Low Mach number limit of nonisentropic inviscid Hookean elastodynamics

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Abstract

The low Mach number limit of the nonisentropic compressible Hookean elastodynamic equations is rigorously proved with respect to well-prepared initial data. We introduce certain suitable seminorms to obtain the uniform estimate of solutions, for which the critical point is to cancel the higher order derivate terms caused by the coupling of velocity and deformation gradient.

Key Words: Low Mach number limit; nonisentropic; elastodynamics

1 Introduction

1.1 The model

We consider the equations of the nonisentropic compressible inviscid flows in elastodynamics(see [2])

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0,\\
&\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \nabla \cdot \boldsymbol{T},\\
&\partial_t E + \nabla \cdot (E u) = \nabla \cdot (\boldsymbol{T} u),\\
&\partial_t \boldsymbol{F} + u \cdot \nabla \boldsymbol{F} = \nabla u \boldsymbol{F},
\end{aligned} \tag{1.1}$$

where ρ denotes the density, $u \in \mathbb{R}^d (d = 2, 3)$ is the velocity, $\boldsymbol{T}$ is the Cauchy stress, F_j is the j-th column of the deformation gradient $\boldsymbol{F} = (F_{ij}) \in \mathbb{M}^{d\times d}$, and E is the total energy. For the neo-Hookean elastodynamics, the total energy takes the form

$$E = \rho \tilde{E} + \frac{1}{2}\rho |u|^2, \quad \tilde{E} = \sum_{i,j=1}^{d} \frac{1}{2} F_{ij}^2 + e(\rho, S),$$

where $e(\rho, S)$ is the thermodynamic energy and S is the entropy. The Cauchy stress $\boldsymbol{T}$ and the pressure p are given by

$$\boldsymbol{T} = \rho \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} - p I_d, \quad p = p(\rho, S) = \rho^2 \frac{\partial e(\rho, S)}{\partial \rho}, \quad \rho = \rho(p, S),$$

where I_d is the $d \times d$ identity matrix, $p_\rho > 0$ for $\rho > 0$ and $\rho_p > 0$ for $p > 0$. We also impose the divergence constraints on the deformation gradient

$$\nabla \cdot (\rho F_j) = 0, \quad j = 1, \cdots, d. \tag{1.2}$$

This will not make the system (1.1) be over-determined because we only require it holds for the initial data and it automatically propagates to any time (cf. [25, Proposition 2.1]). Considering the physical explanation of the incompressible limit, we introduce the dimensionless parameter ε, the Mach number, and make the following changes of variables:

$$p(x,t) = p^\varepsilon(x, \varepsilon t), \quad S(x,t) = S^\varepsilon(x, \varepsilon t), \quad u(x,t) = \varepsilon u^\varepsilon(x, \varepsilon t), \quad F_j(x,t) = \varepsilon F_j^\varepsilon(x, \varepsilon t).$$

The system (1.1) can be transformed into the following equations:

$$\begin{aligned}
&a(S^\varepsilon, p^\varepsilon)\left(\partial_t p^\varepsilon + u^\varepsilon \cdot \nabla p^\varepsilon\right) + \nabla \cdot u^\varepsilon = 0, \\
&\rho(S^\varepsilon, p^\varepsilon)\left(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon\right) + \varepsilon^{-2} \nabla p^\varepsilon = \rho(S^\varepsilon, p^\varepsilon) \sum_{j=1}^d F_j^\varepsilon \cdot \nabla F_j^\varepsilon, \\
&\partial_t F_j^\varepsilon + u^\varepsilon \cdot \nabla F_j^\varepsilon = F_j^\varepsilon \cdot \nabla u^\varepsilon, \quad j = 1, \cdots, d, \\
&\partial_t S^\varepsilon + u^\varepsilon \cdot \nabla S^\varepsilon = 0,
\end{aligned} \tag{1.3}$$

where $a = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$. Using the transformation $p = 1 + \varepsilon q$ yields

$$\begin{aligned}
&a(S^\varepsilon, \varepsilon q^\varepsilon)\left(\partial_t q^\varepsilon + u^\varepsilon \cdot \nabla q^\varepsilon\right) + \frac{1}{\varepsilon} \nabla \cdot u^\varepsilon = 0, \\
&\rho(S^\varepsilon, \varepsilon q^\varepsilon)\left(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon\right) + \frac{1}{\varepsilon} \nabla q^\varepsilon = \rho(S^\varepsilon, \varepsilon q^\varepsilon) \sum_{j=1}^d F_j^\varepsilon \cdot \nabla F_j^\varepsilon, \\
&\partial_t F_j^\varepsilon + u^\varepsilon \cdot \nabla F_j^\varepsilon = F_j^\varepsilon \cdot \nabla u^\varepsilon, \quad j = 1, \cdots, d, \\
&\partial_t S^\varepsilon + u^\varepsilon \cdot \nabla S^\varepsilon = 0.
\end{aligned} \tag{1.4}$$

Obviously, if we set $V^\varepsilon = (q^\varepsilon, u^\varepsilon, S^\varepsilon, F_1^\varepsilon, \cdots, F_d^\varepsilon)$, we can reformulate (1.4) into the following compact form

$$A_0(V^\varepsilon, \varepsilon V^\varepsilon)\partial_t V^\varepsilon + \sum_{k=1}^d A_k(V^\varepsilon, \varepsilon V^\varepsilon)\partial_k V^\varepsilon + \frac{1}{\varepsilon}\mathscr{L}(\partial_x) V^\varepsilon = 0, \tag{1.5}$$

where $A_0 = \mathrm{diag}(a, \rho I_d, 1, \rho I_{d^2})$,

$$\mathscr{L}(\partial_x) = \begin{pmatrix} 0 & \nabla \cdot & 0 \\ \nabla & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_k = \begin{pmatrix} a u_k^\varepsilon & 0 & 0 & 0 & \cdots & 0 \\ 0 & \rho u_k^\varepsilon I_d & 0 & -\rho F_{k1}^\varepsilon I_d & \cdots & -\rho F_{kd}^\varepsilon I_d \\ 0 & 0 & u_k^\varepsilon & 0 & \cdots & 0 \\ 0 & -\rho F_{k1}^\varepsilon I_d & 0 & & & \\ \vdots & \vdots & \vdots & & B_k & \\ 0 & -\rho F_{kd}^\varepsilon I_d & 0 & & & \end{pmatrix},$$

and $B_k = \rho u_k^\varepsilon I_{d^2}$.

1.2 Previous results

More discussions on the physical background and equations of elastodynamics can be found in the book of Dafermos [2]. The mathematical theory of both compressible and incompressible elastodynamics have been studied extensively in the last few decades, for example, [1, 3, 4, 6, 8, 12, 13, 14, 19, 24] and references therein. In this paper, we will focus mainly on the low Mach number limit of (1.5).

For the inviscid flows in elastodynamics, Schochet [20] rigorously justified the incompressible limit in nonlinear elasticity for local solution in the whole space by applying the energy methods of Klainerman and Majda [10, 11] originating in the study of the equations of fluid dynamics, and Sideris and Thomases [23] proved the limit for global small smooth solutions. Recently, Liu and Xu [16] studied the incompressible limit in a bounded domain for well-prepared initial data, and Zhang [27] proved the local well-posedness and incompressible limit of the free-boundary problem in the compressible elastodynamic equations.

However, to our best knowledge, there are no results on the low Mach number limit to the nonisentropic elastodynamic equations. We supplement the system (1.4) with initial data

$$(S^\varepsilon, q^\varepsilon, u^\varepsilon, F_1^\varepsilon, \cdots, F_d^\varepsilon)\big|_{t=0} = (S_0^\varepsilon, q_0^\varepsilon, u_0^\varepsilon, F_{1,0}^\varepsilon, \cdots, F_{d,0}^\varepsilon). \tag{1.6}$$

We shall study the low Mach number limit of nonisentropic equations (1.4) with (1.6) in the torus $\mathbb{T}^d$ and the whole space $\mathbb{R}^d$.

1.3 Notations

We shall give some notations used throughout the present paper. We denote by $C(\cdot)$ the continuous nondecreasing functions on $[0,\infty)$, which may vary from line to line. $L^p(\Omega)$ $(1 \le p < \infty)$ denotes the space of measurable fuctions whose p-powers are integrable with the norm $|\cdot|_p$, and $L^\infty(\Omega)$ is the space of bounded measurable functions with the norm $|\cdot|_\infty$. We also donote $|\cdot|_2$ by $\|\cdot\|_0$. We denote by $\langle\cdot,\cdot\rangle_0$ the standard inner product in $L^2(\Omega)$ with norm $\|u\|_0^2 = \langle u, u\rangle_0$, and by H^k the usual Sobolev space $W^{k,2}(\Omega)$ with norm $\|\cdot\|_k$. The notation $\|(A_1, \cdots, A_k)\|_0$ means the summation of $\|A_i\|_0$ $(i = 1, \cdots, k)$, and it also applies to other norms. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$, we define $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$. We also simply denote ∂_{x_j} by ∂_j. We use the symbols K to denote generic positive constants. In this paper, we consider Ω is the torus $\mathbb{T}^d$ or the whole space $\mathbb{R}^d$.

