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Taylor Collocation Method for Solving Two dimensional Partial Volterra Integro-differential Equations

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We consider a two dimensional partial Volterra integro-differential equations (PVIDEs). We extend the Taylor collocation method of one dimensional Volterra integral equations to approximate the solution of two dimensional partial Volterra integro-differential equations. The method is based on the use of Taylor polynomials in two dimensional. The approximate solution is given by using explicit schemes and the method is proved to be high order convergent with respect to the maximum norm. Some numerical examples are given to verify the theoretical results. Copyright © 0000 John Wiley & Sons, Ltd.

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1. Introduction

Partial integro-differential equations arise in problems of applied sciences and engineering to model dynamical systems, they can be found in financial mathematics [1]-[18], biological models [9], fluid dynamics [3], and many other areas. PVIDEs have many different types such as elliptic [19], hyperbolic [7] and parabolic [11] in one or multi-dimension. In this article, we investigate a numerical method for the two dimensional linear partial Volterra integro-differential equation:

$$\frac{\partial u}{\partial x}(x,y) = g(x,y) + \int_0^x \int_0^y K(x,y,t,s)u(t,s)dsdt, \quad (x,y) \in D, \tag{1.1}$$

with the initial condition:

$$u(0,y) = u_0(y) \tag{1.2}$$

where u, stands for the real unknown function to be determined and the functions g and K are known and assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of the solution on $D := [0,a] \times [0,b] \subset \mathbb{R}^2$ and $S := \{(x,y,t,s) : 0 \le t \le x \le a, 0 \le s \le y \le b\}$ respectively.

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The two-dimensional partial integro-differential equations are one of the most challenging problems to be solved analytically and even approximately. Thereupon, efficient numerical methods for solving such types of problems appear to be needed. This issue has been studied by many authors, and as a result, a notable number of methods have been propounded. For example, Aziz et al. [2] developed a collocation method based on Haar wavelet for a numerical solution of diffusion and reaction-diffusion PVIDEs. Haar wavelets are also applied by Babaghaie [4] and Harish Kumar [12] to solve nonlinear-type. In [16] Khan et al. used Laplace decomposition for solving nonlinear system of fractional order partial differential equations.
The main aim of this paper is to extend the Taylor collocation method of one dimensional Volterra integral and integro-differential in [5, 6] to solve two-dimensional PVIDEs of type (1.1). For that, by direct integration, the Equation (1.1) with the initial condition (1.2) can be transformed to the following two dimensional Volterra integral equation

$$u(x,y) = f(x,y) + \int_0^x \int_0^z \int_0^y K(z,y,t,s)u(t,s)dsdtdz, \quad (x,y) \in D, \tag{1.3}$$

where $f(x,y) = u_0(y) + \int_0^x g(z,y)dz$. It follows from the classical theory of Volterra that (1.3) possesses a unique solution $u \in C(D)$.

Taylor collocation methods found their way into approximation theory, and have been used to find numerical solutions of different types of equations, such as integral equations [20], delay integral equations [5], integro-differential equations [10, 13, 14], delay integro-differential equations [6]. This method has a high order of convergence and its approximate solution is given by using explicit formulas which makes the proposed algorithm easy to implement and very effective.

The remainder of the paper is organized as follows, Section 2 is devoted to the discussion of the proposed method. Convergence analysis and its order of convergence is investigated in Section 3. In Section 4 we illustrate the accuracy of the proposed method and the theoretical results by considering several numerical examples. Finally, a conclusion is given in Section 5.

2. Description of the Method

In this section we develop the Taylor collocation method to approximate the solution of (1.3) in the real polynomial spline space

$$S_{p,p}^{(-1)}(\Pi_{N,M}) = \{v : D \to \mathbb{R} : \; v_{n,m} = v|_{D_{n,m}} \in \pi_{p,p}, \;\; n = 0, ..., N-1; \; m = 0, 1, .., M-1\},$$

where

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(x_n, y_m), \; 0 \leq n \leq N, \; 0 \leq m \leq M\},$$

in which $\Pi_N = \{x_i = ih, i = 0, 1, .., N\}$ and $\Pi_M = \{y_j = jk, j = 0, 1, .., M\}$, uniform partitions of the intervals $[0, a]$ and $[0, b]$ respectively, with the stepsizes are given by $h = \frac{a}{N}$ and $k = \frac{b}{M}$. Along with

$$D_{n,m} := \kappa_n \times \delta_m \qquad\qquad (n = 0, 1, .., N-1; m = 0, 1, .., M-1),$$

a defined grid for D, with the subintervals

$$\kappa_n = [x_n; x_{n+1}), n = 0, 1, .., N-1; \qquad\qquad \kappa_{N-1} = [x_{N-1}, x_N],$$

$$\delta_m = [y_m; y_{m+1}), m = 0, 1, .., M-1; \qquad\qquad \delta_{M-1} = [y_{M-1}, y_M].$$

$\pi_{p,p}$ is the set of all real polynomials of degree not exceeding p in x and y.

$S_{p,p}^{(-1)}(\Pi_{N,M})$ is the space of bivariate polynomial spline functions of degree (at most) p in x and y, its dimension is $NM(p+1)^2$, i.e., the same as the total number of the coefficients of the polynomials $v_{n,m}$, $n = 0, ..., N-1; m = 0, 1, .., M-1$.

To find these coefficients, we use Taylor polynomial on each rectangle.

- **In the rectangle** $D_{0,0}$

  We approximate u by the Taylor polynomial

$$v_{0,0}(x,y)=\sum_{i+j=0}^{p}\frac{1}{i!j!}\frac{\partial^{i+j}u(0,0)}{\partial x^i\partial y^j}x^iy^j\ ;\quad (x,y)\in D_{0,0}, \tag{2.1}$$

where $\frac{\partial^{i+j}u(0,0)}{\partial x^i\partial y^j}$ is the exact value of $\frac{\partial^{i+j}u}{\partial x^i\partial y^j}$ at point $(0,0)$.
To find $\frac{\partial^j u(x,y)}{\partial y^j}$, we differentiate Equation (1.3) j-times with respect to y

$$\begin{aligned}\frac{\partial^j u(x,y)}{\partial y^j}&=\partial_2^{(j)}f(x,y)+\int_0^x\int_0^z\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}k(z,y,t,y)u(t,y)]dtdz\\&\quad+\int_0^x\int_0^z\int_0^y\partial_2^{(j)}k(z,y,t,s)u(t,s)dsdtdz\\&=\partial_2^{(j)}f(x,y)+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x\int_0^z\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(z,y,t,y)]\frac{\partial^l u(t,y)}{\partial y^l}dtdz\\&\quad+\int_0^x\int_0^z\int_0^y\partial_2^{(j)}k(z,y,t,s)u(t,s)dsdtdz.\end{aligned}\tag{2.2}$$

By differentiating (2.2) with respect to x, we obtain

$$\begin{aligned}\frac{\partial^{1+j}u(x,y)}{\partial x\partial y^j}&=\partial_1^{(1)}\partial_2^{(j)}f(x,y)+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\frac{\partial^l u(t,y)}{\partial y^l}dt\\&\quad+\int_0^x\int_0^y\partial_2^{(j)}k(x,y,t,s)u(t,s)dsdt.\end{aligned}\tag{2.3}$$

Now, we differentiate Equation (2.3) i-times with respect to x, we obtain

$$\begin{aligned}\frac{\partial^{i+1+j}u(x,y)}{\partial x^{i+1}\partial y^j}=&\partial_1^{(i+1)}\partial_2^{(j)}f(x,y)\\&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(x,y,t,y)\right]\right)\frac{\partial^l u(x,y)}{\partial y^l}\right]\\&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]\frac{\partial^l u(t,y)}{\partial y^l}dt\\&+\int_0^y\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}k(x,y,x,s)u(x,s)\right]ds\\&+\int_0^x\int_0^y\partial_1^{(i)}\partial_2^{(j)}k(x,y,t,s)u(t,s)dsdt,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{\partial^{i+1+j}u(x,y)}{\partial x^{i+1}\partial y^j}=\partial_1^{(i+1)}\partial_2^{(j)}f(x,y)\\&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}u(x,y)}{\partial x^\eta\partial y^l}\\&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]\frac{\partial^l u(t,y)}{\partial y^l}dt\end{aligned}$$

$$+\int_0^y \sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}k(x,y,x,s)\right]\frac{\partial^\eta u(x,s)}{\partial x^\eta}ds$$
$$+\int_0^x\int_0^y \partial_1^{(i)}\partial_2^{(j)}k(x,y,t,s)u(t,s)dsdt.$$

