Statistical solutions and degenerate regularity for the non-Newtonian micropolar fluids*

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Abstract

This paper studies the non-autonomous non-Newtonian micropolar fluids in two-dimensional bounded domains. We first establish that the generated continuous process of the solutions operator possesses a pullback attractor. Then we verify the existence of statistical solutions by constructing the invariant Borel probability measures. Further, we prove that the statistical solutions possess the degenerated regularity of Lusin's type provided that the Grasshof number associated to the external forces is small enough.

Keywords: Statistical solution; Non-Newtonian micropolar fluids; Pullback attractor; Degenerate regularity; Grashof number.

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1 Introduction

In the theory of fluid mechanics, the invariant measures and statistical solutions have proven to be very useful in the understanding of turbulence (see Foias *et al.* [12]). Statistical solutions have been introduced as a rigorous mathematical notion to formalize the object of ensemble average in the conventional statistical theory of turbulence. Nowadays, invariant measures and statistical solutions are widely used to describe certain characteristics of the fluids. There are several references concerning invariant measures and statistical solutions (see [5–8, 13, 15, 16, 18, 19, 23, 30, 32–39, 42, 43]), statistical solutions and trajectory statistical solutions for deterministic equations, [40] for impulsive lattice system and [41] for 2D stochastic Navier-Stokes equations.

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The asymptotic behavior of solutions for the non-Newtonian fluids and micropolar fluids has been extensively studied, see, e.g. [1–3, 22, 25–29, 31]. Concerning statistical solutions, Zhao, Li and Sang studied trajectory statistical solutions for 3D incompressible micropolar fluids in [36], and Zhao, Li and Łukaszewicz studied the statistical solutions and partial degenerate regularity for the 2D non-autonomous magneto-micropolar fluids in [35].

In this paper we prove the existence of statistical solutions and their partial degenerate regularity of the Lusin's type of the following problem for the two-dimensional (2D) non-autonomous, viscous incompressible non-Newtonian micropolar fluids,

$$\begin{cases} \partial_t u - \nabla \cdot (2\mu_0(\varepsilon + |e|^2)^{-\frac{\alpha}{2}} e(u) - 2\mu_1 \Delta e(u)) + u \cdot \nabla u + \nabla p = 2\nu_r \nabla \times \omega + f(x,t), \\ \nabla \cdot u = 0, \\ \partial_t \omega - \nu_1 \Delta \omega + u \cdot \nabla \omega + 4\nu_r \omega = 2\nu_r \nabla \times u + g(x,t), \end{cases} \tag{1.1}$$

in $\Omega \times (\tau, +\infty)$, where Ω is a smooth and bounded domain of $\mathbb{R}^2$ with boundary $\partial\Omega$, with the initial-boundary conditions

$$u(x,\tau) = u_\tau; \ \ \omega(x,\tau) = \omega_\tau, \ \ x \in \Omega, \tag{1.2}$$

$$u(x,t)|_{\partial\Omega} = 0, \ \ \omega(x,t)|_{\partial\Omega} = 0; \quad \mathrm{T}^u_{ijk} n_j n_k|_{\partial\Omega} = 0, \ \ i,j,k = 1,2, \ \ t \geqslant \tau, \tag{1.3}$$

where $\mathrm{T}^u_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}$ and (n_1, n_2) is the exterior unit normal to $\partial\Omega$. The unknown functions $u = u(x,t)$ and $\omega = \omega(x,t)$ denote the velocity and microrotational scalar field of the fluids, respectively, $\nabla \cdot u = 0$ describes the incompressibility of the fluids, f and g are the external forces and moments, respectively. The first two conditions in (1.3) represent the usual no-slip condition associated with a viscous fluid, while the latter one expresses the fact that the first moments of the traction vanish on $\partial\Omega$. In system (1.1), the microrotational effect of the fluids are taken into consideration, the microrotational scalar field ω is interpreted as the angular velocity of the rotation of the particles.

The expression

$$2\mu_0(\varepsilon + |e|^2)^{-\alpha/2} e(u) - 2\mu_1 \Delta e(u) := \mathrm{T}(u) \tag{1.4}$$

in $(1.1)_1$ is the viscous extra stress tensor, where $\varepsilon, \mu_0, \mu_1 > 0$ and $\alpha \in (0,1)$ are constitutive parameters. It was introduced by O. Ladyzhenskaya in [17] to modify the Navier-Stokes equations in such a way as to be able to deal, among other things, with the gradient $|\nabla u|$ of the velocity field in the 3D situation, thus ensuring the well posedness of the considered problem, cf. also [20]. We refer to [17, 20] and the references therein for the detailed physical background. In addition, we have used the notations

$$\nabla \times \omega = \Big(\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}\Big), \ \ \nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \ \ \nabla \times u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

The first aim of this paper is to prove the existence of statistical solutions for system (1.1). We first establish the global well-posedness of the weak solutions for problem (1.1)-(1.3). Then we show that the solutions operator of problem (1.1)-(1.3) forms a continuous process $\{U(t,\tau)\}_{t \geqslant \tau}$ in the phase space E and that $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}(t)=\{\mathcal{A}(t):t\in\mathbb{R}\}$ in E. Afterwards, we prove that $\{U(t,\tau)\}_{t\geqslant\tau}$ meet the so-called τ-continuity in E and construct a family of invariant Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ for $\{U(t,\tau)\}_{t\geqslant\tau}$ in E. Finally, we verify that this family of invariant Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ satisfies the Liouville type theorem and is a statistical solution of system (1.1).

The second purpose of this paper is to verify the partial degenerate regularity of the statistical solutions for system (1.1). This issue is akin to the degeneration of attractors, which is closely related to the question of dimensionality of the attractor. For the Navier-Stokes equations, there are several references investigating the Grashof number which controls the dimension of the attractor, one can refer to [9–11]. We will use the following form of generalized Grashof number

$$\langle\|F\|^2\rangle_{\leqslant t}:=\limsup_{\gamma\to-\infty}\frac{1}{t-\gamma}\int_\gamma^t\|F(\theta)\|^2\mathrm{d}\theta,\quad \text{where } F(t)=(f(t),g(t)), \tag{1.5}$$

to discuss the partial degenerate regularity of the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ for system (1.1). The partial regularity of $\{m_t\}_{t\in\mathbb{R}}$ means that it is supported on a set in which all weak solutions are partial strong solutions. From the construction of the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ it follows that for each $t\in\mathbb{R}$, the support of m_t is contained in $\mathcal{A}(t)$. The pullback attractor $\hat{\mathcal{A}}(t)=\{\mathcal{A}(t):t\in\mathbb{R}\}$ itself consists of weak, bounded and complete trajectories of system (1.1).

We will prove that if $\langle\|F\|^2\rangle_{\leqslant t}$ given by (1.5) is small enough then $\{m_t\}_{t\in\mathbb{R}}$ possesses partial degenerate regularity of the Lusin type in the following sense: $\forall\,\epsilon>0$ and $\forall\,\varsigma\in\mathbb{R}$, there is a subset $E(\epsilon,\varsigma)\subset[\varsigma,\varsigma+\sigma^{-1}]$ with Lebesgue measure mes $(E(\epsilon,\varsigma))<\epsilon$, such that the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ is regular for $t\in[\varsigma+\sigma^{-1}]\setminus E(\epsilon,\varsigma)$, where σ is a fixed and positive constant that will be specified later.

The rest of this paper is organized as follows. In the next section we show that problem (1.1)-(1.3) is globally well-posed and the associated solutions operator forms a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in the phase space E. In Section 3, we prove that $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}(t)=\{\mathcal{A}(t):t\in\mathbb{R}\}$ by establishing that $\{U(t,\tau)\}_{t\geqslant\tau}$ has a bounded pullback $\mathcal{D}_\sigma$-absorbing set and is pullback $\mathcal{D}_\sigma$-asymptotically compact in E. In Section 4, we prove that $\{U(t,\tau)\}_{t\geqslant\tau}$ satisfies the so-called τ-continuity in E and construct a family of invariant Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ for $\{U(t,\tau)\}_{t\geqslant\tau}$ on E. Then we verify that $\{m_t\}_{t\in\mathbb{R}}$ meets the Liouville type theorem and is a statistical solution of system (1.1), and in Section 5 we prove the partial degenerate regularity of $\{m_t\}_{t\in\mathbb{R}}$.

2 Global well-posedness

In this section we first introduce the mathematical settings for problem (1.1)-(1.3) and then prove the global well-posedness of the latter.

As usual, $L^p(\Omega)$ and $W^{m,p}(\Omega)$ stand for the scalar Lebesgue space and Sobolev space with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. We denote by $W_0^{m,p}(\Omega)$ the closure of $\{\varphi|\varphi\in\mathcal{C}_0^\infty(\Omega)\}$ in $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}$. When $p=2$, we write $W^{m,p}(\Omega)=H^m(\Omega)$, $W_0^{m,p}(\Omega)=H_0^m(\Omega)$ and $\|\cdot\|_2=\|\cdot\|$.

We also use the following vector spaces

$\mathcal{V} = \{\varphi = (\varphi_1, \varphi_2) \in (\mathcal{C}_0^\infty(\Omega))^2 : \nabla \cdot \varphi = 0\}$,

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with norm $\|\cdot\|_H = \|\cdot\|$ and inner product $(\cdot,\cdot)$,

$V =$ the closure of $\mathcal{V}$ in $(H^2(\Omega))^2$ with norm $\|\cdot\|_V = \|\cdot\|_{2,2}$,

$\mathbb{H} = H \times L^2(\Omega)$ with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_{\mathbb{H}} = \|\cdot\|$ defined as

$$(\Phi, \Psi) = (\varphi, \psi) + (\phi, \xi), \;\; \Phi = (\varphi, \phi),\ \Psi = (\psi, \xi) \in \mathbb{H},$$
$$\|\Phi\| = (\|\varphi\|^2 + \|\phi\|^2)^{1/2}, \;\; \Phi = (\varphi, \phi) \in \mathbb{H},$$

$\mathbb{V} = V \times H_0^1(\Omega)$ with the norm $\|\cdot\|_{\mathbb{V}}$ defined as

$$\|\Phi\|_{\mathbb{V}} = (\|\varphi\|_V^2 + \|\phi\|_{1,2}^2)^{1/2}, \;\; \Phi = (\varphi, \phi) \in \mathbb{V}.$$

In addition, we use H^*, V^*, $\mathbb{H}^* = H^* \times L^2(\Omega)$ and $\mathbb{V}^* = V^* \times H^{-1}(\Omega)$ to denote the dual spaces of H, V, $\mathbb{H}$ and $\mathbb{V}$, respectively, where $L^2(\Omega)^* = L^2(\Omega)$ and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. Then we have $V \hookrightarrow H = H^* \hookrightarrow V^*$, $\mathbb{V} \hookrightarrow \mathbb{H} = \mathbb{H}^* \hookrightarrow \mathbb{V}^*$ and these embeddings are compact. Note that we have used the same notations $(\cdot,\cdot)$ to denote the inner product in the spaces $L^2(\Omega)$, H and $\mathbb{H}$. We will also use the same notations $\langle\cdot,\cdot\rangle$ to denote the dual pairing between the spaces V and V^*, $\mathbb{V}$ and $\mathbb{V}^*$, $H_0^m(\Omega)$ and $H^{-1}(\Omega)$ provided that there is no confusion.