Before starting our main results, we introduce the following weighted norms:

Definition 1.1 *Given $\mu \in \mathbb{N}$, we set*

$$X^\mu([0,T]\times\Omega) := \bigcap_{k=0}^{\mu} C^k([0,T], H^{\mu-k}(\Omega)).$$

We define the weighted norms

$$\|w(t)\|_{\mu,\varepsilon}^2 := \|w(t)\|_\mu^2 + \sum_{k=1}^{\mu} \left\|\varepsilon^{k-1}\partial_t^k w(t)\right\|_{\mu-k}^2,$$

$$\|w(t)\|_{\mu,\varepsilon,T} := \sup_{t\in[0,T]} \|w(t)\|_{\mu,\varepsilon}.$$

Next, we introduce seminorms $\|\cdot\|_{E_1}$ and $\|\cdot\|_{E_2}$.

Definition 1.2 *For $V=(S,W)$ and $W=(q,u,F_1,\cdots,F_d)$,*

$$\|V\|_{E_1}^2 := \sum_{k=1}^{s} \left\|\varepsilon^{k-1}\partial_t^k W\right\|_0^2 + \|W\|_s^2 + \|S\|_{s,\varepsilon}^2, \tag{1.7}$$

$$\begin{aligned}\|V\|_{E_2} := \sum_{k=1}^{s-1} \Bigg(& \|\varepsilon^{k-1}\partial_t^k \nabla q\|_{s-k-1} + \|\varepsilon^{k-1}\partial_t^k \nabla\cdot u\|_{s-k-1} + \|\varepsilon^{k-1}\partial_t^k \nabla\times u\|_{s-k-1} \\ & + \sum_{\ell=1}^{d} \|\varepsilon^{k-1}\partial_t^k \nabla F_\ell\|_{s-k-1}\Bigg).\end{aligned} \tag{1.8}$$

One can easily check that $\|V\|_{E_1}+\|V\|_{E_2}$ is equivalent to $\|V\|_{s,\varepsilon}$.

1.4 Main results

We first state the uniform exitence of the local solutions with respect to ε.

Theorem 1.1 (Uniform existence) *Let $s>d/2+1$ be an integer. Suppose that the initial data V_0^ε are well-prepared, i.e.,*

$$\|V_0^\varepsilon\|_s + \frac{1}{\varepsilon}\left(\|\nabla q_0^\varepsilon\|_{s-1} + \|\nabla\cdot u_0^\varepsilon\|_{s-1}\right) \le M_0, \tag{1.9}$$

for some constant $M_0>0$. Then there exist constants $T>0$ and $\varepsilon_0\in(0,1)$ such that for any $\varepsilon\in(0,\varepsilon_0]$ the initial value problem (1.4) and (1.6) has a unique solution $V^\varepsilon\in C([0,T],H^s(\Omega))$, and there exists a positive constant N, depending only on T, ε_0 and M_0, such that

$$\|V^\varepsilon\|_{s,\varepsilon,T} \le N. \tag{1.10}$$

Remark 1.1 *The well-prepared initial data condition (1.9) yields that $|\partial_t V^\varepsilon(t=0)|_\infty$ has a uniform bound independent of ε through (1.4), which implies there is no oscillation produced by the initial data.*

Theorem 1.2 (Low Mach number limit) *Let $\Omega=\mathbb{T}^d$. Suppose that the assumption in Theorem 1.1 holds with $s>d/2+1$. We assume further that the initial data*

$$V_0^\varepsilon \to V_0^0 = (S_0^0,q_0^0,u_0^0,F_{1,0}^0,\cdots,F_{d,0}^0) \text{ in } H^s(\mathbb{T}^d) \text{ as } \varepsilon\to 0.$$

Then there is $V^0=(S^0,q^0,u^0,F_1^0,\cdots,F_d^0)\in C([0,T];H^{s'}(\mathbb{T}^d))$ for any $s'<s$, such that the solution of (1.4), (1.6) satisfies

$$\begin{aligned}&V^\varepsilon \rightharpoonup V^0 \text{ weakly-* in } L^\infty([0,T];H^s(\mathbb{T}^d)),\\ &V^\varepsilon \to V^0 \text{ strongly in } C([0,T];H^{s'}(\mathbb{T}^d)) \ \forall s'<s,\end{aligned}$$

as $\varepsilon \to 0$. Moreover, $(S^0, u^0, F_1^0, \cdots, F_d^0) \in C([0,T]; H^s(\mathbb{T}^d))$ solves the following incompressible nonisentropic Hookean elastodynamic equations:

$$
\begin{aligned}
&\nabla \cdot u^0 = 0, \\
&\rho(S^0, 0)\left(\partial_t u^0 + u^0 \cdot \nabla u^0\right) + \nabla \pi = \rho(S^0, 0) \sum_{j=1}^{d} F_j^0 \cdot \nabla F_j^0, \\
&\partial_t F_j^0 + u^0 \cdot \nabla F_j^0 = F_j^0 \cdot \nabla u^0, \quad j = 1, \cdots, d, \\
&\nabla \cdot (\rho(S^0, 0) F_j^0) = 0, \quad j = 1, \cdots, d, \\
&\partial_t S^0 + u^0 \cdot \nabla S^0 = 0,
\end{aligned}
\tag{1.11}
$$

with the initial data $(S^0, u^0, F_1^0, \cdots, F_d^0)|_{t=0} = (S_0^0, u_0^0, F_{1,0}^0, \cdots, F_{d,0}^0)$. Here $\pi \in C([0,T]; H^s(\mathbb{T}^d))$.

Remark 1.2 *If $\Omega = \mathbb{R}^d$, we can obtain the same convergence result as Theorem 1.2. The only difference is the solutions converge strongly in $C([0,T]; H_{loc}^{s'}(\mathbb{R}^d))$ rather than $C([0,T]; H^{s'}(\mathbb{R}^d))$.*

Remark 1.3 *Setting $\bar{\rho} = \rho(S^0, 0)$, we can rewrite (1.11) as the following incompressible inhomogeneous Hookean elastodynamic equations*

$$
\begin{aligned}
&\nabla \cdot u^0 = 0, \\
&\partial_t \bar{\rho} + u^0 \cdot \nabla \bar{\rho} = 0, \\
&\bar{\rho}\left(\partial_t u^0 + u^0 \cdot \nabla u^0\right) + \nabla \pi = \bar{\rho} \sum_{j=1}^{d} F_j^0 \cdot \nabla F_j^0, \\
&\partial_t F_j^0 + u^0 \cdot \nabla F_j^0 = F_j^0 \cdot \nabla u^0, \quad j = 1, \cdots, d, \\
&\nabla \cdot (\bar{\rho} F_j^0) = 0, \quad j = 1, \cdots, d.
\end{aligned}
$$

Recalling the system (1.5), we note that the matrix A_0 depends on $(S^\varepsilon, \varepsilon q^\varepsilon)$. As Métivier and Schochet stated in [18], the reason why the low Mach number limit is more difficult to analyse in the nonisentropic case is that the matrix multiplying the time derivatives then depends strongly on the dependent variables. More precisely, for the isentropic case, when establishing the energy estimates for solutions and their derivatives, we can factor out ε in $\partial^\alpha A_0$ to balance the weighted norms because A_0 depends only on $\varepsilon V^\varepsilon$(see [16, 22] for example), which does not work for the nonisentropic case. To surmount this difficulty, they developed an quasi-linearization method. For the nonisentropic Hookean elastodynamic equations, because of the strong intertwist between velocity and deformation gradient, one order higher spatial derivatives arise for the deformation gradient when we use the quasi-linearization method developed by [18]. Based on the above analysis, we do not expect to control these troublesome "terms of highest order derivatives". Hence, we should find new relationships to cancel these troublesome terms.

In Theorem 1.1 we obtain the uniform existence of smooth solutions to the nonisentropic compressible Hookean elastodynamic equations in $\mathbb{T}^d$ and $\mathbb{R}^d$. Inspired by [21] we introduce suitable seminorms, and similar applications can also refer to [9, 15, 16, 22]. The key point inspired by [15] in the derivation of this estimate is to cancel the higher order derivate terms caused by the coupling of velocity and deformation gradient. Once the uniform estimate (1.10) is established, we can show the convergence of the solutions by using the Aubin-Lions Lemma and obtain the limitting equations.