Hence

$$\frac{\partial^{i+j+1}u(0,0)}{\partial x^{i+1}\partial y^j}=\partial_1^{(i+1)}\partial_2^{(j)}f(0,0)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(x,y,t,y)\right]\right)\right]_{x=y=0}\frac{\partial^{l+\eta}u(0,0)}{\partial x^\eta\partial y^l}.$$

Moreover, from (2.2) and (2.3), we deduce that

$$\frac{\partial^{1+j}u(0,0)}{\partial x\partial y^j}=\partial_1^{(1)}\partial_2^{(j)}f(0,0),\ \frac{\partial^{j}u(0,0)}{\partial y^j}=\partial_2^{(j)}f(0,0).$$

- **In the rectangle** $D_{n,0},\ n=1,\dots,N-1$

  We approximate u by the Taylor polynomial

$$v_{n,0}(x,y)=\sum_{i+j=0}^{p}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,0}(x_n,0)}{\partial x^i\partial y^j}(x-x_n)^iy^j\ ;\quad (x,y)\in D_{n,0}, \tag{2.4}$$

where $\hat{v}_{n,0}$ is the exact solution of the integral equation:

$$\begin{aligned}\hat{v}_{n,0}(x,y)=f(x,y)&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_0^y K(z,y,t,s)v_{\sigma,0}(t,s)dsdtdz\\&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_0^y K(z,y,t,s)v_{\xi,0}(t,s)dsdtdz\\&+\sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_0^y K(z,y,t,s)v_{\sigma,0}(t,s)dsdtdz\\&+\int_{x_n}^{x}\int_{x_n}^{z}\int_0^y K(z,y,t,s)\hat{v}_{n,0}(t,s)dsdtdz.\end{aligned} \tag{2.5}$$

To find $\frac{\partial^j\hat{v}_{n,0}(x,y)}{\partial y^j}$, we differentiate Equation (2.5) j-times with respect to y

$$\begin{aligned}\frac{\partial^j\hat{v}_{n,0}(x,y)}{\partial y^j}=\partial_2^{(j)}f(x,y)&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}k(z,y,t,y)v_{\sigma,0}(t,y)]dtdz\\&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_0^y\partial_2^{(j)}k(z,y,t,s)v_{\sigma,0}(t,s)dsdtdz\\&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}k(z,y,t,y)v_{\xi,0}(t,y)]dtdz\\&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_0^y\partial_2^{(j)}k(z,y,t,s)v_{\xi,0}(t,s)dsdtdz\\&+\sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}k(z,y,t,y)v_{\sigma,0}(t,y)]dtdz\\&+\sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_0^y\partial_2^{(j)}k(z,y,t,s)v_{\sigma,0}(t,s)dsdtdz\end{aligned}$$

$$
\begin{aligned}
&+\int_{x_n}^{x}\int_{x_n}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}k(z,y,t,y)\hat{v}_{n,0}(t,y)]dtdz\\
&+\int_{x_n}^{x}\int_{x_n}^{z}\int_{0}^{y}\partial_2^{(j)}k(z,y,t,s)\hat{v}_{n,0}(t,s)dsdtdz\\
=\partial_2^{(j)}f(x,y)&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(z,y,t,y)]\frac{\partial^l v_{\sigma,0}}{\partial y^l}(t,y)dtdz\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{0}^{y}\partial_2^{(j)}k(z,y,t,s)v_{\sigma,0}(t,s)dsdtdz\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(z,y,t,y)]\frac{\partial^l v_{\xi,0}}{\partial y^l}(t,y)dtdz\\
&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{0}^{y}\partial_2^{(j)}k(z,y,t,s)v_{\xi,0}(t,s)dsdtdz\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(z,y,t,y)]\frac{\partial^l v_{\sigma,0}}{\partial y^l}(t,y)dtdz\\
&+\sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{0}^{y}\partial_2^{(j)}k(z,y,t,s)v_{\sigma,0}(t,s)dsdtdz\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\int_{x_n}^{z}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(z,y,t,y)]\frac{\partial^l \hat{v}_{n,0}}{\partial y^l}(t,y)dtdz\\
&+\int_{x_n}^{x}\int_{x_n}^{z}\int_{0}^{y}\partial_2^{(j)}k(z,y,t,s)\hat{v}_{n,0}(t,s)dsdtdz. \qquad (2.6)
\end{aligned}
$$

We differentiate Equation (2.6) with respect to x, we obtain

$$
\begin{aligned}
\frac{\partial^{j+1}\hat{v}_{n,0}(x,y)}{\partial x\partial y^j}=\partial_1^{(1)}\partial_2^{(j)}f(x,y)&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\frac{\partial^l v_{\sigma,0}(t,y)}{\partial y^l}dt\\
&+\sum_{\sigma=0}^{n-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{0}^{y}\partial_2^{(j)}k(x,y,t,s)v_{\sigma,0}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\frac{\partial^l \hat{v}_{n,0}(t,y)}{\partial y^l}dt\\
&+\int_{x_n}^{x}\int_{0}^{y}\partial_2^{(j)}k(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt. \qquad (2.7)
\end{aligned}
$$

Now, we differentiate Equation (2.7) i-times with respect to x, we obtain

$$
\begin{aligned}
\frac{\partial^{i+j+1}\hat{v}_{n,0}(x,y)}{\partial x^{i+1}\partial y^j}=&\partial_1^{(i+1)}\partial_2^{(j)}f(x,y)\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]\frac{\partial^l v_{\sigma,0}(t,y)}{\partial y^l}dt\\
&+\sum_{\sigma=0}^{n-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{0}^{y}\partial_1^{(i)}\partial_2^{(j)}k(x,y,t,s)v_{\sigma,0}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(x,y,t,y)\right]\right)\frac{\partial^l \hat{v}_{n,0}(x,y)}{\partial y^l}\right]\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]\frac{\partial^l \hat{v}_{n,0}(t,y)}{\partial y^l}dt\\
&+\int_{0}^{y}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}k(x,y,x,s)\hat{v}_{n,0}(x,s)\right]ds
\end{aligned}
$$

$$+\int_{x_n}^{x}\int_0^y \partial_1^{(i)}\partial_2^{(j)}k(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt,$$

which implies

$$\begin{aligned}
\frac{\partial^{i+j+1}\hat{v}_{n,0}(x,y)}{\partial x^{i+1}\partial y^j} &= \partial_1^{(i+1)}\partial_2^{(j)}f(x,y)\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]\frac{\partial^l v_{\sigma,0}(t,y)}{\partial y^l}dt\\
&+\sum_{\sigma=0}^{n-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_0^y \partial_1^{(i)}\partial_2^{(j)}k(x,y,t,s)v_{\sigma,0}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}\hat{v}_{n,0}(x,y)}{\partial x^\eta\partial y^l} \qquad (2.8)\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]\frac{\partial^l \hat{v}_{n,0}(t,y)}{\partial y^l}\\
&+\int_0^y\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}k(x,y,x,s)\right]\frac{\partial^\eta\hat{v}_{n,0}(x,s)}{\partial x^\eta}ds\\
&+\int_{x_n}^{x}\int_0^y \partial_1^{(i)}\partial_2^{(j)}k(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial^{i+j+1}\hat{v}_{n,0}(x_n,0)}{\partial x^{i+1}\partial y^j} &= \partial_1^{(i+1)}\partial_2^{(j)}f(x_n,0)\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]_{x=x_n,y=0}\frac{\partial^l v_{\sigma,0}(t,0)}{\partial y^l}dt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(x,y,t,y)\right]\right)\right]_{x=x_n,y=0}\frac{\partial^{l+\eta}\hat{v}_{n,0}}{\partial x^\eta\partial y^l}(x_n,0).
\end{aligned}$$