Next, we introduce some operators. First, we define $a(u, v)$ via

$$a(u, v) = \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k} \frac{\partial e_{ij}(v)}{\partial x_k} \mathrm{d}x, \;\; \forall u, v \in V. \tag{2.1}$$

Then there exists a positive constant $c_1 (< 1/4)$ such that (cf. [28, Lemma 2.1])

$$c_1 \|u\|_V^2 \leqslant a(u, u) \leqslant \|u\|_V^2, \;\; \forall u \in V, \tag{2.2}$$

and (2.1) and (2.2) imply that $a(\cdot,\cdot)$ defines a positive, symmetric bilinear form on V. By Lax-Milgram's lemma, we see that there is a linear and bounded operator $A_1 : V \mapsto V^*$ such that

$$\langle A_1 u, v\rangle = a(u, v), \;\; \forall\, u, v \in V.$$

It is not difficult to check that $A_1 = P\Delta^2$, where P is the Leray projector from $(L^2(\Omega))^2$ into H. In fact, $D(A_1) = \{\varphi \in V : A_1\varphi \in H\} = (H^4(\Omega))^2 \cap V$ is a Hilbert space which is compactly and densely injected into V. Similarly, we define $A_2 : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ as

$$\langle A_2 \omega, \xi\rangle = (\nabla\omega, \nabla\xi), \;\; \forall\, \omega, \xi \in H_0^1(\Omega).$$

We can also check that $A_2 = -\Delta$, and $D(A_2) = H^2(\Omega) \cap H_0^1(\Omega)$ is compactly and densely injected into $H_0^1(\Omega)$. Secondly, we define the following trilinear forms $b_1(\cdot,\cdot,\cdot)$ and $b_2(\cdot,\cdot,\cdot)$ by

$$b_1(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x, \;\; \forall\, u,\ v,\ w \in (H_0^1(\Omega))^2,$$
$$b_2(u, \omega, \xi) = \sum_{i=1}^{2} \int_\Omega u_i \frac{\partial \omega}{\partial x_i} \xi \mathrm{d}x, \;\; \forall\, u \in (H_0^1(\Omega))^2,\ \omega,\ \xi \in H_0^1(\Omega).$$

Since V is a closed subspace of $(H_0^1(\Omega))^2$, we can check that the trilinear forms $b_1(\cdot,\cdot,\cdot)$ and $b_2(\cdot,\cdot,\cdot)$ are continuous on $V \times V \times V$ and $V \times H_0^1(\Omega) \times H_0^1(\Omega)$, respectively. By some simple computations, we have

$$\begin{cases} b_1(u,v,w) = -b_1(u,w,v), \;\; b_1(u,v,v) = 0, \;\; \forall\, u,\, v,\, w \in V, \\ b_2(u,\omega,\xi) = -b_2(u,\xi,\omega), \;\; b_2(u,\omega,\omega) = 0, \;\; \forall\, u \in V,\, \omega,\, \xi \in H_0^1(\Omega). \end{cases} \tag{2.3}$$

For every $u, v \in V$, the operator $B_1(u,v) : V \times V \mapsto V^*$ defined as

$$\langle B_1(u,v), w\rangle = b_1(u,v,w), \;\; \forall\, w \in V,$$

is continuous. Similarly, for every $u \in V$ and $\omega \in H_0^1(\Omega)$, the operator $B_2(u,\omega) : V \times H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ defined via

$$\langle B_2(u,\omega), \xi\rangle = b_2(u,\omega,\xi), \;\; \forall\, \xi \in H_0^1(\Omega),$$

is continuous. Thirdly, for every $u \in V$, we define $N_1(u) : V \mapsto V^*$ by

$$\langle N_1(u), v\rangle = \sum_{i,j=1}^{2} \int_\Omega \mu(u) e_{ij}(u) e_{ij}(v) \mathrm{d}x, \;\; \forall v \in V,$$

with $\mu(u) = \mu_0(\varepsilon + |e(u)|^2)^{-\alpha/2}$. We can also check that $N_1(u) : V \mapsto V^*$ is continuous.

Now, for any $\varphi \in V$ and $\phi \in H_0^1(\Omega)$, the weak form of system (1.1) can be written as

$$\begin{cases} (\frac{\mathrm{d}u}{\mathrm{d}t}, \varphi) + 2\mu_1\langle A_1 u, \varphi\rangle + \langle B_1(u,u), \varphi\rangle + \langle N_1(u), \varphi\rangle = 2\nu_r(\nabla \times \omega, \varphi) + (f, \varphi), \\ (\frac{\mathrm{d}\omega}{\mathrm{d}t}, \phi) + \nu_1\langle A_2\omega, \phi\rangle + \langle B_2(u,\omega), \phi\rangle + 4\nu_r(\omega, \phi) = 2\nu_r(\nabla \times u, \phi) + (g, \phi). \end{cases} \tag{2.4}$$

Further, for any $z = (u,\omega) \in \mathbb{V}$ we define the following vector-valued operators

$$\begin{cases} Az = \big(2\mu_1 A_1 u, \nu_1 A_2 \omega\big), \\ Bz = \big(B_1(u,u), B_2(u,\omega)\big), \\ Nz = \big(N_1(u), 0\big), \\ Rz = \big(-2\nu_r \nabla \times \omega, 4\nu_r\omega - 2\nu_r\nabla \times u\big). \end{cases} \tag{2.5}$$

Using the above notations and setting $F(t) = (f(t), g(t))$, we can write the weak form of problem (1.1)-(1.3) as

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} + Az(t) + Bz(t) + Rz(t) + Nz(t) = F(t), \;\; \text{in } \mathcal{D}'((\tau, +\infty), \mathbb{V}^*), \tag{2.6}$$

$$z(\tau) = z_\tau = (u_\tau, \omega_\tau), \; \tau \in \mathbb{R}. \tag{2.7}$$

Definition 2.1. *Let $z_\tau \in \mathbb{H}$, and let $z = (u,\omega) \in L^\infty(\tau, T; \mathbb{H}) \cap L^2(\tau, T; \mathbb{V})$ for any $T > \tau$, with $z|_{t=\tau} = z_\tau$. Then z is called a global weak solution of problem* (2.6)-(2.7)*, if for any $T > \tau$ and $t \in [\tau, T]$,*

$$\langle \frac{\mathrm{d}z(t)}{\mathrm{d}t}, \Phi\rangle + \langle Az(t), \Phi\rangle + \langle Bz(t), \Phi\rangle + \langle Nz(t), \Phi\rangle + \langle Rz(t), \Phi\rangle = \langle F(t), \Phi\rangle, \;\; \forall \Phi \in \mathbb{V},$$

holds in the sense of distributions in $\mathcal{D}'(\tau, T)$. A weak solution $z = (u,\omega)$ belonging to $L^\infty(\tau, T; \mathbb{V}) \cap L^2(\tau, T; D(A))$ for all $T > \tau$ is called a strong solution of problem (2.6)-(2.7).

We denote by λ_1 the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions, so that the following Poincaré's inequality

$$\lambda_1\|\xi\|^2 \leqslant \|\nabla\xi\|^2, \ \ \forall \xi \in H_0^1(\Omega) \tag{2.8}$$

holds. Besides the parameters appearing in system (1.1), c_1 in inequality (2.2) and λ_1 in (2.8), we define the following constants,

$$\delta_1 = \min\{2\mu_1 c_1 - \frac{\nu_r}{\lambda_1}, \frac{\nu_1}{2}, \frac{\nu_1\lambda_1}{2}\}, \ \ \delta_2 = 2\mu_1 + \nu_1 + 8\nu_r + \frac{12\nu_r}{\lambda_1}, \ \ \sigma = \lambda_1\delta_1. \tag{2.9}$$

We prove the global well-posedness of problem (2.6)-(2.7) under the following assumptions,

(H1) The parameters μ_1, ν_r, c_1 and λ_1 satisfy $2c_1\mu_1\lambda_1 > \nu_r$.

(H2) $F = (f, g)$ is in $L^2_{\text{loc}}(-\infty, +\infty; \mathbb{H})$ and

$$\int_{-\infty}^{t} e^{\sigma s}\|F(s)\|^2 \mathrm{d}s < +\infty, \ \ \text{for each} \ \ t \in \mathbb{R}. \tag{2.10}$$

We have the following lemma.