This paper is arranged as follows. The uniform existence of solutions(Theorem 1.1) is proved in Section 2; the low Mach number limit(Theorem 1.2) is proved in Section 3. We recall some basic results in the appendix.

2 Uniform existence

Throughout this section, $(S^\varepsilon, q^\varepsilon, u^\varepsilon, F_1^\varepsilon, \cdots, F_d^\varepsilon)$ will be denoted by $(S, q, u, F_1, \cdots, F_d)$, and the corresponding superscript ε used in other notations is omitted for simplicity of presentation.

In view of the classical local existence result for the symmetric hyperbolic system in [17], we have

Theorem 2.1 (see [17]) *For fixed $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for initial data satisfying (1.9), the initial value problem (1.5) and (1.6) has a unique solution on $C([0, T_\varepsilon); H^s(\Omega))$.*

The key point in the proof of Theorem 1.1 is to establish the uniform estimate (1.10).

Remark 2.1 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2 + 1$. According to the local existence theory, for all $t \in [0, T_\varepsilon)$, there exists a constant $m > 1$ such that*

$$m^{-1} \le a, \rho \le m. \tag{2.1}$$

Lemma 2.1 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2 + 1$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|S\|_{s,\varepsilon}^2 \le C(\|V\|_{s,\varepsilon}). \tag{2.2}$$

Proof. Taking $\partial_t^k \partial^\alpha$ $(0 \le k + |\alpha| \le s)$ of $(1.4)_4$, we have

$$\partial_t^{k+1} \partial^\alpha S + u \cdot \nabla \partial_t^k \partial^\alpha S = \mathscr{C}_S, \tag{2.3}$$

where

$$\mathscr{C}_S = -[\partial_t^k \partial^\alpha, u \cdot \nabla] S.$$

In the case of $k = 0$ we have

$$(\partial_t + u \cdot \nabla) \partial^\alpha S = \mathscr{C}_S = -[\partial^\alpha, u \cdot \nabla] S. \tag{2.4}$$

The force term $\mathscr{C}_S$ is a sum of terms $\partial^\beta u \partial^\gamma S$ with $|\beta| + |\gamma| \le s + 1$, $\beta > 0$ and $\gamma > 0$. By using the estimate (4.1) in appendix we get

$$\|\mathscr{C}_S\|_0 \le C(\|V\|_{s,\varepsilon}).$$

The usual L^2 energy estimate for the transport equation (2.4) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\partial^\alpha S\|_0^2 \le C(\|V\|_{s,\varepsilon}).$$

Taking summation with respect to α yields (2.2).

In the case of $k > 0$, multiplying (2.3) by $\varepsilon^{2(k-1)} \partial_t^k \partial^\alpha S$, integrating the result over Ω and using integration by parts give

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\varepsilon^{k-1} \partial_t^k \partial^\alpha S|^2 \mathrm{d}x \le K |\nabla \cdot u|_\infty \|\varepsilon^{k-1} \partial_t^k \partial^\alpha S\|_0^2 + \|\varepsilon^{k-1} \mathscr{C}_S\|_0 \|\varepsilon^{k-1} \partial_t^k \partial^\alpha S\|_0,$$

where

$$
\begin{aligned}
\|\varepsilon^{k-1}\mathscr{C}_S\|_0 \le& K\varepsilon^{k-1}\sum_{j=1}^{d}\sum_{p=1}^{k}\sum_{|\beta|+|\gamma|\le|\alpha|}\|\partial_t^p\partial^\beta u_j\partial^{k-p}\partial^\gamma\partial_j S\|_0\\
\le& K\varepsilon\sum_{j=1}^{d}\sum_{p=1}^{k}\sum_{|\beta|+|\gamma|\le|\alpha|}\|\varepsilon^{p-1}\partial_t^p\partial^\beta u_j\|_{s-p-|\beta|}\|\varepsilon^{k-p-1}\partial^{k-p}\partial^\gamma\partial_j S\|_{s-(k-p-1)-|\gamma|-1}\\
\le& C(\|V\|_{s,\varepsilon}).
\end{aligned}
$$

We put the estimates together and take summation with respect to k and α to yield (2.2). □

Lemma 2.2 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2+1$ with $s > d/2+1$, we have*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla q\|_{s-1}^2+\|\nabla\cdot u\|_{s-1}^2+\|\nabla\times u\|_{s-1}^2+\sum_{\ell=1}^{d}\left(\|\nabla\cdot F_\ell\|_{s-1}^2+\|\nabla\times F_\ell\|_{s-1}^2\right)\right)\le C(\|V\|_{s,\varepsilon}). \tag{2.5}
$$

Proof. Applying the operator ∇ to $(1.4)_1$, applying the operator $\nabla\cdot$ to $(1.4)_2$ and $(1.4)_3$, and applying the operator $\nabla\times$ to $(1.4)_2$ and $(1.4)_3$, we can get the equations of

$$
\mathscr{U} := (\nabla q, \nabla\cdot u, \nabla\times u, \nabla\cdot F_1,\cdots,\nabla\cdot F_d,\nabla\times F_1,\cdots,\nabla\times F_d)
$$

as follows

$$
\begin{aligned}
&a(\partial_t+u\cdot\nabla)\nabla q+\frac{1}{\varepsilon}\nabla(\nabla\cdot u)=-[\nabla,a(\partial_t+u\cdot\nabla)]q,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\cdot u+\frac{1}{\varepsilon}\Delta q=\sum_{\ell=1}^{d}\nabla\cdot(\rho F_\ell\cdot\nabla F_\ell)-[\nabla\cdot,\rho(\partial_t+u\cdot\nabla)]u,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\times u=\sum_{\ell=1}^{d}\nabla\times(\rho F_\ell\cdot\nabla F_\ell)-[\nabla\times,\rho(\partial_t+u\cdot\nabla)]u,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\cdot F_\ell=\nabla\cdot(\rho F_\ell\cdot\nabla u)-[\nabla\cdot,\rho(\partial_t+u\cdot\nabla)]F_\ell,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\times F_\ell=\nabla\times(\rho F_\ell\cdot\nabla u)-[\nabla\times,\rho(\partial_t+u\cdot\nabla)]F_\ell.
\end{aligned} \tag{2.6}
$$

For $|\alpha|\le s-1$, we take ∂^α of (2.6) to obtain that

$$
\begin{aligned}
&a(\partial_t+u\cdot\nabla)\nabla\partial^\alpha q+\frac{1}{\varepsilon}\nabla(\nabla\cdot\partial^\alpha u)=\mathscr{C}_1,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\cdot\partial^\alpha u+\frac{1}{\varepsilon}\Delta\partial^\alpha q=\sum_{\ell=1}^{d}\nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)+\mathscr{C}_2,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\times\partial^\alpha u=\sum_{\ell=1}^{d}\nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)+\mathscr{C}_3,\\
&\rho(\partial_t+u\cdot\nabla)\nabla\cdot\partial^\alpha F_\ell=\nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha u)+\mathscr{C}_{4\ell},\\
&\rho(\partial_t+u\cdot\nabla)\nabla\times\partial^\alpha F_\ell=\nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha u)+\mathscr{C}_{5\ell}.
\end{aligned} \tag{2.7}
$$

where the commutators $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_{4\ell}$ and $\mathscr{C}_{5\ell}$ are given by

$$\begin{aligned}
\mathscr{C}_1 =& -[\partial^\alpha\nabla, a(\partial_t + u\cdot\nabla)]q,\\
\mathscr{C}_2 =& -[\partial^\alpha\nabla\cdot, \rho(\partial_t + u\cdot\nabla)]u + \sum_{\ell=1}^d[\partial^\alpha, \nabla\cdot(\rho F_\ell\cdot\nabla)]F_\ell,\\
\mathscr{C}_3 =& -[\partial^\alpha\nabla\times, \rho(\partial_t + u\cdot\nabla)]u + \sum_{\ell=1}^d[\partial^\alpha, \nabla\times(\rho F_\ell\cdot\nabla)]F_\ell,\\
\mathscr{C}_{4\ell} =&[\partial^\alpha, \nabla\cdot(\rho F_\ell\cdot\nabla)]u,\\
\mathscr{C}_{5\ell} =&[\partial^\alpha, \nabla\times(\rho F_\ell\cdot\nabla)]u.
\end{aligned}$$

Multiplying (2.7) by $\partial^\alpha\mathscr{U}$, integrating the result over Ω and using integration by parts give