Moreover, from (2.6), we deduce that

$$\frac{\partial^j\hat{v}_{n,0}(x_n,0)}{\partial y^j} = \partial_2^{(j)}f(x,y)+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}k(z,y,t,y)\right]_{x=x_n,y=0}\frac{\partial^l v_{\sigma,0}}{\partial y^l}(t,0)dtdz,$$

and from (2.7), we get

$$\frac{\partial^{j+1}\hat{v}_{n,0}(x_n,0)}{\partial x^1\partial y^j} = \partial_1^{(i+1)}\partial_2^{(j)}f(x_n,0)+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}k(x,y,t,y)]\right]_{x=x_n,y=0}\frac{\partial^l v_{\sigma,0}(t,0)}{\partial y^l}dt.$$

- **In the rectangle** $D_{n,m},\ n=0,\dots,N-1,\ m=1,\dots,M-1$
  We approximate u by by the Taylor polynomial

$$v_{n,m}(x,y)=\sum_{i+j=0}^{p}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,m}(x_n,y_m)}{\partial x^i\partial y^j}(x-x_n)^i(y-y_m)^j\ ;\quad (x,y)\in D_{n,m}, \qquad (2.9)$$

where $\hat{v}_{n,m}$ is the exact solution of the integral equation:

$$
\begin{aligned}
\hat{v}_{n,m}(x,y) = f(x,y) &+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} K(z,y,t,s)v_{\sigma,\rho}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} K(z,y,t,s)v_{\sigma,m}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{y_\rho}^{y_{\rho+1}} K(z,y,t,s)v_{\xi,\rho}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{y_m}^{y} K(z,y,t,s)v_{\xi,m}(t,s)dsdtdz \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} K(z,y,t,s)v_{\sigma,\rho}(t,s)dsdtdz \\
&+ \sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} K(z,y,t,s)v_{\sigma,m}(t,s)dsdtdz \\
&+ \sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{x_n}^{z}\int_{y_\rho}^{y_{\rho+1}} K(z,y,t,s)v_{n,\rho}(t,s)dsdtdz \\
&+ \int_{x_n}^{x}\int_{x_n}^{z}\int_{y_m}^{y} K(z,y,t,s)\hat{v}_{n,m}(t,s)dsdtdz,
\end{aligned}
\tag{2.10}
$$

for $(x, y) \in D_{n,m}$, $n = 0, ..., N-1$ and $m = 1, ..., M-1$.
To find $\frac{\partial^j \hat{v}_{n,m}(x,y)}{\partial y^j}$, we differentiate equation (2.10) j-times with respect to y

$$
\begin{aligned}
\frac{\partial^j \hat{v}_{n,m}(x,y)}{\partial y^j} = \partial_2^{(j)} f(x,y) &+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,\rho}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)} K(z,y,t,y) v_{\sigma,m}(t,y)]dtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,m}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{\xi,\rho}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)} K(z,y,t,y) v_{\xi,m}(t,y)]dtdz \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) v_{\xi,m}(t,s)dsdtdz \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,\rho}(t,s)dsdtdz \\
&+ \sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)} K(z,y,t,y) v_{\sigma,m}(t,y)]dtdz \\
&+ \sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,m}(t,s)dsdtdz \\
&+ \sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{x_n}^{z}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{n,\rho}(t,s)dsdtdz \\
&+ \int_{x_n}^{x}\int_{x_n}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)} K(z,y,t,y) \hat{v}_{n,m}(t,y)]dtdz \\
&+ \int_{x_n}^{x}\int_{x_n}^{z}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) \hat{v}_{n,m}(t,s)dsdtdz
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&= \partial_2^{(j)} f(x,y) + \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,\rho}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}} \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(z,y,t,y)]\frac{\partial^l v_{\sigma,m}(t,y)}{\partial y^l}dtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,m}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{\xi,\rho}(t,s)dsdtdz \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z} \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(z,y,t,y)]\frac{\partial^l v_{\xi,m}(t,y)}{\partial y^l}dtdz \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\xi}^{z}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) v_{\xi,m}(t,s)dsdtdz \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,\rho}(t,s)dsdtdz \\
&+ \sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}} \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(z,y,t,y)]\frac{\partial^l v_{\sigma,m}(t,y)}{\partial y^l}dtdz \\
&+ \sum_{\sigma=0}^{n-1}\int_{x_n}^{x}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) v_{\sigma,m}(t,s)dsdtdz \\
&+ \sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{x_n}^{z}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(z,y,t,s) v_{n,\rho}(t,s)dsdtdz \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\int_{x_n}^{z} \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(z,y,t,y)]\frac{\partial^l \hat{v}_{n,m}(t,y)}{\partial y^l}dtdz \\
&+ \int_{x_n}^{x}\int_{x_n}^{z}\int_{y_m}^{y} \partial_2^{(j)} K(z,y,t,s) \hat{v}_{n,m}(t,s)dsdtdz.
\end{aligned}
\tag{2.12}
$$

By differentiating the equation with respect to x, we obtain

$$
\begin{aligned}
\frac{\partial^{j+1}\hat{v}_{n,m}(x,y)}{\partial x\partial y^j} &= \partial_1^{(1)}\partial_2^{(j)} f(x,y) + \sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(x,y,t,s) v_{\sigma,\rho}(t,s)dsdt \\
&+ \sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}} \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l v_{\sigma,m}(t,y)}{\partial y^l}dt \\
&+ \sum_{\sigma=0}^{n-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y} \partial_2^{(j)} K(x,y,t,s) v_{\sigma,m}(t,s)dsdt \\
&+ \sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}} \partial_2^{(j)} K(x,y,t,s) v_{n,\rho}(t,s)dsdt \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x} \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l \hat{v}_{n,m}(t,y)}{\partial y^l}dt \\
&+ \int_{x_n}^{x}\int_{y_m}^{y} \partial_2^{(j)} K(x,y,t,s) \hat{v}_{n,m}(t,s)dsdt.
\end{aligned}
\tag{2.13}
$$

Now, we differentiate equation 2.15 i-times with respect to x, we obtain

$$
\begin{aligned}
\frac{\partial^{i+j+1}\hat{v}_{n,m}(x,y)}{\partial x^{i+1}\partial y^{j}} =& \partial_1^{(i+1)}\partial_2^{(j)}f(x,y)+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{\sigma,\rho}(t,s)dsdt\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^i}{\partial x^i}[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l v_{\sigma,m}(t,y)}{\partial y^l}]dt\\
&+\sum_{\sigma=0}^{n-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{\sigma,m}(t,s)dsdt\\
&+\sum_{\rho=0}^{m-1}\int_{y_\rho}^{y_{\rho+1}}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\left(\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)v_{n,\rho}(x,s)\right)\right]ds\\
&+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}[\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{n,\rho}(t,s)]dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\Big|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l\hat{v}_{n,m}(x,y)}{\partial y^l}\right)\right]\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l\hat{v}_{n,m}(t,y)}{\partial y^l}\right)dt\\
&+\int_{y_m}^{y}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\left(\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\hat{v}_{n,m}(x,s)\right)\right]ds\\
&+\int_{x_n}^{x}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,m}(t,s)dsdt\\
=&\partial_1^{(i+1)}\partial_2^{(j)}f(x,y)+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{\sigma,\rho}(t,s)dsdt\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^i}{\partial x^i}[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l v_{\sigma,m}(t,y)}{\partial y^l}]dt\\
&+\sum_{\sigma=0}^{n-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{\sigma,m}(t,s)dsdt\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_\rho}^{y_{\rho+1}}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left(\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\right)\right]\frac{\partial^\eta v_{n,\rho}(x,s)}{\partial x^\eta}ds\\
&+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{n,\rho}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\Big|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right)\right]\frac{\partial^{l+\eta}\hat{v}_{n,m}(x,y)}{\partial x^\eta\partial y^l}ds\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l\hat{v}_{n,m}(t,y)}{\partial y^l}\right)dt\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_m}^{y}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left(\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\right)\right]\frac{\partial^\eta\hat{v}_{n,m}(x,s)}{\partial x^\eta}ds\\
&+\int_{x_n}^{x}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,m}(t,s)dsdt.
\end{aligned}
\tag{2.14}
$$