Lemma 2.1. *For the constants δ_1 and δ_2 given in (2.9) there holds*

$$\delta_1\|z\|^2_{\mathbb{V}} \leqslant \langle Az, z\rangle + \langle Rz, z\rangle \leqslant \delta_2\|z\|^2_{\mathbb{V}}, \ \ \forall z \in \mathbb{V}. \tag{2.11}$$

Proof. First, for any $u \in V$, we have by direct computations that

$$\nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \Delta u = -\Delta u,$$

which means that $\|\nabla \times u\|^2 = \|\nabla u\|^2$, $\forall u \in V$. Integrating by parts and using Cauchy's inequality we obtain

$$2\nu_r \int_\Omega \nabla \times \omega \cdot u \mathrm{d}x = 2\nu_r \int_\Omega (\nabla \times u)\omega \mathrm{d}x \leqslant 2\nu_r\|\omega\|^2 + \frac{\nu_r}{2}\|\nabla u\|^2, \ \ z = (u, \omega) \in \mathbb{V}. \tag{2.12}$$

Now, using (2.2) and the definition of operators A and R, we have for $z = (u, \omega) \in \mathbb{V}$ that

$$\langle Az, z\rangle = 2\mu_1\langle A_1 u, u\rangle + \nu_1(\nabla\omega, \nabla\omega) \geqslant 2\mu_1 c_1\|u\|^2_V + \nu_1\|\nabla\omega\|^2, \tag{2.13}$$

$$\langle Rz, z\rangle = -4\nu_r(\nabla \times u, \omega) + 4\nu_r\|\omega\|^2 \geqslant -\nu_r\|\nabla u\|^2. \tag{2.14}$$

It then follows from (2.2), (2.8) and (2.13)-(2.14) that

$$\begin{aligned}\langle Az, z\rangle + \langle Rz, z\rangle \geqslant& 2\mu_1 c_1\|u\|^2_V - \nu_r\|\nabla u\|^2 + \nu_1\|\nabla\omega\|^2 \\ \geqslant& \frac{1}{2}(\nu_1\|\nabla\omega\|^2 + \nu_1\lambda_1\|\omega\|^2) + (2\mu_1 c_1 - \frac{\nu_r}{\lambda_1})\|u\|^2_V \geqslant \delta_1\|z\|^2_{\mathbb{V}}.\end{aligned} \tag{2.15}$$

Finally, using Cauchy's and Poincaré's inequalities we obtain the second inequality of the lemma. $\square$

The following theorem is the main result of this section.

Theorem 2.1. *Suppose that assumptions (H1)-(H2) hold. Then for every $\tau \in \mathbb{R}$ and any $z_\tau \in \mathbb{H}$, there corresponds a unique global weak solution $z(t) = z(t, \tau; z_\tau)$ to problem (2.6)-(2.7) satisfying*

$$z(\cdot) \in L^\infty(\tau, T; \mathbb{H}) \cap L^2(\tau, T; \mathbb{V}) \cap \mathcal{C}([\tau, T]; \mathbb{H}), \ \ \forall T > \tau. \tag{2.16}$$

Moreover, the solution $z(t, \tau; z_\tau)$ depends continuously on the initial value z_τ.

Proof. The proof of the existence of weak solutions is standard. We first prove the local existence of the Galerkin approximate solutions, and then establish some *a priori* estimates of the Galerkin approximate solutions. Afterwards, we prove that the limit function of the Galerkin approximate solutions is a weak solution of problem (2.6)-(2.7) which satisfies (2.16). These are the usual procedures for proving the existence of weak solution via the Galerkin method. We omit these details here and only prove the continuously dependence of the solution on the initial value.

Let $z_1(t,\tau;z_{1\tau}) = (u_1(t,\tau;u_{1\tau}), \omega_1(t,\tau;\omega_{1\tau}))$ and $z_2(t,\tau;z_{2\tau}) = (u_2(t,\tau;u_{2\tau}), \omega_2(t,\tau;\omega_{2\tau}))$ be two weak solutions of problem (2.6)-(2.7) corresponding the initial values $z_{1\tau} = (u_{1\tau}, \omega_{1\tau})$ and $z_{2\tau} = (u_{2\tau}, \omega_{2\tau})$, respectively. Set $\tilde{z} = \tilde{z}(t) = z_1(t) - z_2(t) = (u_1(t) - u_2(t), \omega_1(t) - \omega_2(t))$. Then $\tilde{z}$ is a solution of the following problem

$$\frac{\mathrm{d}\tilde{z}(t)}{\mathrm{d}t} + A\tilde{z}(t) + Bz_1(t) - Bz_2(t) + R\tilde{z}(t) + N(z_1(t)) - N(z_2(t)) = 0, \ \text{ in } \mathcal{D}'((\tau,+\infty),\mathbb{V}^*), \tag{2.17}$$

$$\tilde{z}(\tau) = z_{1\tau} - z_{2\tau} = (u_{1\tau} - u_{2\tau}, \omega_{1\tau} - \omega_{2\tau}). \tag{2.18}$$

Using $\tilde{z}$ to take duality with (2.17) and then using (2.11), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}\|^2 + \delta_1\|\tilde{z}\|_{\mathbb{V}}^2 + \langle Bz_1 - Bz_2, \tilde{z}\rangle + \langle Nz_1 - Nz_2, \tilde{z}\rangle \leqslant 0. \tag{2.19}$$

We next estimate the nonlinear terms $\langle Bz_1 - Bz_2, \tilde{z}\rangle$ and $\langle Nz_1 - Nz_2, \tilde{z}\rangle$. In fact, by (2.3) we have

$$\begin{aligned}\langle Bz_1 - Bz_2, \tilde{z}\rangle =& \langle B_1(u_1,u_1) - B_1(u_2,u_2), u_1 - u_2\rangle + \langle B_2(u_1,\omega_1) - B_2(u_2,\omega_2), \omega_1 - \omega_2\rangle \\ =& b_1(u_1 - u_2, u_2, u_1 - u_2) + b_2(u_1 - u_2, \omega_2, \omega_1 - \omega_2).\end{aligned} \tag{2.20}$$

Now, from a classical estimation (see [12, (A.46d)]) we conclude that there is a positive constant c_2 such that

$$\begin{cases} |b_1(u_1 - u_2, u_2, u_1 - u_2)| \leqslant c_2\|u_1 - u_2\|_V^2\|u_2\|_V, \\ |b_2(u_1 - u_2, \omega_2, \omega_1 - \omega_2)| \leqslant c_2\|u_1 - u_2\|_V\|\omega_2\|_{1,2}\|\omega_1 - \omega_2\|_{1,2}. \end{cases} \tag{2.21}$$

We thus obtain from (2.20) and (2.21) that

$$|\langle Bz_1 - B(z_2), \tilde{z}\rangle| \leqslant c_2(\|u_2\|_V + \|\omega_2\|_{1,2})\|\tilde{z}\|_{\mathbb{V}}^2. \tag{2.22}$$

At the same time, by [4, (3.5)] we have

$$\langle Nz_1 - Nz_2, \tilde{z}\rangle = \langle N_1(u_1) - N_1(u_2), u_1 - u_2\rangle \geqslant 0. \tag{2.23}$$

It then follows from (2.19) and (2.22)-(2.23) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}\|^2 \leqslant \big(c_2(\|u_2\|_V + \|\omega_2\|_{1,2}) - \delta_1\big)\|\tilde{z}\|_{\mathbb{V}}^2. \tag{2.24}$$

Applying Gronwall's inequality to (2.24) and then using (2.18), we arrive at

$$\|\tilde{z}(t)\|^2 \leqslant \|z_{1\tau} - z_{2\tau}\|^2 \exp\int_\tau^t \big(c_2(\|u_2(s)\|_V + \|\omega_2(s)\|_{1,2}) - \delta_1\big)\mathrm{d}s. \tag{2.25}$$

The continuous dependence of the solution $z(t,\tau;z_\tau)$ on its initial value z_τ is deduced from (2.16) and (2.25). The proof is complete. □

Theorem 2.1 implies that the maps of solutions operator

$$U(t,\tau): z_\tau = (u_\tau, \omega_\tau) \in \mathbb{H} \longmapsto z(x,t) = (u(x,t), \omega(x,t)) \in \mathbb{H}, \;\; \forall t \geqslant \tau, \tag{2.26}$$

form a continuous process in space $\mathbb{H}$. In the next section, we will prove the existence of the pullback $\mathcal{D}_\sigma$-attractor for $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\mathbb{H}$.

3 Existence of the pullback $\mathcal{D}_\sigma$-attractor

In the sequel, we denote by $2^{\mathbb{H}}$ the collection of all subsets of $\mathbb{H}$ and consider families of non-empty sets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset 2^{\mathbb{H}}$ which is parameterized by time t. Especially, we use $\mathcal{D}_\sigma$ to denote the class of families $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subseteq 2^{\mathbb{H}}$ such that

$$\lim_{\tau\to-\infty} (e^{\sigma\tau} \sup_{w\in D(\tau)} \|w\|^2) = 0. \tag{3.1}$$

In addition, we will use the notation $a \lesssim b$ (also $a \gtrsim b$) to mean that $a \leqslant cb$ (also $a \geqslant cb$) for a universal constant $c > 0$ that only depends on the parameters coming from the problem.

We first recall some definitions concerning the pullback $\mathcal{D}_\sigma$-attractor.

Definition 3.1.

(1) *A family of sets $\widehat{D}_0 = \{D_0(s) : s \in \mathbb{R}\} \subseteq 2^{\mathbb{H}}$, with $D_0(s) \subset \mathbb{H}$ bounded for every $s \in \mathbb{R}$, is called a bounded pullback $\mathcal{D}_\sigma$-absorbing set for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\mathbb{H}$ if, for each $t \in \mathbb{R}$ and any $\widehat{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}_\sigma$ there exists a $\tau_0(t,\widehat{D}) \leqslant t$ such that $U(t,\tau)D(\tau) \subseteq D_0(t)$ for all $\tau \leqslant \tau_0(t,\widehat{D})$.*

(2) *The process $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be pullback $\mathcal{D}_\sigma$-asymptotically compact in $\mathbb{H}$ if, for each given $t \in \mathbb{R}$, any $\widehat{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}_\sigma$ and any sequence $\{\tau_n\}_{n\geqslant1}$ in $(-\infty,t]$ with $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{U(t,\tau_n)w(\tau_n)\}$ with any $w(\tau_n) \in D(\tau_n)$ possesses a convergent subsequence.*

(3) *A family of sets $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\} \subseteq 2^{\mathbb{H}}$ is said to be a pullback $\mathcal{D}_\sigma$-attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\mathbb{H}$ if it satisfies the following properties*

  (a) *Compactness: for any $t \in \mathbb{R}$, $\mathcal{A}(t)$ is a nonempty compact subset of $\mathbb{H}$;*

  (b) *Invariance: $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, $\forall\, \tau \leqslant t$;*

  (c) *Pullback attraction: $\hat{\mathcal{A}}$ is pullback $\mathcal{D}_\sigma$-attracting in the following sense*

$$\lim_{\tau\to-\infty} \mathrm{dist}_{\mathbb{H}}(U(t,\tau)D(\tau), \mathcal{A}(t)) = 0, \;\; \forall\, \widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\sigma, \;\; t \in \mathbb{R},$$

  where $\mathrm{dist}_{\mathbb{H}}(\cdot,\cdot)$ denotes the Hausdorff semidistance in $\mathbb{H}$.