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a|\nabla\partial^\alpha q|^2 + \rho\left(|\nabla\cdot\partial^\alpha u|^2 + |\nabla\times\partial^\alpha u|^2 + \sum_{\ell=1}^d(|\nabla\cdot\partial^\alpha F_\ell|^2 + |\nabla\times\partial^\alpha F_\ell|^2)\right)\mathrm{d}x\\
=&\frac{1}{2}\int_\Omega \partial_t a|\nabla\partial^\alpha q|^2 + \partial_t\rho\left(|\nabla\cdot\partial^\alpha u|^2 + |\nabla\times\partial^\alpha u|^2 + \sum_{\ell=1}^d(|\nabla\cdot\partial^\alpha F_\ell|^2 + |\nabla\times\partial^\alpha F_\ell|^2)\right)\mathrm{d}x\\
&+\frac{1}{2}\int_\Omega \nabla\cdot(au)|\nabla\partial^\alpha q|^2\\
&+\nabla\cdot(\rho u)\left(|\nabla\cdot\partial^\alpha u|^2 + |\nabla\times\partial^\alpha u|^2 + \sum_{\ell=1}^d(|\nabla\cdot\partial^\alpha F_\ell|^2 + |\nabla\times\partial^\alpha F_\ell|^2)\right)\mathrm{d}x\\
&-\frac{1}{\varepsilon}\left(\langle\nabla\partial^\alpha q, \nabla(\nabla\cdot\partial^\alpha u)\rangle_0 + \langle\nabla\cdot\partial^\alpha u, \Delta\partial^\alpha q\rangle_0\right)\\
&+\sum_{\ell=1}^d\Bigg(\langle\nabla\cdot\partial^\alpha u, \nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0 + \langle\nabla\times\partial^\alpha u, \nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0\\
&+\langle\nabla\cdot\partial^\alpha F_\ell, \nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0 + \langle\nabla\times\partial^\alpha F_\ell, \nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0\Bigg)\\
&+\langle\nabla\partial^\alpha q, \mathscr{C}_1\rangle_0 + \langle\nabla\cdot\partial^\alpha u, \mathscr{C}_2\rangle_0 + \langle\nabla\times\partial^\alpha u, \mathscr{C}_3\rangle_0\\
&+\sum_{\ell=1}^d\left(\langle\nabla\cdot\partial^\alpha F_\ell, \mathscr{C}_{4\ell}\rangle_0 + \langle\nabla\times\partial^\alpha F_\ell, \mathscr{C}_{5\ell}\rangle_0\right).
\end{aligned}\tag{2.8}$$

Here the singular terms with the factor $\frac{1}{\varepsilon}$ are cancelled. We claim that all $|\alpha|+2$ order derivative terms in (2.8) cancel each other out. The key is to simplify the following:

$$\begin{aligned}
&\langle\nabla\cdot\partial^\alpha u, \nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0 + \langle\nabla\times\partial^\alpha u, \nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0\\
&+\langle\nabla\cdot\partial^\alpha F_\ell, \nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0 + \langle\nabla\times\partial^\alpha F_\ell, \nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0.
\end{aligned}\tag{2.9}$$

Recall the vector analysis formula

$$\nabla\times\nabla\times v = \nabla\nabla\cdot v - \Delta v.$$

For (2.9), using integration by parts, Lemma 4.6 in the appendix and above formulas, we see that

$$\begin{aligned}
&\langle\nabla\cdot\partial^\alpha u,\nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0+\langle\nabla\times\partial^\alpha u,\nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0\\
&\quad+\langle\nabla\cdot\partial^\alpha F_\ell,\nabla\cdot(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0+\langle\nabla\times\partial^\alpha F_\ell,\nabla\times(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0\\
=&\langle-\nabla\nabla\cdot\partial^\alpha u,\rho F_\ell\cdot\nabla\partial^\alpha F_\ell\rangle_0+\langle\nabla\times\nabla\times\partial^\alpha u,\rho F_\ell\cdot\nabla\partial^\alpha F_\ell\rangle_0\\
&\quad+\langle-\nabla\nabla\cdot\partial^\alpha F_\ell,\rho F_\ell\cdot\nabla\partial^\alpha u\rangle_0+\langle\nabla\times\nabla\times\partial^\alpha F_\ell,\rho F_\ell\cdot\nabla\partial^\alpha u\rangle_0\\
=&\langle-\Delta\partial^\alpha u,\rho F_\ell\cdot\nabla\partial^\alpha F_\ell\rangle_0+\langle-\Delta\partial^\alpha F_\ell,\rho F_\ell\cdot\nabla\partial^\alpha u\rangle_0\\
=&\langle\nabla\partial^\alpha u,\nabla(\rho F_\ell\cdot\nabla\partial^\alpha F_\ell)\rangle_0+\langle\nabla\partial^\alpha F_\ell,\nabla(\rho F_\ell\cdot\nabla\partial^\alpha u)\rangle_0\\
=&\int_\Omega\partial_j\partial^\alpha u_i\partial_j(\rho F_{k\ell}\partial_k\partial^\alpha F_{i\ell})+\partial_j\partial^\alpha F_{i\ell}\partial_j(\rho F_{k\ell}\partial_k\partial^\alpha u_i)\mathrm{d}x\\
=&\int_\Omega\rho\left(\partial_j\partial^\alpha u_iF_{k\ell}\partial_{jk}\partial^\alpha F_{i\ell}+\partial_j\partial^\alpha F_{i\ell}F_{k\ell}\partial_{jk}\partial^\alpha u_i\right)+C(\partial^{|\alpha|+1}V)\mathrm{d}x\\
\le&C(\|V\|_s).
\end{aligned}$$

Based on the above analysis, we get from (2.8) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a|\nabla\partial^\alpha q|^2+\rho\left(|\nabla\cdot\partial^\alpha u|^2+|\nabla\times\partial^\alpha u|^2+\sum_{\ell=1}^d(|\nabla\cdot\partial^\alpha F_\ell|^2+|\nabla\times\partial^\alpha F_\ell|^2)\right)\mathrm{d}x\\
\le&K|(\partial_ta,\partial_t\rho,\nabla\cdot(au),\nabla\cdot(\rho u))|_\infty\|\partial^\alpha\mathscr{U}\|^2\\
&+\|\nabla\partial^\alpha q\|_0\|\mathscr{C}_1\|_0+\|\nabla\cdot\partial^\alpha u\|_0\|\mathscr{C}_2\|_0+\|\nabla\times\partial^\alpha u\|_0\|\mathscr{C}_3\|_0\\
&+\sum_{\ell=1}^d\left(\|\nabla\cdot\partial^\alpha F_\ell\|_0\|\mathscr{C}_{4\ell}\|_0+\|\nabla\times\partial^\alpha F_\ell\|_0\|\mathscr{C}_{5\ell}\|_0\right). \qquad (2.10)
\end{aligned}$$

Next, we shall get the estimates of the commutators. On account of their forms, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_{4\ell}$ and $\mathscr{C}_{5\ell}$, we study the general comutators

$$\begin{aligned}
&\mathrm{Com}(G_1)\eta_1:=[\partial^\sigma,G_1\partial_t]\eta_1, && |\sigma|\le s,\\
&\mathrm{Com}(G_2)\eta_2:=[\partial^\sigma,G_2\partial_j]\eta_2, && |\sigma|\le s,\ j=1,\cdots,d,\\
&\mathrm{Com}(G_1)\eta_3:=[\partial^\alpha,\partial^\zeta(G_3\partial_j)]\eta_3, && |\alpha|\le s-1,\ |\zeta|=1,\ j=1,\cdots,d,
\end{aligned}$$

where

$$G_1\in\{a,\rho\},\quad G_2=\rho u_j,\quad G_3=\rho F_{j\ell},\quad \eta_1,\eta_2\in\{q,u\},\quad \eta_3\in\{u,F_\ell\}.$$

We use Moser-type inequalities Lemma 4.4 in the appendix to give

$$\begin{aligned}
&\|\mathrm{Com}(G_1)\eta_1\|_0\le K\left(|\nabla G_1|_\infty\|\partial^{s-1}\partial_t\eta_1\|_0+|\partial_t\eta_1|_\infty\|\partial^sG_1\|_0\right)\le C(\|V\|_{s,\varepsilon}),\\
&\|\mathrm{Com}(G_2)\eta_2\|_0\le K\left(|\nabla G_2|_\infty\|\partial^{s-1}\partial_j\eta_2\|_0+|\partial_j\eta_2|_\infty\|\partial^sG_2\|_0\right)\le C(\|V\|_{s,\varepsilon}),
\end{aligned}$$

and by some direct calculation we have

$$\|\mathrm{Com}(G_3)\eta_3\|_0\le K\sum_{|\beta|\le s,|\gamma|\le s}\|\partial^\beta G_3\partial^\gamma\eta_3\|_0\le K\|G_3\|_s\|\eta_3\|_s\le C(\|V\|_{s,\varepsilon}).$$

