 we know that $\int_{\mathbb{R}^3} \frac{u_n^2}{|x|} dx$ is also bounded.

In the next we will show that $\{t_n\}_n$ is bounded. Assume by contradiction that $t_n \to +\infty$. Let

$$Q(u) = \int_{\mathbb{R}^3} |\nabla_A u|^2 dx + \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx.$$

Since $t_n u_n \in \mathcal{N}_0$, we have

$$\begin{aligned} \mathcal{E}_0'(t_n u_n)(t_n u_n) =& t_n^2 Q(u_n) - \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{t_n^4 |u_n(x)|^2|u_n(y)|^2}{|x-y|} dxdy \\ &+ t_n^q \int_{\mathbb{R}^3} K(x) |u_n|^q dx = 0. \end{aligned}$$

Hence,

$$\frac{Q(u_n)}{t_n^{q-2}} = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{t_n^4 |u_n(x)|^2|u_n(y)|^2}{t_n^q |x-y|} dxdy - \int_{\mathbb{R}^3} K(x) |u_n|^q dx.$$

Obviously, $\int_{\mathbb{R}^3} K(x) |u_n|^q dx$ and $\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} dxdy$ is bounded, and

$$\frac{Q(u_n)}{t_n^{q-2}} \to 0,$$

it follows that $\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{t_n^4|u_n(x)|^2|u_n(y)|^2}{t_n^q|x-y|}dxdy$ is bounded. By Lemma 2.7 we have $0<c_1\le c_n\le c_{n+1}$, and then

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|}dxdy\nrightarrow 0.$$

This is a contradiction. Combining with (5.3) and the boundedness of t_n, we obtain

$$\lim_{n\to+\infty}c_n=c_0.$$

This completes the proof of Lemma 5.2. $\square$

Proof of Theorem 1.3. Suppose that

$$\lim_{n\to+\infty}\sup_{z\in\mathbb{R}^3}\int_{B(z,1)}|u_n|^2\,dx=0.$$

From the well known lemma of Lions we obtain

$$u_n\to 0 \text{ in } L^t\left(\mathbb{R}^3\right) \text{ for all } t\in(2,6).$$

Similar to (3.10), we have

$$\mathcal{D}'\left(u_n\right)\left(u_n\right)\to 0 \quad \text{ as } n\to+\infty.$$

Due to $K\in L^\infty\left(\mathbb{R}^3\right)$ we have

$$\int_{\mathbb{R}^3}K(x)|u_n|^q dx\le\|K\|_\infty\|u_n\|_q^q\to 0 \quad \text{ as } n\to+\infty.$$

Recall that

$$\|u_n\|_{\mu_n}^2=\mathcal{D}'\left(u_n\right)\left(u_n\right)-\int_{\mathbb{R}^3}K(x)\left|u_n\right|^q dx,$$

it results from the above estimates that $\|u_n\|_{\mu_n}\to 0$, and then we have

$$\lim_{n\to+\infty}\mathcal{E}_n\left(u_n\right)=0.$$

From Lemma 5.1 and $c_0>0$, we get

$$\lim_{n\to+\infty}\mathcal{E}_n\left(u_n\right)=c_0>0,$$

which is a contradiction. Hence, there is a sequence $\{z_n\}_n\subset\mathbb{Z}^3$ such that

$$\liminf_{n\to+\infty}\int_{B\left(z_n,1+\sqrt{3}\right)}|u_n|^2\,dx>0.$$

Since $g_{z_n}^{-1}u_n$ is bounded, there is $u_0\in H_A^1\left(\mathbb{R}^3\right)\setminus\{0\}$ such that

$$\begin{aligned}
&g_{z_n}^{-1}u_n\rightharpoonup u_0 \text{ in } H_A^1\left(\mathbb{R}^3\right),\\
&g_{z_n}^{-1}u_n\to u_0 \text{ in } L_{\mathrm{loc}}^2\left(\mathbb{R}^3\right),\\
&g_{z_n}^{-1}u_n\to u_0 \text{ for a.e. } x\in\mathbb{R}^3.
\end{aligned}$$

Let $w_n=g_{z_n}^{-1}u_n$. For any fixed $\varphi\in\mathcal{C}_0^\infty\left(\mathbb{R}^3\right)$,

$$\mathcal{E}_0'\left(w_n\right)(\varphi)=\mathcal{E}_n'\left(u_n\right)\left(g_{z_n}\varphi\right)+\mu_n\Re\int_{\mathbb{R}^3}\frac{u_n\overline{\left(g_{z_n}\varphi\right)}}{|x|}dx=\mu_n\Re\int_{\mathbb{R}^3}\frac{u_n\overline{\left(g_{z_n}\varphi\right)}}{|x|}dx.$$

By [9, Lemma 2.5] and Hölder's inequality,

$$\mu_n\Re\int_{\mathbb{R}^3}\frac{u_n\overline{\left(g_{z_n}\varphi\right)}}{|x|}dx\to 0 \quad \text{ as } n\to+\infty.$$

Hence $\mathcal{E}_0'(w_n)(\varphi) \to 0$. It follows from Corollary 3.1 that

$$\mathcal{E}_0'(w_n)(\varphi) \to \mathcal{E}_0'(u_0)(\varphi).$$

Thus u_0 is a nontrivial critical point of $\mathcal{E}_0$. In particular, $u_0 \in \mathcal{N}_0$. By Lemma 5.2 we have

$$\begin{aligned}
c_0 = &\lim_{n\to+\infty} \mathcal{E}_n(u_n) = \lim_{n\to+\infty}\left(\mathcal{E}_n(u_n) - \frac{1}{q}\mathcal{E}_n'(u_n)(u_n)\right)\\
= &\lim_{n\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{q}\right)Q(u_n) + \left(\frac{1}{q}-\frac{1}{2}\right)\mu_n\int_{\mathbb{R}^3}\frac{|u_n|^2}{|x|}dx\right.\\
&\left.+(\frac{1}{q}-\frac{1}{4})\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|}dxdy\right]\\
= &\lim_{n\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{q}\right)Q(w_n) + \left(\frac{1}{q}-\frac{1}{2}\right)\mu_n\int_{\mathbb{R}^3}\frac{|u_n|^2}{|x|}dx\right.\\
&\left.+(\frac{1}{q}-\frac{1}{4})\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|w_n(x)|^2|w_n(y)|^2}{|x-y|}dxdy\right]\\
\geq &\left(\frac{1}{2}-\frac{1}{q}\right)Q(u_0) + (\frac{1}{q}-\frac{1}{4})\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u_0(x)|^2|u_0(y)|^2}{|x-y|}dxdy\\
= &\mathcal{E}_0(u_0) - \frac{1}{q}\mathcal{E}_0'(u_0)(u_0) = \mathcal{E}_0(u_0) \geq c_0.
\end{aligned}$$

Hence, $\mathcal{E}_0(u_0) = c$ and $u_0 \in H_A^1(\mathbb{R}^3)$ is a ground state solution for $\mathcal{E}_0$. This ends the proof. □

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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