Lemma 3.1. *Suppose that assumptions* (H1)-(H2) *hold. Then for any $z_\tau \in \mathbb{H}$, the corresponding solution $z = z(t;\tau,z_\tau)$ of problem* (2.6)-(2.7) *satisfies*

$$\|z(t;\tau,z_\tau)\|^2 \leqslant e^{-\sigma(t-\tau)}\|z_\tau\|^2 + \frac{e^{-\sigma t}}{\sigma}\int_\tau^t e^{\sigma s}\|F(s)\|^2 \mathrm{d}s, \ \ \forall t \geqslant \tau. \tag{3.2}$$

Proof. Let $z = z(t;\tau,z_\tau)$ be the solution of problem (2.6)-(2.7) corresponding to the initial value z_τ. Taking the dual pairing between z and (2.6) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + \langle Az, z\rangle + \langle Bz, z\rangle + \langle Rz, z\rangle + \langle Nz, z\rangle = \langle F(t), z\rangle.$$

Now, using (2.11), (2.23), the fact that $\langle Bz, z\rangle = 0$, and Cauchy's and Poincaré's inequalities we obtain

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 + \sigma\|z(t)\|^2 \leqslant \frac{\|F(t)\|^2}{\sigma}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 + \delta_1\|z(t)\|_{\mathbb{V}}^2 \leqslant \frac{\|F(t)\|^2}{\sigma}, \end{cases} \tag{3.3}$$

Then, applying Gronwall's inequality to $(3.3)_1$ we obtain (3.2). This ends the proof. □

From Lemma 3.1, we have

Lemma 3.2. *Let assumptions* (H1)-(H2) *hold. Then the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a bounded pullback $\mathcal{D}_\sigma$-absorbing set in $\mathbb{H}$.*

Proof. For each $t \in \mathbb{R}$, we define

$$r_0(t) = 1 + \frac{e^{-\sigma t}}{\sigma}\int_{-\infty}^t e^{\sigma s}\|F(s)\|^2 \mathrm{d}s.$$

Then from (3.1) and (3.2) we deduce that $\hat{\mathcal{B}}_0 = \{\mathcal{B}(0, \sqrt{r_0(t)}) : t \in \mathbb{R}\}$ is a bounded pullback $\mathcal{D}_\sigma$-absorbing set for $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\mathbb{H}$, where $\mathcal{B}(0, r_\sigma^{1/2}(t))$ is the ball in $\mathbb{H}$ which is centered at zero with radius $\sqrt{r_0(t)}$. □

To establish that $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback $\mathcal{D}_\sigma$-asymptotically compact in $\mathbb{H}$, we need some more estimates of the solutions.

Lemma 3.3. *Let assumptions* (H1)-(H2) *hold. Then for any $t \in \mathbb{R}$ and $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\sigma$, there exists some $\tau_1(\hat{D}, t) < t$ such that for any $\tau \leqslant \tau_1(\hat{D}, t)$ and $z_\tau \in D(\tau)$ the corresponding solution $z(\cdot;\tau,z_\tau)$ satisfies*

$$\|z(r;\tau,z_\tau)\|^2 \lesssim \eta_1(t) := 1 + e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\|F(s)\|^2 \mathrm{d}s, \ \ \forall r \in [t-2, t], \tag{3.4}$$

$$\int_{r-1}^r \|z(s;\tau,z_\tau)\|_{\mathbb{V}}^2 \mathrm{d}s \lesssim \eta_2(t) := \max_{\theta\in[t-2,t]} \eta_1(\theta) + \int_{t-2}^t \|F(s)\|^2 \mathrm{d}s, \ \ \forall r \in [t-1, t], \tag{3.5}$$

$$\int_{r-1}^r \|\frac{\mathrm{d}z(s;\tau,z_\tau)}{\mathrm{d}s}\|_{\mathbb{V}^*}^2 \mathrm{d}s \lesssim \eta_3(t) := \big(1 + \max_{\theta\in[t-2,t]} \eta_1(\theta)\big)\eta_2(t) + \int_{t-2}^t \|F(s)\|^2 \mathrm{d}s, \ \ \forall r \in [t-1, t]. \tag{3.6}$$

Proof. The estimation (3.4) is directly deduced from (3.1) and (3.2). Now from $(3.3)_2$ we can derive

$$\|z(r)\|^2 + \delta_1\int_{r-1}^r \|z(s)\|_{\mathbb{V}}^2 \mathrm{d}s \lesssim \|z(r-1)\|^2 + \int_{r-1}^r \|F(s)\|^2 \mathrm{d}s,$$

which gives (3.5). In order to prove (3.6), we need to estimate the terms $\|Az\|_{\mathbb{V}^*}$, $\|Bz\|_{\mathbb{V}^*}$, $\|Rz\|_{\mathbb{V}^*}$ and $\|Nz\|_{\mathbb{V}^*}$. In fact, for any solution $z(\cdot;\tau,z_\tau)$ satisfying (2.16), we have by $(2.5)_1$ and (2.1) that for any $\Phi=(\varphi,\phi)\in\mathbb{V}$,

$$|\langle Az,\Phi\rangle|\leqslant|\langle 2\mu_1A_1u,\varphi\rangle+\langle\nu_1A_2\omega,\phi\rangle|\lesssim\|u\|_V\|\varphi\|_V+\|\omega\|_{1,2}\|\phi\|_{1,2}\lesssim\|z\|_{\mathbb{V}}\|\Phi\|_{\mathbb{V}}. \tag{3.7}$$

By $(2.5)_2$ and [12, (A.26d), (A.26e)], we deduce

$$|\langle Bz,\Phi\rangle|\leqslant|\langle B_1(u,u),\varphi\rangle+\langle B_2(u,\omega),\phi\rangle|\lesssim\|u\|(\|u\|_V\|\varphi\|_V+\|\omega\|_{1,2}\|\phi\|_{1,2})\lesssim\|z\|\|z\|_{\mathbb{V}}\|\Phi\|_{\mathbb{V}}. \tag{3.8}$$

From $(2.5)_3$ and the definition of $N_1(\cdot)$ we obtain

$$|\langle Nz,\Phi\rangle|=|\langle N_1(u),\varphi\rangle|\lesssim\|u\|_V\|\varphi\|_V\lesssim\|z\|_{\mathbb{V}}\|\Phi\|_{\mathbb{V}}, \tag{3.9}$$

and from $(2.5)_4$,

$$|\langle Rz,\Phi\rangle|\leqslant|\langle\nabla\times\omega,\varphi\rangle|+|\langle\omega,\phi\rangle|+|\langle\nabla\times u,\phi\rangle|\lesssim\|z\|_{\mathbb{V}}\|\Phi\|_{\mathbb{V}}. \tag{3.10}$$

It then follows from (3.7)-(3.10) that

$$\begin{cases}\|Az\|_{\mathbb{V}^*}\lesssim\|z\|_{\mathbb{V}},\\ \|Bz\|_{\mathbb{V}^*}\lesssim\|z\|\|z\|_{\mathbb{V}},\\ \|Nz\|_{\mathbb{V}^*}\lesssim\|z\|_{\mathbb{V}},\\ \|Rz\|_{\mathbb{V}^*}\lesssim\|z\|_{\mathbb{V}}.\end{cases} \tag{3.11}$$

Since $\mathbb{V}^*\hookrightarrow\mathbb{H}$, (2.6) and (3.11) imply that

$$\begin{aligned}\|\frac{\mathrm{d}z(\theta)}{\mathrm{d}t}\|_{\mathbb{V}^*}\lesssim&\|Az(\theta)\|_{\mathbb{V}^*}+\|Bz(\theta)\|_{\mathbb{V}^*}+\|Rz(\theta)\|_{\mathbb{V}^*}+\|Nz(\theta)\|_{\mathbb{V}^*}+\|F(\theta)\|_{\mathbb{V}^*}\\ \lesssim&\|z(\theta)\|_{\mathbb{V}}+\|z(\theta)\|\|z(\theta)\|_{\mathbb{V}}+\|F(\theta)\|,\end{aligned}$$

which gives

$$\|\frac{\mathrm{d}z(\theta)}{\mathrm{d}t}\|^2_{\mathbb{V}^*}\lesssim\|z(\theta)\|^2_{\mathbb{V}}+\|z(\theta)\|^2\|z(\theta)\|^2_{\mathbb{V}}+\|F(\theta)\|^2. \tag{3.12}$$

Using (3.4) and (3.5), we obtain (3.6) by integrating (3.12) over $[r-1,r]$. The proof is complete. □

Lemma 3.4. *Let assumptions* (H1)-(H2) *hold. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\mathcal{D}_\sigma$*-asymptotically compact in* $\mathbb{H}$.

Proof. Consider given $t\in\mathbb{R}$, $\hat{D}=\{D(t):t\in\mathbb{R}\}\in D_\sigma$, $\{\tau_n\}\subset(-\infty,t]$ with $\tau_n\to-\infty$ as $n\to\infty$, and $\{z_{\tau_n}\}\subset\mathbb{H}$ with $z_{\tau_n}\in D(\tau_n)$ for each n. Set $z^{(n)}=z^{(n)}(\cdot)=z^{(n)}(\cdot,\tau_n;z_{\tau_n})$. We shall prove that the sequence $\{z^{(n)}(t)\}$ possesses a convergent subsequence in $\mathbb{H}$.