Hence

$$
\frac{\partial^{i+j+1}\hat{v}_{n,m}(x_n,y_m)}{\partial x^{i+1}\partial y^{j}}=\partial_1^{(i+1)}\partial_2^{(j)}f(x_n,y_m)
$$

$$
\begin{aligned}
&+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}K(x_n,y_m,t,s)v_{\sigma,\rho}(t,s)dsdt\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]_{x=x_n,y=y_m}\frac{\partial^l v_{\sigma,m}(t,y_m)}{\partial y^l}dt\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_\rho}^{y_{\rho+1}}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\partial_2^{(j)}K(x,y,x,s)\right)\right]_{x=x_n,y=y_m}\frac{\partial^\eta v_{n,\rho}(x_n,s)}{\partial x^\eta}ds\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right)\right]_{x=x_n,y=y_m}\frac{\partial^{l+\eta}\hat{v}_{n,m}(x_n,y_m)}{\partial x^\eta\partial y^l}ds.
\end{aligned}
$$

Moreover, from (2.6), we deduce that

$$
\begin{aligned}
\frac{\partial^j\hat{v}_{n,m}(x_n,y_m)}{\partial y^j}=&\partial_2^{(j)}f(x_n,y_m)+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}}[\partial_2^{(j)}K(z,y,t,s)]_{x=x_n,y=y_m}v_{\sigma,\rho}(t,s)dsdtdz\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(z,y,t,y)]_{x=x_n,y=y_m}\frac{\partial^l v_{\sigma,m}(t,y_m)}{\partial y^l}dtdz,
\end{aligned}
$$

and from (2.7), we get

$$
\begin{aligned}
\frac{\partial^{j+1}\hat{v}_{n,m}(x_n,y_m)}{\partial x\partial y^j}=&\partial_1^{(1)}\partial_2^{(j)}f(x_n,y_m)+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\sigma}^{x_{\sigma+1}}\int_{y_\rho}^{y_{\rho+1}}[\partial_2^{(j)}K(x,y,t,s)]_{x=x_n,y=y_m}v_{\sigma,\rho}(t,s)dsdt\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\sigma}^{x_{\sigma+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]_{x=x_n,y=y_m}\frac{\partial^l v_{\sigma,m}(t,y_m)}{\partial y^l}dt.
\end{aligned}
\tag{2.15}
$$

3. Study of convergence and error of the numerical method

The following lemmas will be used in proving the convergence of the presented method. The Lemmas 3.2-Lemma 3.5 are the generalized Gronwall inequality and its discrete forms. To facilitate our proof, a technical lemma is given in Lemma 3.6.

Lemma 3.1 (*Taylor's Theorem for functions of two independent variables [8]*) *consider ω a p times continuously differentiable on $D=[a,b]\times[c,d]$ and let $(x_0,y_0)\in D$. Then for all $(x,y)\in D$, we have*

$$\omega(x,y)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\omega(x_0,y_0)}{\partial x^i\partial y^j}(x-x_0)^i(y-y_0)^j+\sum_{i+j=p}\frac{1}{i!j!}\frac{\partial^{i+j}\omega(x_1,y_1)}{\partial x^i\partial y^j}(x-x_0)^i(y-y_0)^j,$$

such that

$$\begin{cases} x_1=\varepsilon x+(1-\varepsilon)x_0\in[a,b],\\ y_1=\varepsilon y+(1-\varepsilon)y_0\in[c,d],\end{cases}\quad \varepsilon\in(0,1),$$

Lemma 3.2 *[15] (Gronwall-type inequality) Let β, α_i, $i=1,2,3$ a non negative constants, and ω be a bounded, integrable, and non-negative function satisfies*

$$\omega(x,y)\leq\beta+\alpha_1\int_0^x\omega(u,y)du+\alpha_2\int_0^y\omega(x,v)dv+\alpha_3\int_0^x\int_0^y\omega(u,v)dudv,$$

for $(x, y) \in [0, b] \times [0, d]$, *then* ω *satisfies*

$$\omega(x, y) \leq \beta e^{\zeta(x+y)},$$

where ζ *is defined by*

$$\zeta = \frac{1}{2}\left(\alpha_1 + \alpha_2 + \sqrt{(\alpha_1 + \alpha_2)^2 + 4\alpha_3}\right).$$

Lemma 3.3 (*Generalization of Lemma 3.2*) *Let* β, α_i, $i = 1, 2, 3$ *a non negative constants, and* ω *be a bounded, integrable, and nonnegative function defined on* $D = [a, b] \times [c, d]$ *satisfies*

$$\omega(x, y) \leq \beta + \alpha_1 \int_a^x \omega(u, y)du + \alpha_2 \int_c^y \omega(x, v)dv + \alpha_3 \int_a^x \int_c^y \omega(u, v)dudv, \quad (x, y) \in D, \tag{3.1}$$

then, we have

$$\omega(x, y) \leq \beta e^{\zeta(x+y)},$$

where ζ *is defined by*

$$\zeta = \frac{1}{2}\left(\alpha_1 + \alpha_2 + \sqrt{(\alpha_1 + \alpha_2)^2 + 4\alpha_3}\right).$$

Proof. By applying a similar argument to that employed in [15], from (3.1) there exists a bounded and non positive function G such that

$$\omega(x, y) = \beta + \alpha_1 \int_a^x \omega(u, y)du + \alpha_2 \int_c^y \omega(x, v)dv + \alpha_3 \int_a^x \int_c^y \omega(u, v)dudv + G(x, y), \quad (x, y) \in D, \tag{3.2}$$

we define the iterated sequence, for all $n = 0, 1, 2, \ldots$

$$\omega_{n+1}(x, y) = \alpha_1 \int_a^x \omega_n(u, y)du + \alpha_2 \int_c^y \omega_n(x, v)dv + \alpha_3 \int_a^x \int_c^y \omega_n(u, v)dudv + \beta + G(x, y), \tag{3.3}$$

$$\omega_0(x, y) = 0, \tag{3.4}$$

Using similar argument used in Section 2 in [15], the sequence (ω_n) converges and $\lim_{n\to\infty} \omega_n(x, y)$ is the solution of (3.2). We want to prove that:

$$\omega_n(x, y) \leq \beta e^{\zeta(x+y)}, \qquad for\ (x, y) \in D,\ n = 0, 1, 2, \ldots, \tag{3.5}$$

where ζ is defined by

$$\zeta = \frac{1}{2}\left(\alpha_1 + \alpha_2 + \sqrt{(\alpha_1 + \alpha_2)^2 + 4\alpha_3}\right). \tag{3.6}$$

By recurrence, Equation (3.5) is verified for $n = 0$. Assume that Equation (3.5) holds for N. It follows from the assumptions (3.3) and (3.4) that

$$\begin{aligned}
\omega_{N+1}(x, y) \leq & \alpha_1 \int_a^x \omega_N(u, y)du + \alpha_2 \int_c^y \omega_N(x, v)dv + \alpha_3 \int_a^x \int_c^y \omega_N(u, v)dudv + \beta \\
& \leq \beta\alpha_1 \int_a^x e^{\zeta(u+y)}du + \beta\alpha_2 \int_c^y e^{\zeta(x+v)}dv + \beta\alpha_3 \int_a^x \int_c^y e^{\zeta(u+v)}dudv + \beta \\
& = \beta\left(\frac{\alpha_1}{\zeta} + \frac{\alpha_2}{\zeta} + \frac{\alpha_3}{\zeta^2}\right) e^{\zeta(x+y)} + \beta\frac{\alpha_3}{\zeta^2} e^{\zeta(a+c)} + \beta \\
& - \beta\left(\frac{\alpha_1}{\zeta} e^{\zeta(a+y)} + \frac{\alpha_2}{\zeta} e^{\zeta(x+c)} + \frac{\alpha_3}{\zeta^2} e^{\zeta(x+c)} + \frac{\alpha_3}{\zeta^2} e^{\zeta(a+y)}\right).
\end{aligned}$$

By using the relation $\frac{\alpha_1}{\zeta}+\frac{\alpha_2}{\zeta}+\frac{\alpha_3}{\zeta^2}=1$, we get

$$\begin{aligned}\omega_{N+1}(x,y) &\leq \beta e^{\zeta(x+y)}+\beta\frac{\alpha_3}{\zeta^2}e^{\zeta(a+c)}+\beta-\beta\left(\frac{\alpha_1}{\zeta}+\frac{\alpha_2}{\zeta}+\frac{\alpha_3}{\zeta^2}\right)-\beta\frac{\alpha_3}{\zeta^2}e^{\zeta(a+c)}\\ &=\beta e^{\zeta(x+y)}+\beta-\beta\left(\frac{\alpha_1}{\zeta}+\frac{\alpha_2}{\zeta}+\frac{\alpha_3}{\zeta^2}\right)\\ &=\beta e^{\zeta(x+y)}.\end{aligned}$$