These get the estimates of $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_{4\ell}$ and $\mathscr{C}_{5\ell}$ as

$$\|\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_{4\ell}, \mathscr{C}_{5\ell}\|_0 \leq C(\|V\|_{s,\varepsilon}).$$

Since a and ρ are smooth functions of $(S, \varepsilon q)$, using Sobolev's inequality, we have

$$\begin{aligned} |(\partial_t a, \partial_t \rho)|_\infty \leq C(|(q, S, \partial_t q, \partial_t S)|_\infty) \leq & C(\|V\|_{s,\varepsilon}), \\ |(\nabla \cdot (au), \nabla \cdot (\rho u))|_\infty \leq & C(\|V\|_{s,\varepsilon}). \end{aligned}$$

Recalling the definition of $\partial^\alpha \mathscr{U}$ and one can see that

$$\|\partial^\alpha \mathscr{U}\|_0^2 \leq K \|V\|_s^2.$$

Thus, the first term on the right-hand side of (2.10) can be bounded by

$$|(\partial_t a, \partial_t \rho, \nabla \cdot (au), \nabla \cdot (\rho u))|_\infty \|\partial^\alpha \mathscr{U}\|_0^2 \leq C(\|V\|_{s,\varepsilon}).$$

Then we get from (2.10) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega a|\nabla \partial^\alpha q|^2 + \rho \left(|\nabla \cdot \partial^\alpha u|^2 + |\nabla \times \partial^\alpha u|^2 + \sum_{\ell=1}^d (|\nabla \cdot \partial^\alpha F_\ell|^2 + |\nabla \times \partial^\alpha F_\ell|^2) \right) \mathrm{d}x \\ \leq & C(\|V\|_{s,\varepsilon}). \end{aligned} \tag{2.11}$$

Using (2.1) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \partial^\alpha q|^2 + |\nabla \cdot \partial^\alpha u|^2 + |\nabla \times \partial^\alpha u|^2 + \sum_{\ell=1}^d (|\nabla \cdot \partial^\alpha F_\ell|^2 + |\nabla \times \partial^\alpha F_\ell|^2) \mathrm{d}x \leq C(\|V\|_{s,\varepsilon}).$$

Then summing up above inequality for all $|\alpha| \leq s-1$, we obtain (2.5). □

Next, we give the L^2-estimate and the time derivatives estimates of $W = (q, u, F_1, \cdots, F_d)$.

Lemma 2.3 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2 + 1$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W\|_0^2 \leq C(\|V\|_{s,\varepsilon}), \tag{2.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=1}^s \left\| \varepsilon^{k-1} \partial_t^k W \right\|_0^2 \leq C(\|V\|_{s,\varepsilon}). \tag{2.13}$$

Proof. Multiplying $(1.4)_1$-$(1.4)_3$ by W, integrating over Ω, and integrating by parts yield

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(a|q|^2 + \rho|u|^2 + \sum_{j=1}^d \rho |F_j|^2 \right) = & \frac{1}{2} \int_\Omega \left(\partial_t a|q|^2 + \partial_t \rho |u|^2 + \sum_{j=1}^d \partial_t \rho |F_j|^2 \right) \\ & + \frac{1}{2} \int_\Omega \left(\nabla \cdot (au)|q|^2 + \nabla \cdot (\rho u)|u|^2 + \sum_{j=1}^d \nabla \cdot (\rho u)|F_j|^2 \right) \\ & + \int_\Omega \left(\rho \sum_{j=1}^d F_j \cdot \nabla F_j \cdot u + \rho \sum_{j=1}^d F_j \cdot \nabla u \cdot F_j \right) \\ := & I_1 + I_2 + I_3. \end{aligned}$$

Recalling that

$$|(\partial_t a, \partial_t \rho)|_\infty + |(\nabla\cdot(au), \nabla\cdot(\rho u))|_\infty \le C(\|V\|_{s,\varepsilon}).$$

Then we obtain

$$I_1 + I_2 \le C(\|V\|_{s,\varepsilon}).$$

Next, using the divergence constrains of (1.2), and integrating by parts, we have

$$I_3 = \int_\Omega \left(-\sum_{j=1}^d \nabla\cdot(\rho F_j)(F_j\cdot u) - \sum_{j=1}^d \rho(F_j\cdot\nabla)u\cdot F_j + \sum_{j=1}^d \rho(F_j\cdot\nabla)u\cdot F_j \right) = 0,$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(a|q|^2 + \rho|u|^2 + \sum_{j=1}^d \rho|F_j|^2 \right) \mathrm{d}x \le C(\|V\|_{s,\varepsilon}),$$

which implies (2.12) by (2.1).

Taking ∂_t^ℓ with $1 \le \ell \le s$ to $(1.4)_1$-$(1.4)_3$, we have

$$\begin{aligned} a\left(\partial_t + u\cdot\nabla\right)\partial_t^\ell q + \frac{1}{\varepsilon}\nabla\cdot\partial_t^\ell u =& \mathscr{C}_q, \\ \rho\left(\partial_t + u\cdot\nabla\right)\partial_t^\ell u + \frac{1}{\varepsilon}\nabla\partial_t^\ell q =& \rho\sum_{j=1}^d F_j\cdot\nabla\partial_t^\ell F_j + \mathscr{C}_u, \\ (\partial_t + u\cdot\nabla)\partial_t^\ell F_j =& F_j\cdot\nabla\partial_t^\ell u + \mathscr{C}_{F_j}, \quad j = 1,\cdots,d, \end{aligned} \tag{2.14}$$

where the commutators are

$$\begin{aligned} \mathscr{C}_q =& -[\partial_t^\ell, a]\partial_t q - [\partial_t^\ell, a(u\cdot\nabla)]q, \\ \mathscr{C}_u =& -[\partial_t^\ell, \rho]\partial_t u - [\partial_t^\ell, \rho(u\cdot\nabla)]u + \sum_{j=1}^d [\partial_t^\ell, \rho(F_j\cdot\nabla)]F_j, \\ \mathscr{C}_{F_j} =& -[\partial_t^\ell, (u\cdot\nabla)]F_j + [\partial_t^\ell, (F_j\cdot\nabla)]u. \end{aligned}$$

Multiplying (2.14) by $\varepsilon^{2(\ell-1)}\partial_t^\ell(q,u)$, $\varepsilon^{2(\ell-1)}\rho\partial_t^\ell F_j$, integrating over Ω, integrating by parts give

$$\begin{aligned} &\frac{\varepsilon^{2(\ell-1)}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(a|\partial_t^\ell q|^2 + \rho|\partial_t^\ell u|^2 + \sum_{j=1}^d \rho|\partial_t^\ell F_j|^2 \right) \\ =&\frac{\varepsilon^{2(\ell-1)}}{2}\int_\Omega \left(\partial_t a|\partial_t^\ell q|^2 + \partial_t\rho|\partial_t^\ell u|^2 + \sum_{j=1}^d \partial_t\rho|\partial_t^\ell F_j|^2 \right) \\ &+ \frac{\varepsilon^{2(\ell-1)}}{2}\int_\Omega \left(\nabla\cdot(au)|\partial_t^\ell q|^2 + \nabla\cdot(\rho u)|\partial_t^\ell u|^2 + \sum_{j=1}^d \nabla\cdot(\rho u)|\partial_t^\ell F_j|^2 \right) \\ &+ \varepsilon^{2(\ell-1)}\int_\Omega \left(\mathscr{C}_q\cdot\partial_t^\ell q + \mathscr{C}_u\cdot\partial_t^\ell u + \rho\sum_{j=1}^d \mathscr{C}_{F_j}\cdot\partial_t^\ell F_j \right) \\ :=& I_4 + I_5 + I_6. \end{aligned} \tag{2.15}$$

Note that the estimates of I_4 and I_5 can be given in a similar fashion as I_1 and I_2, so we have

$$I_4 + I_5 \le C(\|V\|_{s,\varepsilon}). \tag{2.16}$$

In order to get the estimate of I_6, we need to consider $\|\varepsilon^{\ell-1}\mathscr{C}_q\|_0$. From the expression of $\mathscr{C}_q$, we have

$$\varepsilon^{\ell-1}\mathscr{C}_q = -\varepsilon^{\ell-1}\sum_{k=1}^{\ell} C_\ell^k \partial_t^k a \partial_t^{\ell-k+1} q - \varepsilon^{\ell-1}\sum_{k=1}^{\ell}\sum_{i=1}^{d} C_\ell^k \partial_t^k (a u_i)\partial_t^{\ell-k}\partial_i q.$$