Indeed, from Lemma 3.3, we see that there exists a $\tau_1(\hat{D},t)<t-2$ such that the sequence $\{z^{(n)}(\cdot):\tau_n\leqslant\tau_1(\hat{D},t)\}$ is uniformly bounded in $L^\infty(t-2,t;\mathbb{H})\cap L^2(t-2,t;\mathbb{V})$, and $\{\frac{\mathrm{d}}{\mathrm{d}s}z^{(n)}(s)\}$ is uniformly bounded in $L^2(t-2,t;\mathbb{V}^*)$. By the Aubin-Lions compactness lemma (see [17]), there

exists an element $z(\cdot) \in L^\infty(t-2,t;\mathbb{H}) \cap L^2(t-2,t;\mathbb{V})$, with $\dfrac{\mathrm{d}z(s)}{\mathrm{d}s} \in L^2(t-2,t;\mathbb{V}^*)$, such that for a subsequence of $z^{(n)}(\cdot)$ we have the following convergent relations:

$$\begin{cases} z^{(n)}(\cdot) \rightharpoonup^* z(\cdot) \ \text{ in } \ L^\infty(t-2,t;\mathbb{V}), \\ z^{(n)}(\cdot) \rightharpoonup z(\cdot) \ \text{ in } \ L^2(t-2,t;\mathbb{V}), \\ \frac{\mathrm{d}}{\mathrm{d}s} z^{(n)}(s) \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}s} z(s) \ \text{ in } \ L^2(t-2,t;\mathbb{V}^*), \\ z^{(n)}(\cdot) \to z(\cdot) \ \text{ in } \ L^2(t-2,t;\mathbb{V}), \\ z^{(n)}(s) \to z(s) \ \text{ in } \ \mathbb{V}, \text{ a.e.}\, s \in [t-2,t]. \end{cases} \tag{3.13}$$

Now, from (3.13) and the embedding theorems [10, P_{32}, Theorem 1.5; P_{33}, Theorem 1.8] we conclude that

$$\begin{cases} z^{(n)}(\cdot) \to z(\cdot) \ \text{ in } \ C([t-2,t];\mathbb{V}^*), \\ z^{(n)}(\cdot) \ \text{ is uniformly bounded in } \ C([t-2,t];\mathbb{V}), \\ z(\cdot) \in C([t-2,t];\mathbb{V}). \end{cases} \tag{3.14}$$

Now, (3.14) implies

$$z^{(n)}(s_n) \rightharpoonup z(s_*) \ \text{ in } \ \mathbb{V}, \ \ \forall \{s_n\} \subset [t-2,t] \ \text{ with } \ s_n \to s_*. \tag{3.15}$$

We next establish that

$$z^{(n)}(\cdot) \to z(\cdot) \ \text{ in } \ C([t-2,t];\mathbb{V}), \ \text{ as } \ n \to \infty. \tag{3.16}$$

Indeed, if (3.16) does not hold, then there exist an $\epsilon_0 > 0$ and a sequence $\{t_n\} \subset [t-2,t]$ converging to some $t_* \in [t-2,t]$ such that

$$\|z^{(n)}(t_n) - z(t_*)\| \geqslant \epsilon_0, \ \ n \geqslant 1. \tag{3.17}$$

Since $\mathbb{H}$ is a Hilbert space, (3.17) will contradict with

$$\begin{cases} \|z(t_*)\| \leqslant \liminf\limits_{n\to\infty} \|z^{(n)}(t_n)\|, \\ \|z(t_*)\| \geqslant \limsup\limits_{n\to\infty} \|z^{(n)}(t_n)\|. \end{cases} \tag{3.18}$$

Now, the first inequality in (3.18) is deduced from the lower semi-continuity of the norm and (3.15). So we just need to prove the second one. From (3.13)-(3.14) we see that $z(\cdot)$ satisfies (2.6) and $(3.3)_2$ on any subinterval of $[t-2,t]$. For $s_1, s_2 \in [t-2,t]$ with $s_1 \leqslant s_2$, we integrate $(3.3)_2$ over $[s_1,s_2]$, for $z^{(n)}(\cdot)$ and $z(\cdot)$ respectively, and deduce

$$\begin{cases} \|z^{(n)}(s_2)\|^2 + \delta_1 \displaystyle\int_{s_1}^{s_2} \|z^{(n)}(s)\|_{\mathbb{V}}^2 \mathrm{d}s \leqslant \frac{1}{\sigma} \int_{s_1}^{s_2} \|F(s)\|^2 \mathrm{d}s + \|z^{(n)}(s_1)\|^2, \\ \|z(s_2)\|^2 + \delta_1 \displaystyle\int_{s_1}^{s_2} \|z(s)\|_{\mathbb{V}}^2 \mathrm{d}s \leqslant \frac{1}{\sigma} \int_{s_1}^{s_2} \|F(s)\|^2 \mathrm{d}s + \|z(s_1)\|^2. \end{cases} \tag{3.19}$$

Set

$$\begin{cases} J_n(s) = \|z^{(n)}(s)\|^2 - \dfrac{1}{\sigma} \displaystyle\int_{t-2}^{s} \|F(\theta)\|^2 \mathrm{d}\theta, \\ J(s) = \|z(s)\|^2 - \dfrac{1}{\sigma} \displaystyle\int_{t-2}^{s} \|F(\theta)\|^2 \mathrm{d}\theta. \end{cases} \tag{3.20}$$

Then $J_n(s)$ and $J(s)$ are obviously continuous on $[t-2,t]$ and (3.19) implies

$$\begin{cases} J_n(s_2)-J_n(s_1)=\|z^{(n)}(s_2)\|^2-\|z^{(n)}(s_1)\|^2-\dfrac{1}{\sigma}\displaystyle\int_{s_1}^{s_2}\|F(s)\|^2\mathrm{d}s\leqslant-\delta_1\int_{s_1}^{s_2}\|z^{(n)}(s)\|_{\mathbb{V}}^2\leqslant 0,\\ J(s_2)-J(s_1)=\|z(s_2)\|^2-\|z(s_1)\|^2-\dfrac{1}{\sigma}\displaystyle\int_{s_1}^{s_2}\|F(s)\|^2\mathrm{d}s\leqslant-\delta_1\int_{s_1}^{s_2}\|z(s)\|_{\mathbb{V}}^2\leqslant 0,\end{cases}$$

which means that both $J_n(\cdot)$ and $J(\cdot)$ are non-increasing functions on $[t-2,t]$. Now, $(3.13)_5$ gives

$$J_n(s)\to J(s),\ \text{ a.e. }\ s\in[t-2,t].$$

Thus, there exists a sequence $\{t_j^*\}\subset[t-2,t_*]$ satisfying $t_j^*\to t_*$ as $j\to\infty$, which yields

$$\lim_{n\to\infty}J_n(t_j^*)=J(t_j^*),\ \text{ for each }\ j.$$

Note that $J(\cdot)$ is continuous on $[t-2,t]$. For any $\kappa>0$, there exists some positive integer N_κ such that

$$J(t_j^*)-J(t_*)<\frac{\kappa}{2},\ \ \forall j\geqslant N_\kappa. \tag{3.21}$$

For above N_κ, we pick $j>N_\kappa$ large enough such that

$$t_j\geqslant t_{N_\kappa}^*\ \text{ and }\ |J_j(t_{N_\kappa}^*)-J(t_{N_\kappa}^*)|<\frac{\kappa}{2}. \tag{3.22}$$

Then, from (3.21)-(3.22) and the fact that $J_j(\cdot)$ is non-increasing for each j we arrive at

$$\begin{aligned} J_j(t_j)-J(t_*)\leqslant& J_j(t_{N_\kappa}^*)-J(t_*)\leqslant|J_j(t_{N_\kappa}^*)-J(t_*)|\\ \leqslant&|J_j(t_{N_\kappa}^*)-J(t_{N_\kappa}^*)|+|J(t_{N_\kappa}^*)-J(t_*)|<\kappa,\ \ \forall j\geqslant N_\kappa,\end{aligned}$$

which, together with the arbitrariness of κ, means that

$$\lim_{j\to\infty}J_j(t_j)\leqslant J(t_*). \tag{3.23}$$

The second inequality in (3.18) is deduced from (3.20) and (3.23). The proof is complete. □

Now, we combine Lemma 3.2, Lemma 3.4 and Theorem 3.11 in [14] to obtain the main result of this section.

Theorem 3.1. *Let assumptions* (H1)-(H2) *hold. Then the continuous process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *possesses a pullback* $\mathcal{D}_\sigma$*-attractor* $\hat{\mathcal{A}}=\{\mathcal{A}(t):t\in\mathbb{R}\}$ *in* $\mathbb{H}$.

4 Existence of invariant measures and statistical solutions

In this section, we will first prove that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a family of invariant Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ in $\mathbb{H}$, and then establish that $\{m_t\}_{t\in\mathbb{R}}$ meets the Liouville-type theorem and is indeed a statistical solution for the non-Newtonian micropolar fluids equations.

We first establish that the $\mathbb{H}$-valued function $\tau\longmapsto U(t_*,\tau)z_*$ satisfies the so-called τ-continuity, that is, $\tau\longmapsto U(t_*,\tau)z_*$ is continuous and bounded on $(-\infty,t_*]$ for each $t_*\in\mathbb{R}$ and $z_*\in\mathbb{H}$. We next establish the following lemma.