This completes the proof. Letting $n\to\infty$ gives the generalized gronwall-type inequality

$$\omega(x,y)\leq \beta e^{\zeta(x+y)}. \tag{3.7}$$

Lemma 3.4 *[17] (Discrete Gronwall-type inequality in one dimension) Let $\{\alpha_j\}_{j=0}^{n}$ be a given non-negative sequence and the sequence $\{\omega_n\}$ satisfies $\omega_0\leq\beta$ and*

$$\omega_n\leq\beta+\sum_{i=0}^{n-1}\alpha_i\omega_i,\quad n\geq 1,$$

with $\beta\geq 0$. Then ω_n can be bounded by

$$\omega_n\leq\beta\exp\left(\sum_{j=0}^{n-1}\alpha_j\right),\quad n\geq 1.$$

Lemma 3.5 *[15] (The discrete Gronwall inequality in two dimension) Consider the sequence $\omega_{nm}\geq 0$ ($n=0,1,...,N$; $m=0,1,...,M$) satisfying*

$$\omega_{nm}\leq\beta+h\alpha_1\sum_{i=0}^{n-1}\omega_{im}+k\alpha_2\sum_{j=0}^{m-1}\omega_{nj}+hk\alpha_3\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\omega_{ij},$$

such that $h=\frac{T}{N}$ and $k=\frac{S}{M}$, where $T,S,\alpha_i\,(i=1,2,3)$ and β are finite strictly positive constants independent of N and M. Then

$$\omega_{nm}\leq\beta e^{\zeta(T+S)},$$

where

$$\zeta=\frac{1}{2}\left(\alpha_1+\alpha_2+\sqrt{(\alpha_1+\alpha_2)^2+4\alpha_3}\right).$$

We consider the space $L^\infty(D)$ with the norm

$$\|\varphi\|_{L^\infty(D)}=\inf\{C\in\mathbb{R}:|\varphi(x,y)|\leq C\ \text{ for all. } (x,y)\in D\}<\infty.$$

Lemma 3.6 *Let g and K be p times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(p)$ such that for all $n=0,\dots,N-1$, $m=0,\dots,M-1$ and $i+j=0,1,...,p$, we have,*

$$\left\|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial x^i\partial y^j}\right\|_{L^\infty(D_{n,m})}\leq\alpha(p),$$

where $\hat{v}_{0,0}(x,y)=u(x,y)$ for $(x,y)\in D_{0,0}$.

Proof. Let $a_{n,m}^{i,j}=\|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial x^i\partial y^j}\|_{L^\infty(D_{n,m})}$, we have for all $i+j=0,1,...,p$,

$$a_{0,0}^{i,j}\leq\max\left\{\left\|\frac{\partial^{i+j}u}{\partial x^i\partial y^j}\right\|_{L^\infty(D_{0,0})},i+j=0,1,...,p\right\}=\alpha_1(p). \tag{3.8}$$

Now, from (2.8), we have for all $n = 1, \ldots, N-1$ and $i + j = 0, 1, \ldots, p$

$$a_{n,0}^{i+1,j} \le c_1 + c_1 h \sum_{\xi=0}^{n-1} \sum_{q+l=0}^{p} a_{\xi,0}^{q,l} + c_1 hk \sum_{\xi=0}^{n-1} \sum_{q+l=0}^{p} a_{\xi,0}^{q,l} + c_1 \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} a_{n,0}^{\eta,l}$$
$$+ c_1 h \sum_{r=0}^{j-1} \sum_{l=0}^{r} a_{n,0}^{0,l} + c_1 k \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} a_{n,0}^{\eta,0} + c_1 hk a_{n,0}^{0,0},$$

where the constant c_1 is positive and independent of N and M, Hence

$$a_{n,0}^{i+1,j} \le c_1 + c_2 h \sum_{\xi=0}^{n-1} \sum_{q+l=0}^{p} a_{\xi,0}^{q,l} + c_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,l} + c_2 h \sum_{l=0}^{j-1} a_{n,0}^{0,l} + c_2 k \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,0} + c_1 hk a_{n,0}^{0,0}. \tag{3.9}$$

Consider, the sequence $\Gamma_n = \max\{a_{n,0}^{i,j}, i + j = 0, \ldots, p\}$, $n = 0, 1, \ldots, N-1$, then by (3.9), the sequence Γ_n satisfies

$$a_{n,0}^{i+1,j} \le c_1 + c_2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,l} + c_2 h \sum_{l=0}^{j-1} a_{n,0}^{0,l} + c_2 k \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,0} + c_1 hk a_{n,0}^{0,0}$$
$$\le c_1 + c_2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} a_{n,0}^{\eta,l} + c_2 h \sum_{l=0}^{j-1} a_{n,0}^{0,l} + c_2 k \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,0} + c_1 hk a_{n,0}^{0,0}.$$

Which implies for all $i = 1, \ldots, p$ and $j = 0, \ldots, p$,

$$a_{n,0}^{i,j} \le c_1 + c_2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,l} + c_2 h \sum_{l=0}^{j-1} a_{n,0}^{0,l} + c_2 k \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,0} + c_1 hk a_{n,0}^{0,0}. \tag{3.10}$$

Moreover, from (2.6), we obtain for all $j = 0, \ldots, p$ and $n = 0, \ldots, N-1$,

$$a_{n,0}^{0,j} \le c_1 + c_3 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_3 \sum_{l=0}^{j-1} a_{n,0}^{0,l} + c_3 hk a_{n,0}^{0,0}. \tag{3.11}$$

On the other hand, from Equation (2.5), we obtain for all $n = 0, \ldots, N-1$,

$$|\hat{v}_{n,0}(x,y)| \le c_1 + c_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_4 \int_{x_n}^{x} \int_0^y |\hat{v}_{n,0}(t,s)| ds dt.$$

Hence by Lemma 3.3, we obtain for all $n = 0, \ldots, N-1$,

$$\begin{aligned} a_{n,0}^{0,0} &\le \left(c_1 + c_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) e^{c_4(a+b)} \\ &\le c_1 e^{c_5(a+b)} + c_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi e^{c_4(a+b)} \\ &\le c_5 + c_5 h \sum_{\xi=0}^{n-1} \Gamma_\xi. \end{aligned} \tag{3.12}$$

From (3.10), (3.11) and (3.12), we deduce that for all $i, j = 0, \ldots, p$ and $n = 0, \ldots, N-1$,

$$a_{n,0}^{i,j} \le c_6 + c_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_6 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,l} + c_6 \sum_{l=0}^{j-1} a_{n,0}^{0,l} + c_6 \sum_{\eta=0}^{i-1} a_{n,0}^{\eta,0}, \tag{3.13}$$

where the constant c_6 is positive and independent of N and M.
Using the notations of Lemma 3.5, we put

$$\omega_{ij} = a^{i,j}_{n,0}, \beta = c_6 + c_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi, \alpha_1 = \alpha_2 = pc_6, \alpha_3 = p^2 c_6, T = S = 1.$$

Then, by Lemma 3.5, we obtain from (3.13)

$$a^{i,j}_{n,0} \leq \left(c_6 + c_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) e^{2p(c_6 + \sqrt{c_6 + c_6^2})}. \tag{3.14}$$

Which implies that,

$$\Gamma_n \leq c_7 + c_7 h \sum_{\xi=0}^{n-1} \Gamma_\xi. \tag{3.15}$$

It follows, by Lemma 3.4, for all $n = 0, 1, ..., N-1$

$$\Gamma_n \leq c_7 e^{ac_7}. \tag{3.16}$$

On the other hand, we have, from (2.14), for all $n = 0, ..., N-1$, $m = 1, ..., N-1$ and $i + j = 0, ..., p$

$$\begin{aligned} a^{i+1,j}_{n,m} \leq\ & b_1 + b_1 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a^{s,t}_{\xi,\rho} + b_1 h \sum_{\xi=0}^{n-1} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{s+t=0}^{p-1} a^{s,t}_{\xi,m} \\ & + b_1 hk \sum_{\xi=0}^{n-1} \sum_{s+t=0}^{p-1} a^{s,t}_{\xi,m} + b_1 k \sum_{\rho=0}^{m-1} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \sum_{s+t=0}^{p-1} a^{s,t}_{n,\rho} \\ & + kb_1 h \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a^{s,t}_{n,\rho}(t,s)dsdt + b_1 \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} a^{\eta,l}_{n,m} \\ & + b_1 h \sum_{r=0}^{j-1} \sum_{l=0}^{r} a^{0,l}_{n,m} + b_1 k \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} a^{\eta,0}_{n,m} + kb_1 h a^{0,0}_{n,m}. \end{aligned} \tag{3.17}$$