The product estimate (4.2) yields

$$\begin{aligned}
\left\|\varepsilon^{\ell-1}\mathscr{C}_q\right\|_0 \le& \varepsilon^{\ell-1} K \sum_{k=1}^{\ell}\left\|\partial_t^k a \partial_t^{\ell-k+1} q\right\|_0 + \varepsilon^{\ell-1} K \sum_{k=1}^{\ell}\sum_{i=1}^{d}\left\|\partial_t^k (a u_i)\partial_t^{\ell-k}\partial_i q\right\|_0 \\
\le& \varepsilon^{\ell-1} K \sum_{k=1}^{\ell}\left\|\partial_t^k a\right\|_{\ell-k}\left\|\partial_t^{\ell-k+1} q\right\|_{s-(\ell-k)-1} \\
&+ \varepsilon^{\ell-1} K \sum_{k=1}^{\ell}\sum_{i=1}^{d}\left\|\partial_t^k (a u_i)\right\|_{\ell-k}\left\|\partial_t^{\ell-k}\partial_i q\right\|_{s-(\ell-k)-1} \\
\le& C(\|V\|_{s,\varepsilon}).
\end{aligned}$$

The estimates of $\|\varepsilon^{\ell-1}\mathscr{C}_u\|_0$ and $\|\varepsilon^{\ell-1}\mathscr{C}_{F_j}\|_0$ are similar. After a straightforward calculation, we find that

$$I_6 = \varepsilon^{2(\ell-1)}\int_\Omega \left(\mathscr{C}_q\cdot\partial_t^\ell q + \mathscr{C}_u\cdot\partial_t^\ell u + \rho\sum_{j=1}^{d}\mathscr{C}_{F_j}\cdot\partial_t^\ell F_j\right) \le C(\|V\|_{s,\varepsilon}). \tag{2.17}$$

Thus, subsituting (2.16) and (2.17) into (2.15) yield

$$\varepsilon^{2(\ell-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(a|\partial_t^\ell q|^2 + \rho|\partial_t^\ell u|^2 + \sum_{j=1}^{d}\rho|\partial_t^\ell F_j|^2\right) \le C(\|V\|_{s,\varepsilon}),$$

which implies (2.13) by (2.1). $\square$

Remark 2.2 *(2.5) and (2.12) imply that the following estimates of spatial derivatives hold:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W\|_s^2 \le C(\|V\|_{s,\varepsilon}). \tag{2.18}$$

Lemma 2.4 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2+1$, we have*

$$\sum_{k=1}^{s-1}\|\varepsilon^{k-1}\partial_t^k\nabla F_\ell\|_{s-k-1} \le C(\|V\|_{E_1}) + \varepsilon C(\|V\|_{s,\varepsilon}). \tag{2.19}$$

Proof. In the case of $k=1$, we need to estimate $\|\partial_t\nabla F_\ell\|_{s-2}$. Taking $\partial^\alpha\nabla$ of $(1.4)_3$ with $|\alpha|\le s-2$ yields

$$\partial_t\partial^\alpha\nabla F_\ell = \partial^\alpha\nabla\left(-u\cdot\nabla F_\ell + F_\ell\cdot\nabla u\right).$$

Noting that the term $\partial^\alpha \nabla (-u \cdot \nabla F_\ell + F_\ell \cdot \nabla u)$ is a sum of terms

$$-\partial^\beta u_j \partial^\gamma \partial_j F_\ell + \partial^\beta F_{j\ell} \partial^\gamma \partial_j u$$

with $1 \le |\beta| + |\gamma| \le s-1$ and $j = 1, \cdots, d$. Thus

$$\|\partial_t \nabla F_\ell\|_{s-2} \le C(\|V\|_s) \le C(\|V\|_{E_1}).$$

In the case of $1 < k \le s-1$, taking $\varepsilon^{k-1}\partial_t^{k-1}\partial^\alpha \nabla$ of $(1.4)_3$ with $|\alpha| \le s-k-2$ to obtain

$$\varepsilon^{k-1}\partial_t^k \partial^\alpha \nabla F_\ell = \varepsilon^{k-1}\partial_t^{k-1}\partial^\alpha \nabla (-u \cdot \nabla F_\ell + F_\ell \cdot \nabla u).$$

From the following analysis we get an estimate of $\|\varepsilon^{k-1}\partial_t^{k-1}\partial^\alpha \nabla (u \cdot \nabla F_\ell)\|_0$.

$$\begin{aligned}
&\|\varepsilon^{k-1}\partial_t^{k-1}\partial^\alpha \nabla (u \cdot \nabla F_\ell)\|_0 \\
\le & K\varepsilon^{k-1} \sum_{p=0}^{k-1} \sum_{1 \le |\beta|+|\gamma| \le s-k-1} \sum_{j=1}^{d} \|\partial_t^p \partial^\beta u_j \partial_t^{k-p-1}\partial^\gamma \partial_j F_\ell\|_0 \\
\le & K\varepsilon^{k-1} \sum_{p=0}^{k-1} \sum_{1 \le |\beta|+|\gamma| \le s-k-1} \sum_{j=1}^{d} \|\partial_t^p \partial^\beta u_j\|_{s-p-|\beta|} \|\partial_t^{k-p-1}\partial^\gamma \partial_j F_\ell\|_{s-(k-p-1)-|\gamma|-1} \\
\le & \varepsilon C(\|V\|_{s,\varepsilon}),
\end{aligned}$$

where we have used (4.2). The estimate of $\|\varepsilon^{k-1}\partial_t^{k-1}\partial^\alpha \nabla (F_\ell \cdot \nabla u)\|_0$ can be given in a similar fashion, so we have

$$\|\varepsilon^{k-1}\partial_t^k \partial^\alpha \nabla F_\ell\|_0 \le \varepsilon C(\|V\|_{s,\varepsilon}).$$

The proof is completed by putting the estimates together. □

Lemma 2.5 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2+1$, we have*

$$\sum_{k=1}^{s-1} \|\varepsilon^{k-1}\partial_t^k \nabla \times u\|_{s-k-1} \le \left(1 + \sum_{k=1}^{s} \|\varepsilon^{k-1}\partial_t^k (q,u)\|_0\right) C(\|V\|_s) + \varepsilon C(\|V\|_{s,\varepsilon}), \tag{2.20}$$

$$\sum_{k=1}^{s-1} \|\varepsilon^{k-1}\partial_t^k (\nabla q, \nabla \cdot u)\|_{s-k-1} \le \left(1 + \sum_{k=1}^{s} \|\varepsilon^{k-1}\partial_t^k (q,u)\|_0\right) C(\|V\|_s) + \varepsilon C(\|V\|_{s,\varepsilon}). \tag{2.21}$$

Proof. Set

$$\mathscr{M}(V,s,\varepsilon) := \left(1 + \sum_{k=1}^{s} \|\varepsilon^{k-1}\partial_t^k (q,u)\|_0\right) C(\|V\|_s) + \varepsilon C(\|V\|_{s,\varepsilon}).$$

If we obtain

$$\begin{aligned}
\|\varepsilon^{k-1}\partial_t^k \nabla \times u\|_{s-k-1} \le & \mathscr{M}(V,s,\varepsilon), \\
\|\varepsilon^{k-1}\partial_t^k (\nabla q, \nabla \cdot u)\|_{s-k-1} \le & \mathscr{M}(V,s,\varepsilon),
\end{aligned}$$

for $k = 1, \cdots, s-1$, the proof is completed. Next, let's prove the case of $k = s-1$, $k = s-2$ and $k = s-3$, respectively. Applying $\nabla \times$ to $(1.4)_2$, we have

$$\partial_t \nabla \times u = -\rho^{-1}\nabla\rho \times \partial_t u - \rho^{-1}\nabla \times (\rho u \cdot \nabla u) + \sum_{\ell=1}^{d} \rho^{-1}\nabla \times (\rho F_\ell \cdot \nabla F_\ell). \tag{2.22}$$

We rewrite the equations of q and u as

$$\begin{aligned}\nabla\cdot u=&-\varepsilon a(\partial_t q+u\cdot\nabla q),\\ \nabla q=&-\varepsilon\rho\left(\partial_t u+u\cdot\nabla u-\sum_{\ell=1}^{d}F_\ell\cdot\nabla F_\ell\right).\end{aligned}\tag{2.23}$$