Lemma 4.1. *Let assumptions* (H1)-(H2) *hold, and z_* and t_* be given. Then for any $\epsilon > 0$, there exists $\rho = \rho(\epsilon, z_*, t_*) > 0$ such that*

$$\|U(t_*, s)z_* - z_*\|^2 < \epsilon, \ \ \forall s \in (t_* - \rho, t_*). \tag{4.1}$$

Proof. Let z_* and t_* be fixed. Then for any $s \in (t_* - 1, t_*)$, we deduce from $(3.3)_2$ that

$$\int_s^{t_*} \frac{\mathrm{d}}{\mathrm{d}\theta}\|U(\theta, s)z_*\|^2 \mathrm{d}\theta + \delta_1 \int_s^{t_*} \|U(\theta, s)z_*\|_{\mathbb{V}}^2 \mathrm{d}\theta \leqslant \frac{1}{\sigma}\int_s^{t_*} \|F(\theta)\|^2 \mathrm{d}\theta, \tag{4.2}$$

which gives that

$$\begin{cases} \displaystyle\int_s^{t_*} \frac{\mathrm{d}}{\mathrm{d}\theta}\|U(\theta, s)z_*\|^2 \mathrm{d}\theta \leqslant \frac{1}{\sigma}\int_{t_*-1}^{t_*} \|F(\theta)\|^2 \mathrm{d}\theta := c_3, \\ \displaystyle\|U(t_*, s)z_*\|^2 \leqslant \frac{1}{\sigma}\int_{t_*-1}^{t_*} \|F(\theta)\|^2 \mathrm{d}\theta + \|z_*\|^2 := c_4, \\ \displaystyle\int_s^{t_*} \|U(\theta, s)z_*\|_{\mathbb{V}}^2 \mathrm{d}\theta \leqslant \frac{1}{\sigma\delta_1}\int_{t_*-1}^{t_*} \|F(\theta)\|^2 \mathrm{d}\theta + \frac{1}{\delta_1}\|z_*\|^2 := c_5, \end{cases} \tag{4.3}$$

and obviously all the constants c_3, c_4 and c_5 are independent of s. Now we observe that

$$\begin{aligned} \|U(t_*, s)z_* - z_*\|^2 =& \big(U(t_*, s)z_* - z_*, U(t_*, s)z_* - z_*\big) \\ =& \int_s^{t_*} \frac{\mathrm{d}}{\mathrm{d}\theta}\|U(\theta, s)z_*\|^2 \mathrm{d}\theta - 2(U(t_*, s)z_* - z_*, z_*). \end{aligned} \tag{4.4}$$

By (H2), $F(\cdot) \in L^2_{\mathrm{loc}}(-\infty, +\infty; \mathbb{H})$. Hence, for any $\epsilon > 0$ we deduce from (4.2) that there exists $\rho_1 = \rho_1(\epsilon, t_*)$ such that

$$\int_s^{t_*} \frac{\mathrm{d}}{\mathrm{d}\theta}\|U(\theta, s)z_*\|^2 \mathrm{d}\theta \leqslant \frac{1}{\sigma}\int_s^{t_*} \|F(\theta)\|^2 \mathrm{d}\theta \leqslant \frac{1}{\sigma}\int_{t_*-\rho_1}^{t_*} \|F(\theta)\|^2 \mathrm{d}\theta \leqslant \frac{\epsilon}{2}, \ \ \forall s \in (t_* - \rho_1, t_*). \tag{4.5}$$

At the same time, $\mathbb{V}$ is dense in $\mathbb{H}$. Thus, for above $\epsilon > 0$ there exists a $\tilde{z} \in \mathbb{V}$ yielding

$$\|z_* - \tilde{z}\| \leqslant \frac{\epsilon}{8(c_4 + \|z_*\|)}.$$

Therefore, we have

$$\begin{aligned} |-(U(t_*, s)z_* - z_*, z_*)| \leqslant& |(U(t_*, s)z_* - z_*, z_* - \tilde{z})| + |\langle U(t_*, s)z_* - z_*, \tilde{z}\rangle| \\ \leqslant& (\|U(t_*, s)z_*\| + \|z_*\|)\|z_* - \tilde{z}\| + |\langle U(t_*, s)z_* - z_*, \tilde{z}\rangle| \\ \leqslant& (c_4 + \|z_*\|)\|z_* - \tilde{z}\| + |\langle U(t_*, s)z_* - z_*, \tilde{z}\rangle| \\ \leqslant& |\langle U(t_*, s)z_* - z_*, \tilde{z}\rangle| + \frac{\epsilon}{8}. \end{aligned} \tag{4.6}$$

We also observe that

$$\begin{aligned} \big|\langle U(t_*, s)z_* - z_*, \tilde{z}\rangle\big| =& \Big|\Big\langle \int_s^{t_*} \frac{\mathrm{d}U(\theta, s)z_*}{\mathrm{d}\theta}\mathrm{d}\theta, \tilde{z}\Big\rangle\Big| \leqslant \|\tilde{z}\|_{\mathbb{V}} \Big\|\int_s^{t_*} \frac{\mathrm{d}U(\theta, s)z_*}{\mathrm{d}\theta}\mathrm{d}\theta\Big\|_{\mathbb{V}^*} \\ \leqslant& \|\tilde{z}\|_{\mathbb{V}} \int_s^{t_*} \Big\|\frac{\mathrm{d}U(\theta, s)z_*}{\mathrm{d}\theta}\Big\|_{\mathbb{V}^*} \mathrm{d}\theta \\ \leqslant& \|\tilde{z}\|_{\mathbb{V}} (t_* - s)^{\frac{1}{2}} \Big(\int_s^{t_*} \Big\|\frac{\mathrm{d}U(\theta, s)z_*}{\mathrm{d}\theta}\Big\|_{\mathbb{V}^*}^2 \mathrm{d}\theta\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

(3.12) and (4.3) imply that

$$\Big(\int_s^{t_*}\|\frac{\mathrm{d}U(\theta,s)z_*}{\mathrm{d}\theta}\|_{\mathbb{V}^*}^2\mathrm{d}\theta\Big)^{1/2} \leqslant \Big(\int_s^{t_*}[(1+c_4)\|U(\theta,s)z_*\|_{\mathbb{V}}^2+2\|F(\theta)\|^2]\mathrm{d}\theta\Big)^{1/2} \leqslant \big((1+c_4)c_4+2\sigma c_3\big)^{1/2}. \tag{4.8}$$

Inserting (4.8) into (4.7) yields

$$\big|\big\langle U(t_*,s)z_*-z_*,\tilde{z}\big\rangle\big| \leqslant ((1+c_4)c_4+2\sigma c_3)^{1/2}\|\tilde{z}\|_{\mathbb{V}}(t_*-s)^{1/2}. \tag{4.9}$$

Therefore, for above $\epsilon>0$, we have

$$\big|\big\langle U(t_*,s)z_*-z_*,\tilde{z}\big\rangle\big| \leqslant \frac{\epsilon}{8},\ \ \forall s\in(t_*-\rho_2,t_*) \tag{4.10}$$

with $\rho_2=\frac{\epsilon^2}{8((1+c_4)c_4+2\sigma c_3)\|\tilde{z}\|_{\mathbb{V}}^2)}$ being independent of s. We obtain (4.1) from (4.4)-(4.6) and (4.10) by choosing $\rho=\min\{\rho_1,\rho_2\}$. The proof of Lemma 4.1 is complete. □

Lemma 4.2. *Let assumptions* (H1)-(H2) *hold. Then for any given* $z_*\in\mathbb{H}$ *and any* $t_*\in\mathbb{R}$*, the* $\mathbb{H}$*-valued mapping* $\tau\longmapsto U(t_*,\tau)z_*$ *is bounded and continuous on* $(-\infty,t_*]$.

Proof. Let $z_*\in\mathbb{H}$ and $t_*\in\mathbb{R}$ be fixed. Then (3.2) gives

$$\|U(t_*,\tau)z_*\|^2\leqslant\|z_*\|^2+\frac{e^{-\sigma t_*}}{\sigma}\int_{-\infty}^{t_*}e^{-\sigma\theta}\|F(\theta)\|^2\mathrm{d}\theta,\ \ \forall\tau\in(-\infty,t_*]. \tag{4.11}$$

The right-hand side of (4.11) is a bounded quantity which is independent of τ, and therefore the mapping $\tau\longmapsto U(t_*,\tau)z_*$ is bounded on $(-\infty,t_*]$. Next we prove that the mapping $\tau\longmapsto U(t_*,\tau)z_*$ is both left-continuous and right-continuous at each $s_*\in(-\infty,t_*]$.

Firstly, we prove the right-continuity. Let $s_*+1>s>s_*$ and $s<t_*$. By the invariance property of the process and (2.25), we have

$$\begin{aligned}\|U(t_*,s)z_*-U(t_*,s_*)z_*\|^2 =&\|U(t_*,s)U(s_*,s_*)z_*-U(t_*,s)U(s,s_*)z_*\|^2\\ \leqslant&\|U(s,s_*)z_*-z_*\|^2\exp\big\{(2c_2\int_s^{t_*}\|U(\theta,s_*)z_*\|_{\mathbb{V}}-\delta_1)\mathrm{d}\theta\big\}\\ \lesssim&\|U(s,s_*)z_*-z_*\|^2,\end{aligned} \tag{4.12}$$

where we have also used the fact that $\max\limits_{s\in[s_*-1,s_*]}\exp\big\{(2c_2\int_s^{t_*}\|U(\theta,s_*)z_*\|_{\mathbb{V}}-\delta_1)\mathrm{d}\theta\big\}$ is a bounded quantity which is independent of s. Now (2.16) shows that $U(\cdot,s_*)z_*\in C([s_*,t_*];\hat{H})$. Therefore, (4.12) implies that $U(t_*,\cdot)z_*$ is right-continuous at s_*.

Secondly, we establish the left-continuity. Let $s_*-1<s<s_*<t_*$. Again, by using (2.25) and the invariance property of the process, we have

$$\begin{aligned}\|U(t_*,s)z_*-U(t_*,s_*)z_*\|^2 =&\|U(t_*,s_*)U(s_*,s)z_*-U(t_*,s_*)U(s,s)z_*\|^2\\ \leqslant&\|U(s_*,s)z_*-z_*\|^2\exp\big\{(2c_2\int_{s_*}^{t_*}\|U(\theta,s_*)z_*\|_{\mathbb{V}}-\delta_1)\mathrm{d}\theta\big\}\\ \lesssim&\|U(s_*,s)z_*-z_*\|^2,\end{aligned} \tag{4.13}$$

where we have also used the fact that $\exp\big\{(2c_2\int_{s_*}^{t_*}\|U(\theta,s_*)z_*\|_{\mathbb{V}}-\delta_1)\mathrm{d}\theta\big\}$ is a bounded quantity which does not depend on s. Lemma 4.1 and (4.13) show that $U(t_*,\cdot)z_*$ is left-continuous at s_*. The proof of Lemma 4.2 is complete. □

We next reformulate the definition of generalized Banach limit (cf. [12, 19]) such that it can be used directly in our investigations.