Consider, the sequence $\Gamma_{n,m} = \max\{a^{i,j}_{n,m}, i + j = 0, \dots, p\}$, $n = 0, 1, ..., N-1; m = 0, ..., M-1$, then by (3.17), we have

$$\begin{aligned} a^{i+1,j}_{n,m} \leq\ & b_1 + b_1 p^2 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_1 p^4 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ & + b_1 p^2 hk \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_1 p^4 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \\ & + hkb_1 p^2 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_1 p^2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a^{\eta,l}_{n,m} \\ & + b_1 ph \sum_{l=0}^{j-1} a^{0,l}_{n,m} + b_1 pk \sum_{\eta=0}^{i-1} a^{\eta,0}_{n,m} + hkb_1 a^{0,0}_{n,m}. \end{aligned}$$

We obtain

$$\begin{aligned} a^{i+1,j}_{n,m} \leq & b_1 + b_2 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_2 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_2 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a^{\eta,l}_{n,m} \\ & + b_2 h \sum_{l=0}^{j-1} a^{0,l}_{n,m} + b_2 k \sum_{\eta=0}^{i-1} a^{\eta,0}_{n,m} + hkb_2 a^{0,0}_{n,m}. \end{aligned} \tag{3.18}$$

which implies for all $i = 1, \dots, p$ and $j = 0, \dots, p$,

$$\begin{aligned} a^{i,j}_{n,m} \le & b_1 + b_2 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_2 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_2 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_2 \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} a^{\eta,l}_{n,m} \\ & + b_2 h \sum_{l=0}^{j-1} a^{0,l}_{n,m} + b_2 k \sum_{\eta=0}^{i-1} a^{\eta,0}_{n,m} + hkb_2 a^{0,0}_{n,m}. \end{aligned} \tag{3.19}$$

Moreover, from (2.12), we obtain for all $j = 0, \dots, p$ and $n = 0, \dots, N-1$ and $m = 0, \dots, M-1$,

$$a^{0,j}_{n,m} \le b_3 + b_3 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_3 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_3 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_3 \sum_{l=0}^{j-1} a^{0,l}_{n,m} + b_3 a^{0,0}_{n,m}. \tag{3.20}$$

Also, from equation (2.10), we obtain for all $n = 0, \dots, N-1$ and $m = 0, \dots, M-1$

$$|\hat{v}_{n,m}(x,y)| \le b_4 + b_4 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_4 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_4 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_4 \int_{x_n}^{x}\int_{y_m}^{y} |\hat{v}_{n,m}(t,s)| ds dt.$$

Hence by Lemma 3.3, we obtain for all $n = 0, ..., N-1$ and $m = 0, ..., M-1$

$$\begin{aligned} a^{0,0}_{n,m} & \le \left(b_4 + b_4 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_4 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_4 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) e^{(a+b)b_4} \\ & \le b_5 + b_5 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_5 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_5 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}. \end{aligned} \tag{3.21}$$

which implies from (3.19),(3.20) and (3.21) that, for all $i = 0, ..., p$ and $j = 0, ..., p$

$$a^{i,j}_{n,m} \le b_6 + b_6 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_6 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_6 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_7 \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} a^{\eta,l}_{n,m} + b_7 \sum_{l=0}^{j-1} a^{0,l}_{n,m} + b_7 \sum_{\eta=0}^{i-1} a^{\eta,0}_{n,m}. \tag{3.22}$$

Using the notations of Lemma 3.5, we put

$$\omega_{ij} = a^{i,j}_{n,m}, \beta = b_6 + b_6 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_6 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_6 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \alpha_1 = \alpha_2 = pb_7, \alpha_3 = p^2 b_7, T = S = 1.$$

Then, by Lemma 3.5, we obtain from (3.19)

$$a^{i,j}_{n,m} \le \left(b_6 + b_6 hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_6 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_6 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) e^{2p(b_7 + \sqrt{b_7 + b_7^2})}.$$

It follows that, for all $n = 0, 1, ..., N-1$; $m = 0, ..., M-1$,

$$\Gamma_{n,m} \le b_8 + hkb_8 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + hb_8 \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \tag{3.23}$$

by using Lemma (3.5), we obtain

$$\Gamma_{n,m} \le b_8 e^{(a+b)p(b_8 + \sqrt{b_8 + b_8^2})}. \tag{3.24}$$

Hence from (3.8), (3.16) and (3.24) the proof of Lemma 3.6 is completed by setting

$$\alpha(p) = max\{\alpha_1(p), c_7 e^{ac_7}, b_8 e^{(a+b)p(b_8 + \sqrt{b_8 + b_8^2})}\}.$$

The following theorem gives the convergence of the presented method.

Theorem 3.7 *Let g and k be p times continuously differentiable on their respective domains. Then equations (2.1),...,(2.9) define a unique approximation $v \in S^{(-1)}_{p-1,p-1}(\Pi_{N,M})$, and the resulting error function $e(x,y) = u(x,y) - v(x,y)$ satisfies:*

$$\|e\|_{L^\infty(D)} \leq C(h+k)^p,$$

where C is a finite constant independent of h and k.

Proof. Define the error $e(x,y)$ on $D_{n,m}$ by $e_{n,m}(x,y) = u(x,y) - v_{n,m}(x,y)$ for all $n \in \{0,\dots,N\}$ and $m \in \{0,\dots,M\}$.
Claim 1. There exist a constant C_1 independent of h and k such that $\|e_{0,0}\|_{L^\infty(D_{0,0})} \leq C_1(h+k)^p$. Let $(x,y) \in D_{0,0}$, by using Lemma 3.1, we obtain from (2.1)

$$|e_{0,0}(x,y)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\| h^i k^j.$$

Hence by Lemma 3.6, we have

$$|e_{0,0}(x,y)| \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \underbrace{\frac{\alpha(p)}{p!}}_{=C_1}(h+k)^p. \tag{3.25}$$

Claim 2. There exist a constant C_2 independent of h and k such that $\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq C_2(h+k)^p$ for all $n = 1,\dots,N-1$. Let $(x,y) \in D_{n,0}$, we have from (2.5)

$$\begin{aligned}
u(x,y) - \hat{v}_{n,0}(x,y) = & \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1} \int_{x_\xi}^{x_{\xi+1}} \int_{x_\sigma}^{x_{\sigma+1}} \int_0^y K(z,y,t,s) e_{\sigma,0}(t,s) dsdtdz \\
& + \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_{x_\xi}^{z} \int_0^y K(z,y,t,s) e_{\xi,0}(t,s) dsdtdz \\
& + \sum_{\sigma=0}^{n-1} \int_{x_n}^{x} \int_{x_\sigma}^{x_{\sigma+1}} \int_0^y K(z,y,t,s) e_{\sigma,0}(t,s) dsdtdz \\
& + \int_{x_n}^{x} \int_{x_n}^{z} \int_0^y K(z,y,t,s)(u(t,s) - \hat{v}_{n,0}(t,s)) dsdtdz.
\end{aligned}$$

Hence

$$|u(x,y) - \hat{v}_{n,0}(x,y)| \leq \sum_{\xi=0}^{n-1} 3ahk\overline{K} \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \overline{K}a \int_{x_n}^{x} \int_0^y |u(t,s) - \hat{v}_{n,0}(t,s)| dsdt.$$