In the case of $k=s-1$, we need to estimate $\|\varepsilon^{s-2}\partial_t^{s-1}\nabla\times u\|_0$ and $\|\varepsilon^{s-2}\partial_t^{s-1}(\nabla q,\nabla\cdot u)\|_0$. Applying $\varepsilon^{n-1}\partial_t^{n-1}$ to (2.22) with $1\le n\le s-1$ and taking L^2 norm yield

$$\|\varepsilon^{n-1}\partial_t^n\nabla\times u\|_0\le\|\varepsilon^{n-1}\partial_t^n u\|_0C(\|V\|_s)+\varepsilon C(\|V\|_{s,\varepsilon})\le\mathscr{M}(V,s,\varepsilon),\tag{2.24}$$

which implies that

$$\|\varepsilon^{s-2}\partial_t^{s-1}\nabla\times u\|_0\le\mathscr{M}(V,s,\varepsilon).$$

Taking $\varepsilon^{n-1}\partial_t^n$ of (2.23) with $1\le n\le s-1$ and taking the L^2 norm of both sides of the results yield

$$\|\varepsilon^{n-1}\partial_t^n(\nabla q,\nabla\cdot u)\|_0\le\|\varepsilon^{n}\partial_t^{n+1}(q,u)\|_0C(\|V\|_s)+\varepsilon C(\|V\|_{s,\varepsilon})\le\mathscr{M}(V,s,\varepsilon),\tag{2.25}$$

which implies that

$$\|\varepsilon^{s-2}\partial_t^{s-1}(\nabla q,\nabla\cdot u)\|_0\le\mathscr{M}(V,s,\varepsilon).$$

In the case of $k=s-2$, we need to estimate $\|\varepsilon^{s-3}\partial_t^{s-2}\nabla\times u\|_1$ and $\|\varepsilon^{s-3}\partial_t^{s-2}(\nabla q,\nabla\cdot u)\|_1$. Using (2.24) and (2.25) we have

$$\begin{aligned}\|\varepsilon^{s-3}\partial_t^{s-2}\nabla\times u\|_0\le&\mathscr{M}(V,s,\varepsilon),\\ \|\varepsilon^{s-3}\partial_t^{s-2}(\nabla q,\nabla\cdot u)\|_0\le&\mathscr{M}(V,s,\varepsilon).\end{aligned}$$

So we focus on $\|\varepsilon^{s-3}\partial_t^{s-2}\partial_i\nabla\times u\|_0$ and $\|\varepsilon^{s-3}\partial_t^{s-2}\partial_i(\nabla q,\nabla\cdot u)\|_0$ with $1\le i\le d$. Applying $\varepsilon^{n-1}\partial_t^{n-1}\partial_i$ to (2.22) with $1\le n\le s-2$, $1\le i\le d$ and taking L^2 norm yield

$$\begin{aligned}&\|\varepsilon^{n-1}\partial_t^n\partial_i\nabla\times u\|_0\\ \le&\left(\|\varepsilon^{n-1}\partial_t^n u\|_0+\|\varepsilon^{n-1}\partial_t^n\nabla\cdot u\|_0+\|\varepsilon^{n-1}\partial_t^n\nabla\times u\|_0\right)C(\|V\|_s)+\varepsilon C(\|V\|_{s,\varepsilon})\\ \le&\mathscr{M}(V,s,\varepsilon),\end{aligned}\tag{2.26}$$

where we have used estimates in the case of $k=s-1$, which implies

$$\|\varepsilon^{s-3}\partial_t^{s-2}\partial_i\nabla\times u\|_0\le\mathscr{M}(V,s,\varepsilon).$$

Taking $\varepsilon^{n-1}\partial_t^n\partial_i$ of (2.23) with $1\le n\le s-2$, $1\le i\le d$ and taking the L^2 norm of both sides of the results yield

$$\begin{aligned}\|\varepsilon^{n-1}\partial_t^n\partial_i(\nabla q,\nabla\cdot u)\|_0\le&\|\varepsilon^{n}\partial_t^{n+1}(\nabla q,\nabla\cdot u,\nabla\times u)\|_0C(\|V\|_s)+\varepsilon C(\|V\|_{s,\varepsilon})\\ \le&\mathscr{M}(V,s,\varepsilon),\end{aligned}\tag{2.27}$$

where we have used estimates in the case of $k = s-1$, which implies

$$\|\varepsilon^{s-3}\partial_t^{s-2}\partial_i(\nabla q, \nabla\cdot u)\|_0 \le \mathscr{M}(V,s,\varepsilon).$$

In the case of $k = s-3$, we need to estimate $\|\varepsilon^{s-4}\partial_t^{s-3}\nabla\times u\|_2$ and $\|\varepsilon^{s-4}\partial_t^{s-3}(\nabla q, \nabla\cdot u)\|_2$. Using (2.24), (2.25), (2.26) and (2.27) we have

$$\begin{aligned}
\|\varepsilon^{s-4}\partial_t^{s-3}\nabla\times u\|_0 \le&\mathscr{M}(V,s,\varepsilon),\\
\|\varepsilon^{s-4}\partial_t^{s-3}(\nabla q, \nabla\cdot u)\|_0 \le&\mathscr{M}(V,s,\varepsilon),\\
\|\varepsilon^{s-4}\partial_t^{s-3}\partial_i\nabla\times u\|_0 \le&\mathscr{M}(V,s,\varepsilon),\\
\|\varepsilon^{s-4}\partial_t^{s-3}\partial_i(\nabla q, \nabla\cdot u)\|_0 \le&\mathscr{M}(V,s,\varepsilon),
\end{aligned}$$

where $1 \le i \le d$. So we focus on $\|\varepsilon^{s-4}\partial_t^{s-3}\partial^\alpha\nabla\times u\|_0$ and $\|\varepsilon^{s-4}\partial_t^{s-3}\partial^\alpha(\nabla q, \nabla\cdot u)\|_0$ with $|\alpha| = 2$. Applying $\varepsilon^{n-1}\partial_t^{n-1}\partial^\alpha$ to (2.22) with $1 \le n \le s-3$, $|\alpha| = 2$ and taking L^2 norm yields

$$\begin{aligned}
&\|\varepsilon^{n-1}\partial_t^n\partial^\alpha\nabla\times u\|_0\\
\le&\Big(\|\varepsilon^{n-1}\partial_t^n u\|_0 + \|\varepsilon^{n-1}\partial_t^n\nabla\cdot u\|_0 + \|\varepsilon^{n-1}\partial_t^n\nabla\times u\|_0\\
&+ \|\varepsilon^{n-1}\partial_t^n\partial_i\nabla\cdot u\|_0 + \|\varepsilon^{n-1}\partial_t^n\partial_i\nabla\times u\|_0\Big)C(\|V\|_s) + \varepsilon C(\|V\|_{s,\varepsilon})\\
\le&\mathscr{M}(V,s,\varepsilon),
\end{aligned}$$

where we have used estimates in the case of $k = s-1$ and $k = s-2$, which implies

$$\|\varepsilon^{s-4}\partial_t^{s-3}\partial^\alpha\nabla\times u\|_0 \le \mathscr{M}(V,s,\varepsilon).$$

Taking $\varepsilon^{n-1}\partial_t^n\partial^\alpha$ of (2.23) with $1 \le n \le s-3$, $|\alpha| = 2$ and taking the L^2 norm of both sides of the results yield

$$\begin{aligned}
\|\varepsilon^{n-1}\partial_t^n\partial^\alpha(\nabla q, \nabla\cdot u)\|_0 \le&\|\varepsilon^n\partial_t^{n+1}\partial_i(\nabla q, \nabla\cdot u, \nabla\times u)\|_0 C(\|V\|_s) + \varepsilon C(\|V\|_{s,\varepsilon})\\
\le&\mathscr{M}(V,s,\varepsilon),
\end{aligned}$$

where we have used estimates in the case of $k = s-2$, which implies

$$\|\varepsilon^{s-4}\partial_t^{s-3}\partial^\alpha(\nabla q, \nabla\cdot u)\|_0 \le \mathscr{M}(V,s,\varepsilon).$$

In fact, the remaining proof is standard. If we have got the estimates in the case of $1 \le k \le m$ for some $1 \le m < s-1$, we can use standard methods to get estimates in the case of $k = m+1$. The proof is completed. □

Lemma 2.6 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2+1$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V\|_{E_1}^2 \le C(\|V\|_{s,\varepsilon}), \tag{2.28}$$

$$\|V\|_{E_2} \le C(\|V\|_{E_1}) + \varepsilon C(\|V\|_{s,\varepsilon}). \tag{2.29}$$

Proof. The inequality (2.28) comes from Lemma 2.1, Lemma 2.3 and Remark 2.2. The inequality (2.29) comes from Lemma 2.4 and Lemma 2.5. □

Finally, we prove the following lemma, which can deduce the main Theorem 1.1.