Definition 4.1. *A generalized Banach limit is any linear functional, denoted by* $\mathrm{LIM}_{t\to-\infty}$*, defined on the space of all bounded real-valued functions on* $(-\infty,+\infty)$ *and satisfying*

(1) $\mathrm{LIM}_{t\to-\infty}h(t)\geqslant 0$ *for nonnegative functions* $h(\cdot)$ *on* $(-\infty,+\infty)$*;*

(2) $\mathrm{LIM}_{t\to-\infty}h(t)=\lim\limits_{t\to-\infty}h(t)$ *if the usual limit* $\lim\limits_{t\to-\infty}h(t)$ *exists.*

Combining Theorems 2.1 and 3.1, together with Lemma 4.2 and [19, Theorem 3.1], we obtain

Theorem 4.1. *Suppose assumptions* (H1)-(H2) *hold. Let* $\{U(t,\tau)\}_{t\geqslant\tau}$ *be the process defined via* (2.26) *and* $\hat{\mathcal{A}}=\{\mathcal{A}(t):t\in\mathbb{R}\}$ *be the pullback* $\mathcal{D}_\sigma$*-attractor obtained in Theorem 3.1. Then for a given generalized Banach limit* $\mathrm{LIM}_{t\to-\infty}$ *and a continuous map* $\zeta(\cdot):\mathbb{R}\mapsto\mathbb{H}$ *with* $\zeta(\cdot)\in\mathcal{D}_\sigma$*, there exists a unique family of Borel probability measures* $\{m_t\}_{t\in\mathbb{R}}$ *in* $\mathbb{H}$ *such that the support of the measure* m_t *is contained in* $\mathcal{A}(t)$ *and*

$$\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\Upsilon\big(U(t,\theta)\zeta(\theta)\big)\mathrm{d}\theta=\int_{\mathcal{A}(t)}\Upsilon(z)\mathrm{d}m_t(z)=\int_{\mathbb{H}}\Upsilon(z)\mathrm{d}m_t(z) \tag{4.14}$$

$$=\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\int_{\mathbb{H}}\Upsilon\Big(U(t,\theta)z\Big)\mathrm{d}m_\theta(z)\mathrm{d}\theta, \tag{4.15}$$

for any $\Upsilon(\cdot)\in C(\mathbb{H})$ *(the set of all real-valued continuous functions defined on* $\mathbb{H}$*). Furthermore,* m_t *is invariant in the sense that*

$$\int_{\mathcal{A}(t)}\Upsilon(z)\mathrm{d}m_t(z)=\int_{\mathcal{A}(s)}\Upsilon\big(U(t,s)z\big)\mathrm{d}m_s(z),\ \ t\geqslant s. \tag{4.16}$$

To formulate the definition of statistical solution, we first introduce the concept of the class $\mathcal{T}$ of test functions. We write equation (2.6) as

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\mathcal{H}(z,t)=F(t)-Az(t)-Bz(t)-Rz(t)-Nz(t). \tag{4.17}$$

Observe that $\mathcal{H}(z,t):\mathbb{V}\times\mathbb{R}\longmapsto\mathbb{V}^*$.

Definition 4.2. (cf. [34, Definition 3.3]) *The class* $\mathcal{T}$ *of test functions is the set of real-valued functionals* $\Gamma=\Gamma(\cdot)$ *defined on* $\mathbb{H}$ *that are bounded on bounded subset of* $\mathbb{H}$ *and satisfy*

(a) *For any* $z\in\mathbb{V}$*, the Fréchet derivative* $\Gamma'(z)$ *exists, that is, for each* $z\in\mathbb{V}$ *there exists an element* $\Gamma'(z)$ *such that*

$$\frac{|\Gamma(z+w)-\Gamma(z)-(\Gamma'(z),w)|}{\|w\|}\longrightarrow 0\ \ as\ \ \|w\|_{\mathbb{V}}\to 0,\ \ w\in\mathbb{V};$$

(b) $\Gamma'(z)\in\mathbb{V}$ *for all* $z\in\mathbb{V}$*, and the mapping* $z\longmapsto\Gamma'(z)$ *is continuous and bounded as a function from* $\mathbb{V}$ *to* $\mathbb{V}$*;*

(c) *Every global weak solution $z(t)$ of equation (2.6) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(z(t)) = \langle \Gamma'(z(t)), \mathcal{H}(z(t),t)\rangle. \tag{4.18}$$

Now we can prove the following theorem.

Theorem 4.2. *Let assumptions* (H1)-(H2) *hold. Then the family of Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 is a statistical solution of equation* (2.6) *in the following sense:*

(a) *For every $\Upsilon \in C(\mathbb{H})$, the mapping $t \mapsto \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_t(z)$ is continuous;*

(b) *For almost $t \in \mathbb{R}$, the mapping $z \mapsto \big\langle y, \mathcal{H}(z,t)\big\rangle$ is m_t-integrable for every $y \in \mathbb{V}$. Moreover, the map $t \mapsto \int_{\mathbb{H}} \big\langle y, \mathcal{H}(z,t)\big\rangle \mathrm{d}m_t(z)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $y \in \mathbb{V}$;*

(c) *Any test function $\Gamma \in \mathcal{T}$ satisfies*

$$\int_{\mathbb{H}} \Gamma(z)\mathrm{d}m_t(z) - \int_{\mathbb{H}} \Gamma(z)\mathrm{d}m_s(z) = \int_s^t \int_{\mathbb{H}} \big\langle \Gamma'(z), \mathcal{H}(z,\theta)\big\rangle \mathrm{d}m_\theta(z)\mathrm{d}\theta, \ \ \forall\, t \geqslant s.$$

Proof. We first prove item (a). Let $s_* \in \mathbb{R}$ and $\Upsilon \in C(\mathbb{H})$ be given. We need to prove that

$$\begin{cases} \lim\limits_{s\to s_*^+} \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_s(z) = \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_{s_*}(z), \\ \lim\limits_{s\to s_*^-} \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_s(z) = \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_{s_*}(z). \end{cases} \tag{4.19}$$

By (4.14) and (4.16) we have

$$\begin{cases} \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_s(z) - \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_{s_*}(z) = \int_{\mathcal{A}(s_*)} \big(\Upsilon(U(s,s_*)z) - \Upsilon(z)\big)\mathrm{d}m_{s_*}(z), \ \ s > s_*, \\ \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_{s_*}(z) - \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_s(z) = \int_{\mathcal{A}(s)} \big(\Upsilon(U(s_*,s)z) - \Upsilon(z)\big)\mathrm{d}m_s(z), \ \ s < s_*. \end{cases} \tag{4.20}$$

Form (2.16) and Lemma 4.1 we see that

$$\begin{cases} U(s,s_*)z \to z \ \text{ strongly in } \mathbb{H} \text{ as } s \to s_*^+, \\ U(s_*,s)z \to z \ \text{ strongly in } \mathbb{H} \text{ as } s \to s_*^-, \end{cases} \tag{4.21}$$

which, together with the facts that $\mathcal{A}(\cdot)$ is compact in $\mathbb{H}$ and $\Upsilon \in C(\mathbb{H})$, gives (4.19).

We next establish item (b). For every $y = (y_u, y_\omega) \in \mathbb{V}$, we define

$$\Upsilon_y(w) = \big\langle y, \mathcal{H}(w,s)\big\rangle, \ \ \forall\, w = (u,\omega) \in \mathbb{V}. \tag{4.22}$$

We claim that $\Upsilon_y(\cdot) \in C(\mathbb{V})$. Indeed, let $w_* = (u_*, \omega_*) \in \mathbb{V}$ be fixed and consider $w = (u,\omega) \in \mathbb{V}$ with $\|w_* - w\|_{\mathbb{V}} \leqslant 1$. By (3.11) we have

$$\begin{aligned} \big|\Upsilon_y(w_*) - \Upsilon_y(w)\big| &= \big|\big\langle y, \mathcal{H}(w_*,s) - \mathcal{H}(w,s)\big\rangle\big| \\ &\leqslant \big|\big\langle y, A(w_*-w) + R(w_*-w)\big\rangle\big| + \big|\big\langle y, Bw_* - Bw + Nw_* - Nw\big\rangle\big| \\ &\lesssim \|y\|_{\mathbb{V}}\|w_*-w\|_{\mathbb{V}} + \big|\big\langle y, Bw_* - Bw\big\rangle\big| + \big|\big\langle y, Nw_* - Nw\big\rangle\big|. \end{aligned} \tag{4.23}$$

We shall estimate the last two terms on the right-hand side of (4.23). By (2.3), [12, A.26e] and the definitions of the operators $B_1(\cdot,\cdot)$ and $B_2(\cdot,\cdot)$, we obtain

$$\begin{aligned}
|\langle y, Bw_* - Bw\rangle| \leqslant & |b_1(u_*, y_u, u_* - u)| + |b_1(u_* - u, y_u, u)| + |b_2(\omega_*, y_\omega, \omega_* - \omega)| + |b_2(\omega_* - \omega, y_\omega, \omega)| \\
\lesssim & (\|u_*\|_V + \|u\|_V)\|y_u\|_V\|u_* - u\|_V + (\|\omega_*\|_{1,2} + \|\omega\|_{1,2})\|y_\omega\|_{1,2}\|\omega_* - \omega\|_{1,2} \\
\lesssim & (\|w_*\|_{\mathbb{V}} + \|w\|_{\mathbb{V}})\|y\|_{\mathbb{V}}\|w_* - w\|_{\mathbb{V}} \lesssim (1 + 2\|w_*\|_{\mathbb{V}})\|y\|_{\mathbb{V}}\|w_* - w\|_{\mathbb{V}}. \qquad (4.24)
\end{aligned}$$

To estimate the term $|\langle y, Nw_* - Nw\rangle|$, we set

$$G(S) = 2\mu_0(\varepsilon + |S|^2)^{-\alpha/2}S, \;\; S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \in M_{2\times 2}$$

where $M_{2\times 2}$ is the set of matrixes with order 2×2 and

$$|S|^2 = \sum_{i=1}^{4} s_i^2, \quad s_i \in \mathbb{R}, \;\; i = 1, 2, 3, 4.$$

Then the first order Fréchet derivative of $G(S)$ is

$$DG(S) = 2\mu_0\left(\varepsilon + |S|^2\right)^{-\alpha/2}\begin{pmatrix} 1 - \frac{\alpha s_1^2}{\varepsilon+|S|^2} & -\frac{\alpha s_1 s_2}{\varepsilon+|S|^2} & -\frac{\alpha s_1 s_3}{\varepsilon+|S|^2} & -\frac{\alpha s_1 s_4}{\varepsilon+|S|^2} \\ -\frac{\alpha s_1 s_2}{\varepsilon+|S|^2} & 1 - \frac{\alpha s_2^2}{\varepsilon+|S|^2} & -\frac{\alpha s_2 s_3}{\varepsilon+|S|^2} & -\frac{\alpha s_2 s_4}{\varepsilon+|S|^2} \\ -\frac{\alpha s_1 s_3}{\varepsilon+|S|^2} & -\frac{\alpha s_2 s_3}{\varepsilon+|S|^2} & 1 - \frac{\alpha s_3^2}{\varepsilon+|S|^2} & -\frac{\alpha s_3 s_4}{\varepsilon+|S|^2} \\ -\frac{\alpha s_1 s_4}{\varepsilon+|S|^2} & -\frac{\alpha s_2 s_4}{\varepsilon+|S|^2} & -\frac{\alpha s_3 s_4}{\varepsilon+|S|^2} & 1 - \frac{\alpha s_4^2}{\varepsilon+|S|^2} \end{pmatrix}.$$