Then by Lemma 3.3,

$$\begin{aligned}
|u(x,y) - \hat{v}_{n,0}(x,y)| & \leq \sum_{\xi=0}^{n-1} hk3a\overline{K} \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} \exp\left(\overline{K}a(a+b)\right) \\
& \leq \sum_{\xi=0}^{n-1} 3ahb\overline{K} \exp\left(\overline{K}a(a+b)\right) \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} \\
& \leq \sum_{\xi=0}^{n-1} hd_1 \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})}
\end{aligned}$$

which implies, by using Lemma 3.1, that

$$\begin{aligned}
\|e_{n,0}\|_{L^\infty(D_{n,0})} & \leq \|u - \hat{v}_{n,0}\| + \|\hat{v}_{n,0} - v_{n,0}\| \\
& \leq \sum_{\xi=0}^{n-1} hd_1 \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j} \hat{v}_{n,0}}{\partial x^i \partial y^j} \right\| h^i k^j,
\end{aligned}$$

Hence by Lemma 3.6, we obtain

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq \sum_{\xi=0}^{n-1} hd_1 \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \frac{\alpha(p)}{p!}(h+k)^p,$$

then, by Lemma 3.4, we have

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq \frac{\alpha(p)}{p!}(h+k)^p \exp(ad_1),$$

Thus, we take $C_2 = \frac{\alpha(p)}{p!}\exp(ad_1)$.
Claim 3. There exist a constant C_3 independent of h and k such that $\|e_{n,m}\|_{L^\infty(D_{n,m})} \leq C_3(h+k)^p$ for all $n = 0, \ldots, N-1$ and $m = 1, \ldots, M-1$. Let $(x,y) \in D_{n,m}$, we have from (2.10)

$$\begin{aligned}|u(x,y) - \hat{v}_{n,m}(x,y)| \leq & \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hka\overline{K}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hka\overline{K}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hka\overline{K}\|e_{n,\rho}\| \\ & + \overline{K}a\int_{x_n}^{x}\int_{y_m}^{y}|u(t,s) - \hat{v}_{n,m}(t,s)|dsdt,\end{aligned}$$

Then by Lemma 3.3,

$$\begin{aligned}|u(x,y) - \hat{v}_{n,m}(x,y)| &\leq \left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hka\overline{K}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hka\overline{K}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hka\overline{K}\|e_{n,\rho}\|\right)\exp(a\overline{K}(a+b)) \\ &\leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} d_2hk\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkd_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\|\end{aligned}$$

which implies, by using Lemma 3.1, that

$$\begin{aligned}\|e_{n,m}\|_{L^\infty(D_{n,0})} \leq & \|u - \hat{v}_{n,m}\| + \|\hat{v}_{n,m} - v_{n,m}\| \\ \leq & \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkd_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| \\ & + \sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial x^i \partial y^j}\right\| h^i k^j,\end{aligned}$$

Hence, by Lemma 3.6, we obtain

$$\|e_{n,m}\| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkd_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p. \tag{3.26}$$

Then, by Lemma 3.3, we obtain from (3.26)

$$\begin{aligned}\|e_{n,m}\| &\leq \left(\frac{\alpha(p)}{p!}(h+k)^p\right)\exp(d_3(Nh+Mk)) \\ &\leq \underbrace{\frac{\alpha(p)}{p!}\exp(d_3(a+b))}_{=C_3}(h+k)^p\end{aligned} \tag{3.27}$$

Thus, the proof is completed by taking $C = max(C_1, C_2, C_3)$.

4. Numerical Examples

In order to test the efficiency of the suggested Taylor collocation method for solving linear partial Volterra integro-differential equations (PVIDE), three numerical examples for some linear PVIDEs with known exact solutions are given. In the following examples, we set $p = 3$ with three different values of N and M. The error estimation is given to show the accuracy of approximation.

Example 4.1 *Let us dedicate the first example to the case that the desired equation is of form*

$$\frac{\partial u}{\partial x}(x,y) = g(x,y) + \int_0^x \int_0^y (t+s^2)u(t,s)dsdt, \qquad 0 \le x \le 1, \ 0 \le y \le 1, \tag{4.1}$$

with the initial condition

$$u_0(y) = sin(y), \qquad 0 \le y \le 1. \tag{4.2}$$

The exact solution of this problem is given by $u(x,y) = (1-x^2)sin(y)$, *Then, the function* $g(x,y)$ *is calculated using the exact solution as follows:*

$$\begin{aligned} g(x,y) = &\frac{1}{4}x^4(1-cos(y)) + \frac{2}{3}x^3(cos(y) - \frac{1}{2}y^2cos(y) + ysin(y) - 1) + \frac{1}{2}x^2(cos(y)-1) \\ &+ 2x(\frac{1}{2}y^2cos(y) - ysin(y) + sin(y) - cos(y) + 1). \end{aligned} \tag{4.3}$$

Numerical results were thus calculated by applying the proposed Taylor collocation method for different collocation points. The absolute value error is reported in Table 1. It shows that the absolute errors are decreasing by increasing the number of collocation points. In order to illustrate the behavior of the approximate solution, the absolute error function have been depicted in Figure 1 as three dimensional graphs for $0 \le x \le 1$ *and* $0 \le y \le 1$ *using two different values of N,M,* $(N,M) = (10,10)$ *and* $(N,M) = (30,30)$ *for comparison purpose.*

Table 1. Numerical results for Example 4.1

(x,y)	$N=M=10$	$N=M=20$	$N=M=30$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$9.5482e-07$	$5.8051e-07$	$4.0984e-07$
$(0.2, 0.2)$	$2.0284e-05$	$1.0937-05$	$7.4685e-06$
$(0.3, 0.3)$	$1.1512e-04$	$6.0072e-05$	$4.0605e-05$
$(0.4, 0.4)$	$3.8832e-04$	$1.9952e-04$	$1.3420e-04$
$(0.5, 0.5)$	$9.8326e-04$	$5.0065e-04$	$3.3578e-04$
$(0.6, 0.6)$	$2.0681e-03$	$1.0467e-03$	$7.0068e-04$
$(0.7, 0.7)$	$3.8101e-03$	$1.9201e-03$	$1.2835e-03$
$(0.8, 0.8)$	$6.3403e-03$	$3.1852e-03$	$2.1270e-03$
$(0.9, 0.9)$	$9.7160e-03$	$4.8698e-03$	$3.6839e-03$

Example 4.2 *Consider the following linear two-dimensional PVIDE*

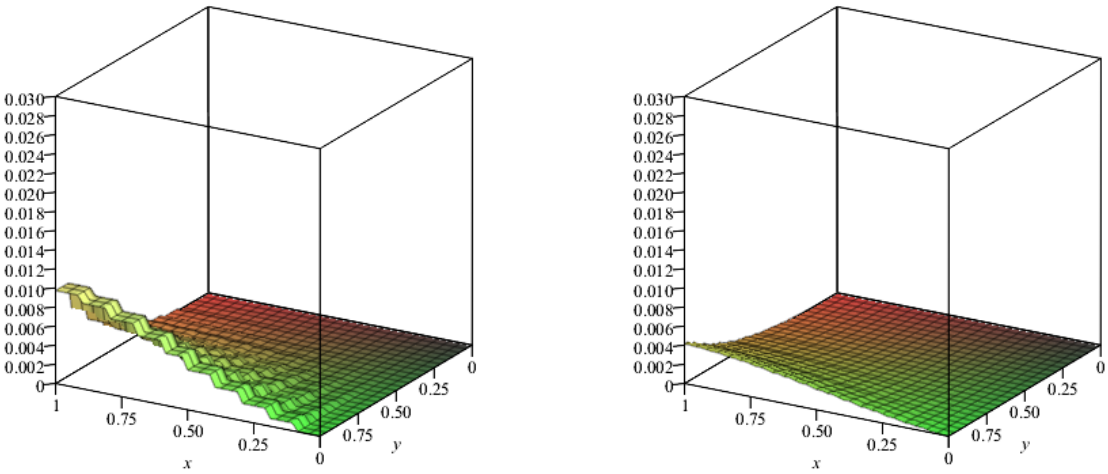
$$\frac{\partial u}{\partial x}(x,y) = g(x,y) + \int_0^x \int_0^y (tcos(s))u(t,s)dsdt, \qquad 0 \le x \le 1, \ 0 \le y \le 1, \tag{4.4}$$

with the initial condition

$$u_0(y) = y, \qquad 0 \le y \le 1. \tag{4.5}$$

where the analytic solution is given by $u(x,y) = ye^{-x}$, *and*

$$g(x,y) = e^{-x}(xysin(y) + xcos(y) + ysin(y) + cos(y) - y - x - 1) - ysin(y) - cos(y) + 1 \tag{4.6}$$



(a) The absolute error function for $N = M = 10$.