Lemma 2.7 *Suppose that the assumption in Theroem 1.1 holds with $s > d/2+1$, then there are constants $T > 0$ and $0 < \varepsilon_0 < 1$, such that for all $\varepsilon \in (0, \varepsilon_0]$,*

$$\|V\|_{s,\varepsilon,T} \leq M. \tag{2.30}$$

Proof. First, it is obvious that $\|V\|_{E_1} + \|V\|_{E_2}$ is equivalent to $\|V\|_{s,\varepsilon}$, and there are $c_1, c_2 > 0$ such that

$$c_1(\|V\|_{E_1} + \|V\|_{E_2}) \leq \|V\|_{s,\varepsilon} \leq c_2(\|V\|_{E_1} + \|V\|_{E_2}).$$

Then (2.29) returns to

$$\|V\|_{E_2} \leq C(\|V\|_{E_1}) + \varepsilon C(\|V\|_{E_2}).$$

Choosing $\varepsilon_0 \in (0,1)$ such that

$$\|V\|_{E_2} \leq C(\|V\|_{E_1}), \quad \forall \varepsilon \in (0, \varepsilon_0].$$

Thus, we can replace $\|V\|_{s,\varepsilon}$ in (2.28), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V\|_{E_1}^2 \leq C(\|V\|_{s,\varepsilon}) \leq C(\|V\|_{E_1} + \|V\|_{E_2}) \leq C(\|V\|_{E_1}), \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{2.31}$$

It is easy to get $\|V(t=0)\|_{E_1} \leq K(M_0)$ by (1.9). Then (2.31) deduces

$$\|V(t)\|_{E_1} \leq K(M_0) + \int_0^t C(\|V(s)\|_{E_1})\mathrm{d}s.$$

Choosing a $T > 0$ small enough, we get

$$\|V(t)\|_{E_1} \leq K(M_0, T), \quad t \in [0, T]. \tag{2.32}$$

The proof is completed. □

3 Low Mach number limit

The uniform estimate (1.10) and Lemma 4.7 imply, after extracting a subsequence, the following limit:

$$\begin{aligned}
&V^\varepsilon \rightharpoonup V^0 \text{ weakly-* in } L^\infty([0,T]; H^s(\mathbb{T}^d)),\\
&V^\varepsilon \to V^0 \text{ strongly in } C([0,T]; H^{s'}(\mathbb{T}^d)) \ \forall s' < s,
\end{aligned}$$

as $\varepsilon \to 0$, where $V^0 = (S^0, q^0, u^0, F_1^0, \cdots, F_d^0) \in C([0,T]; H^{s'}(\mathbb{T}^d))$ for any $s' < s$. Thus, according to (1.4), we can easily bound the singular terms as follows

$$\sup_{t\in[0,T]} (\|\nabla q\|_{s-1} + \|\nabla \cdot u\|_{s-1}) \leq K\varepsilon.$$

Applying the Leray projection $\mathscr{P}$ to $(1.4)_2$, multiplying $(1.4)_1$ by ε and then taking $\varepsilon \to 0$, we find that

$$\begin{aligned}
&\nabla \cdot u^0 = 0,\\
&\mathscr{P}\left(\rho(S^0,0)\left(\partial_t u^0 + u^0\cdot\nabla u^0 - \sum_{j=1}^d F_j^0 \cdot \nabla F_j^0\right)\right) = 0,\\
&\partial_t F_j^0 + u^0 \cdot \nabla F_j^0 = F_j^0 \cdot \nabla u^0, \quad j = 1, \cdots, d,\\
&\partial_t S^0 + u^0 \cdot \nabla S^0 = 0,
\end{aligned}$$

Thus, we obtain (1.11) for some function π. The uniqueness of the solution of (1.11) with initial data $(S_0^0, u_0^0, F_{1,0}^0, \cdots, F_{d,0}^0) \in H^s(\mathbb{T}^d)$ implies the convergence holds as $\varepsilon \to 0$ without restricting to a subsequence. Hence, the proof of Theorem 1.2 is completed.

4 Appendix

Lemma 4.1 (see [5]) *For all $\alpha = (\alpha_1, \cdots, \alpha_d)$, $|\alpha| = k$, $\sigma \geq 0$, and $f, g \in H^{k+\sigma}(\Omega)$, it holds that*

$$\|[f, \partial^\alpha]g\|_0 \leq K\left(\|f\|_{W^{1,\infty}}\|g\|_{\sigma+k-1} + \|f\|_{\sigma+k}\|g\|_{L^\infty}\right). \tag{4.1}$$

Lemma 4.2 (see [5]) *For $k \geq 0$, $l \geq 0$, $k + l \leq \sigma$, and $\sigma > d/2$, the product maps continuously $H^{\sigma-k}(\Omega) \times H^{\sigma-l}(\Omega)$ to $H^{\sigma-k-l}(\Omega)$, and*

$$\|uv\|_{\sigma-k-l} \leq K\|u\|_{\sigma-k}\|v\|_{\sigma-l}. \tag{4.2}$$

Lemma 4.3 (see [5]) *F is a smooth function such that $F(0) = 0$, and $u \in H^\sigma(\Omega)$ with $\sigma > d/2$, then $F(u) \in H^\sigma(\Omega)$ and*

$$\|F(u)\|_\sigma \leq C(\|u\|_\sigma)\|u\|_\sigma. \tag{4.3}$$

Moreover, suppose $v \in H^m(\Omega)$, with $m \in \{0, \cdots, \sigma\}$, then $F(u)v \in H^m(\Omega)$ and

$$\|F(u)v\|_m \leq (|F(0)| + C(\|u\|_\sigma))\|v\|_m. \tag{4.4}$$

Lemma 4.4 (Moser-type Calculus Inequalities, see [17]) *Let $s \geq 1$ ba an integer,*

(a) For $f, g \in H^s \cap L^\infty$ and $|\alpha| \leq s$

$$\|\partial^\alpha(fg)\|_0 \leq K_s\left(|f|_\infty\|\partial^s g\|_0 + |g|_\infty\|\partial^s f\|_0\right).$$

(b) For $f \in H^s$, $\nabla f \in L^\infty$, $g \in H^{s-1} \cap L^\infty$ and $|\alpha| \leq s$

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_0 \leq K_s\left(|\nabla f|_\infty\|\partial^{s-1}g\|_0 + |g|_\infty\|\partial^s f\|_0\right).$$

(c) Assume $g(u)$ is a smooth vector-valued function on G, $u(x)$ is a continuous function with $u(x) \in G_1$, $\overline{G}_1 \subset\subset G$, and $u \in L^\infty \cap H^s$. Then

$$\|\partial^s g(u)\|_0 \leq K_s\left|\frac{\partial g}{\partial u}\right|_{s-1,\overline{G}_1}|u|_\infty^{s-1}\|\partial^s u\|_0.$$

Here $|\cdot|_{r,\overline{G}_1}$ is the C^r norm on the set $\overline{G}_1$.

Lemma 4.5 (see [15, 26]) *For any smooth vector function $v \in H^1(\Omega)$, we have*

$$\|\nabla v\|_0 \leq \|\nabla \cdot v\|_0 + \|\nabla \times v\|_0.$$

Moreover, if $v \in H^s(\Omega)$ $(s > 1)$, we have

$$\|\nabla v\|_{s-1} \leq \|\nabla \cdot v\|_{s-1} + \|\nabla \times v\|_{s-1}.$$

Lemma 4.6 (see [7, 15]) *Assume that g and h are smooth vector functions on Ω. Then the following formula holds*

$$\int_\Omega (\nabla \times g) \cdot h \mathrm{d}x = \int_\Omega g \cdot (\nabla \times h) \mathrm{d}x.$$

Lemma 4.7 (Aubin-Lions compactness Lemma, see [17]) *Let $\{v^\varepsilon\}$ be a sequence of functions satisfying*

$$v^\varepsilon \in C([0,T]; H^s(\Omega)) \cap C^1([0,T]; H^{s-1}(\Omega)),$$

and there exists a constant M independent of ε, such that

$$\|v^\varepsilon\|_s + \|\partial_t v^\varepsilon\|_{s-1} \leq M.$$

Then, by passing to a subsequence, there exists a function v^0 such that

$$\begin{aligned} v^\varepsilon &\rightharpoonup v^0 \ \textit{weakly-* in } L^\infty([0,T]; H^s(\Omega)), \\ v^\varepsilon &\to v^0 \ \textit{strongly in } C([0,T]; H^{s'}(\Omega)) \ \forall s' < s, \\ \partial_t v^\varepsilon &\rightharpoonup \partial_t v^0 \ \textit{weakly-* in } L^\infty([0,T]; H^{s-1}(\Omega)), \end{aligned}$$

as $\varepsilon \to 0$. Moreover,

$$v^0 \in C_w([0,T]; H^s(\Omega)) \cap Lip([0,T]; H^{s-1}(\Omega)).$$

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