(b) The absolute error function for $N = M = 30$.

Figure 1. Plot of the absolute error function for $p = 3$ for Example 4.1

Table 2 shows the absolute error values, and Figure 2 (a) illustrates the behaviors of the exact solution, while Figure 2 (b) depicts the approximate solution of the Example 4.2. The graph of approximate solution is plotted using $N = M = 20$ collocation points and it is in complete agreement with the graph of exact solution.

Table 2. Numerical results for Example 4.2

(x, y)	$N = M = 10$	$N = M = 20$	$N = M = 30$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$7.8981e-07$	$4.8780e-07$	$3.4580e-07$
$(0.2, 0.2)$	$1.4646e-05$	$7.9986e-06$	$5.4816e-06$
$(0.3, 0.3)$	$7.3101e-05$	$3.8588e-05$	$2.6178e-05$
$(0.4, 0.4)$	$2.1930e-04$	$1.1396e-04$	$7.6933e-05$
$(0.5, 0.5)$	$4.9957e-04$	$2.5728e-04$	$1.7318e-04$
$(0.6, 0.6)$	$9.5623e-04$	$4.8967e-04$	$3.2902e-04$
$(0.7, 0.7)$	$1.6220e-03$	$8.2745e-04$	$5.5531e-04$
$(0.8, 0.8)$	$2.5158e-03$	$1.2801e-03$	$9.0808e-04$
$(0.9, 0.9)$	$3.6403e-03$	$1.8490e-03$	$1.0290e-03$

Example 4.3 *The last example is dedicated to equation*

$$\frac{\partial u}{\partial x}(x, y) = g(x, y) + \int_0^x \int_0^y (xt + cos(s))u(t, s)dsdt, \qquad 0 \leq x \leq 1,\ 0 \leq y \leq 1, \tag{4.7}$$

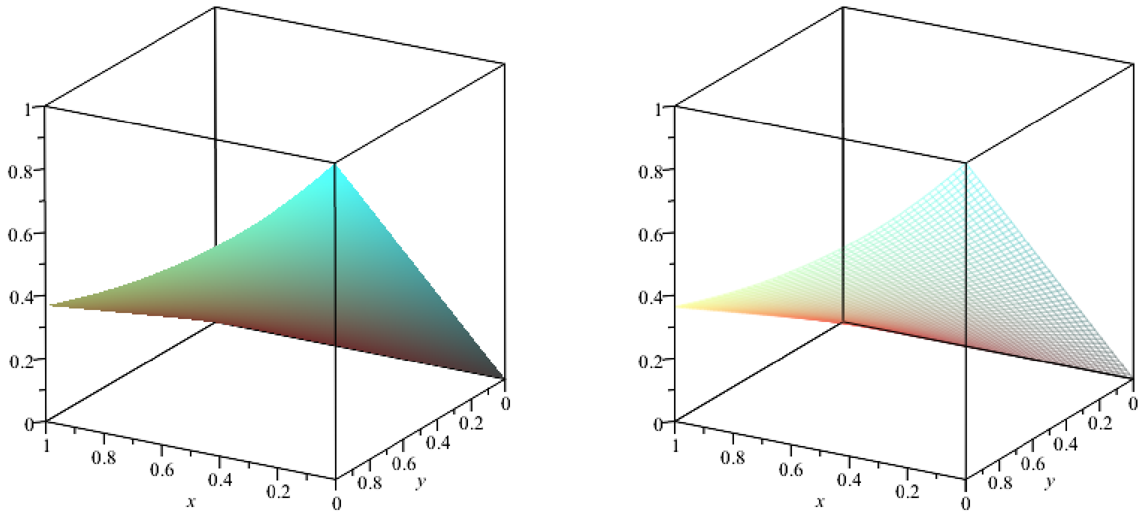
subject to initial condition

$$u_0(y) = 0, \qquad 0 \leq y \leq 1. \tag{4.8}$$

And $u(x, y) = x\sin(y),$. Then, $g(x, y)$ is given by

$$g(x, y) = sin(y) + \frac{x^4}{3}(cos(y) - 1) + \frac{x^2}{4}sin^2(y). \tag{4.9}$$

Numerical results in terms of absolute errors are reported in Table 4.3. Once again an excellent performance of the proposed method is observed from this table. The absolute error function is depicted in 3-dimensional space as shown in Figure 3.



(a) The exact solution.

(b) The approximate solution for $N = M = 20$.

Figure 2. Plot of approximate and exact solution for Example 4.2

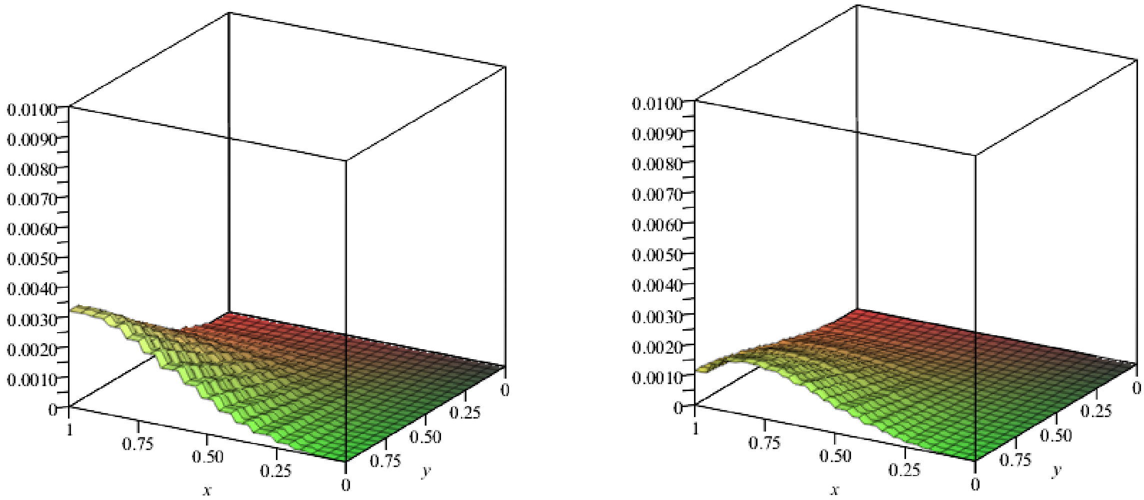
Table 3. Numerical results for Example 4.3

(x, y)	$N = M = 10$	$N = M = 20$	$N = M = 30$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$8.30769e-07$	$5.21368e-07$	$3.7061e-07$
$(0.2, 0.2)$	$1.648471e-05$	$9.17145e-06$	$6.2693e-06$
$(0.3, 0.3)$	$8.810838e-05$	$4.73639e-05$	$3.1640e-05$
$(0.4, 0.4)$	$2.838251e-04$	$1.47792e-04$	$9.5703e-05$
$(0.5, 0.5)$	$6.966309e-04$	$3.44822e-04$	$1.6742e-04$
$(0.6, 0.6)$	$1.4414516e-03$	$6.55191e-04$	$2.35141e-04$
$(0.7, 0.7)$	$2.6493896e-03$	$1.04630e-03$	$8.14990e-04$
$(0.8, 0.8)$	$4.4561125e-03$	$1.39008e-03$	$1.2109e-03$
$(0.9, 0.9)$	$6.9807065e-03$	$1.40116e-03$	$1.30225e-03$

The numerical experiments indicate that the Taylor collocation method is an efficient approximation tool for solving linear PVIDEs, which guarantee our convergence investigation in Section 3. We also find that the error decreases and approaches zero when M, N and p increase.

5. Conclusion

A new algorithm is proposed in this paper to solve two-dimensional linear partial Volterra integro-differential equations of the form (1.1) using a Collocation Method based on Taylor polynomials. The convergence and error analysis are investigated, and several test examples are used to observe the efficiency of the method, in which the obtained results are very accurate in comparison with the exact solution for certain well selected examples.



(a) The absolute error function for $N = M = 15$.

(b) The absolute error function for $N = M = 20$.

Figure 3. Plot of absolute error function for $p = 3$ for Example 4.3

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