Obviously, the norm of the Fréchet derivative of $G(s)$ satisfies

$$\|DG(S)\| \leqslant 2\mu_0(\varepsilon + |S|^2)^{-\alpha/2}\sqrt{4 + \frac{12}{\varepsilon^2}} \lesssim 2\mu_0\varepsilon^{-\alpha/2}\sqrt{4 + \frac{12}{\varepsilon^2}} := c_6, \quad \forall\, S \in M_{2\times 2}. \qquad (4.25)$$

For any $S_1, S_2 \in M_{2\times 2}$, we have

$$G(S_2) - G(S_1) = \int_0^1 DG(S_1 + \theta(S_2 - S_1))(S_2 - S_1)\,\mathrm{d}\theta. \qquad (4.26)$$

Using (4.25) and (4.26), we arrive at

$$\begin{aligned}
|\langle y, Nw_* - Nw\rangle| =& |\langle y_u, N_1(u_*) - N_1(u)\rangle| \\
=& |\langle y_u, G(e(u_*)) - G(e(u))\rangle| = \left|\langle y_u, \int_0^1 DG(e(u_*) + \theta e(u_* - u))e(u_* - u)\,\mathrm{d}\theta\rangle\right| \\
\lesssim & \|DG(e(u_*) + \theta e(u_* - u))\|\|\nabla(u_* - u)\|\|y_u\|_V \\
\lesssim & \|u_* - u\|_V\|y_u\|_V \lesssim \|w_* - w\|_{\mathbb{V}}\|y\|_{\mathbb{V}}. \qquad (4.27)
\end{aligned}$$

Inserting (4.24) and (4.27) into (4.23) yields

$$\left|\Upsilon_y(w_*) - \Upsilon_y(w)\right| \lesssim \|w_* - w\|_{\mathbb{V}}\|y\|_{\mathbb{V}}(3 + 2\|w_*\|_{\mathbb{V}}),$$

which means that the functional $\Upsilon_y(\cdot) \in C(\mathbb{V})$ for each $y \in \mathbb{V}$. Consequently, (4.14) implies that the mapping $z \mapsto \langle y, \mathcal{H}(z(t), t)\rangle = \Upsilon(z)$ is m_t-integrable for every $y \in \mathbb{V}$, and the result of item (a) shows that $t \mapsto \int_{\mathbb{H}} \langle y, \mathcal{H}(z,t)\rangle \mathrm{d}m_t(z) = \int_{\mathbb{H}} \Upsilon(z)\mathrm{d}m_t(z) \in L_{\mathrm{loc}}(\mathbb{R})$.

Lastly, we establish item (c). For any $\Gamma \in \mathcal{T}$, we have from (4.18) that

$$\Gamma(z(t)) - \Gamma(z(s)) = \int_s^t \big\langle \Gamma'(z(\theta)), \mathcal{H}(z(\theta), \theta)\big\rangle \mathrm{d}\theta, \ \ \forall\, t \geqslant s. \tag{4.28}$$

For $\tau \leqslant s \leqslant t$, we let $z_* \in \mathbb{V}$ and $z(\theta) = U(\theta, \tau) z_*$ for $\theta \in [\tau, s]$. Then from (4.28) we have

$$\Gamma(U(t,\tau)z_*) - \Gamma(U(s,\tau)z_*) = \int_\tau^t \big\langle \Gamma'(U(\theta,\tau)z_*), \mathcal{H}(U(\theta,\tau)z_*, \theta)\big\rangle \mathrm{d}\theta. \tag{4.29}$$

Using (4.15), (4.29), Fubini's Theorem and a series of calculations, we derive

$$\begin{aligned}&\int_{\mathbb{H}} \Gamma(z)\mathrm{d}m_t(z) - \int_{\mathbb{H}} \Gamma(z)\mathrm{d}m_s(z) = \int_{\mathcal{A}(t)} \Gamma(z)\mathrm{d}m_t(z) - \int_{\mathcal{A}(s)} \Gamma(z)\mathrm{d}m_s(z)\\ =&\mathrm{LIM}_{\gamma\to-\infty} \frac{1}{s-\gamma}\int_\gamma^s \int_s^t \int_{\mathbb{H}} \big\langle \Gamma'(U(\theta,\tau)z_*), \mathcal{H}(U(\theta,\tau)z_*, \theta)\big\rangle \mathrm{d}m_\tau(z_*)\mathrm{d}\theta\mathrm{d}\tau. \end{aligned} \tag{4.30}$$

By (4.15) and the invariance property of the process, we have

$$\begin{aligned}&\int_{\mathbb{H}} \big\langle \Gamma'(U(\theta,\tau)z_*), \mathcal{H}(U(\theta,\tau)z_*, \theta)\big\rangle \mathrm{d}m_\tau(z_*)\\ =&\int_{\mathbb{H}} \big\langle \Gamma'(U(\theta,s)U(s,\tau)z_*), \mathcal{H}(U(\theta,s)U(s,\tau)z_*, \theta)\big\rangle \mathrm{d}m_\tau(z_*)\\ =&\int_{\mathbb{H}} \big\langle \Gamma'(U(\theta,s)z_*), \mathcal{H}(U(\theta,s)z_*, \theta)\big\rangle \mathrm{d}m_s(z_*). \end{aligned} \tag{4.31}$$

Note that the right-hand side of (4.31) is independent of τ. Inserting (4.31) into (4.30) gives

$$\begin{aligned}&\int_{\mathbb{H}} \Gamma(z)\mathrm{d}m_t(z) - \int_{\mathbb{H}} \Gamma(z)\mathrm{d}m_s(z) = \int_s^t \int_{\mathbb{H}} \big\langle \Gamma'(U(\theta,s)z_*), \mathcal{H}(U(\theta,s)z_*, \theta)\big\rangle \mathrm{d}m_s(z_*)\mathrm{d}\theta\\ =&\int_s^t \int_{\mathbb{H}} \big\langle \Gamma'(z), \mathcal{H}(z(\tau), \tau)\big\rangle \mathrm{d}m_\tau(z)\mathrm{d}\tau. \end{aligned}$$

The proof of Theorem 4.2 is complete. □

5 Lusin's type degenerate regularity of the statistical solution

In the theory of real and complex analysis ([21, P_{55}, Lusin's Theorem]), Lusin's Theorem describes the interesting relations between measurable functions and continuous function. In this section, we first prove that the pullback attractor $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ consists of weak and complete trajectories of equation (2.6), and that all elements $z(t)$ within $\hat{\mathcal{A}}$ are uniformly bounded with respect to $t \in \mathbb{R}$ (see Lemma 5.1) and possess Lusin's type regularity (see Lemma 5.2), provided that the external force $F = (f, g)$ is translationally bounded in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H})$. Afterwards, we establish that the pullback attractor $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ degenerates to a single complete trajectory if the Grashof number of F is small enough. From this degeneration of the pullback attractor and the Lusin's type regularity of the solution, we conclude that the the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ possesses Lusin's type degenerate regularity.

Definition 5.1. *A function $F(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H})$ is called translationally bounded in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H})$ if for some $a > 0$ there holds*

$$\|F\|^2_{L^2_{\mathbb{H}}(a)} := \sup_{t\in\mathbb{R}} \frac{1}{a}\int_t^{t+a} \|F(\theta)\|^2 \mathrm{d}\theta < +\infty. \tag{5.1}$$

Let $L_b^2(\mathbb{H})$ be the set of functions that are translationally bounded in $L^2_{\text{loc}}(\mathbb{R};\mathbb{H})$. By [11, Lemma 2.10] we know that for any $F \in L_b^2(\mathbb{H})$, the norms $\|F\|^2_{L^2_{\mathbb{H}}(a_1)}$ and $\|F\|^2_{L^2_{\mathbb{H}}(a_2)}$ are equivalent for any given $a_1 > 0$ and $a_2 > 0$.

We shall show that the pullback attractor $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ consists of weak and complete trajectories of equation (2.6), and that all elements $z(t)$ within $\hat{\mathcal{A}}$ are uniformly bounded with respect to $t \in \mathbb{R}$ and possess Lusin's type regularity.

Lemma 5.1. *Let assumptions* (H1)-(H2) *hold and* $F(\cdot) \in L_b^2(\mathbb{H})$. *Then the pullback attractor* $\hat{\mathcal{A}}$ *obtained in Theorem 3.1 satisfies*

$$\hat{\mathcal{A}} = \{z(t) : z(t)\,\text{is a complete and weak solution to equation (2.6) and}\,\|z(t)\|^2 \leqslant c_7,\ \ \forall t \in \mathbb{R}\}, \tag{5.2}$$

where $c_7 := \frac{2\|F\|^2_{L^2(\sigma^{-1})}}{\sigma(1-e^{-1})}$. *Moreover, for any* $\varsigma \in \mathbb{R}$ *and any* $\{z(\cdot)\} \in \hat{\mathcal{A}}$, *the measure of the set* $\{\theta \in [\varsigma, \varsigma+\sigma^{-1}] : \|z(\theta)\|^2_{\mathbb{V}} \leqslant c_7 k\}$ *satisfies*

$$\text{mes}\,\{\theta \in [\varsigma, \varsigma+\sigma^{-1}] : \|z(\theta)\|^2_{\mathbb{V}} \leqslant c_7 k\} \geqslant (1-\frac{1}{k})\sigma^{-1},\ \ k = 2, 3, \cdots. \tag{5.3}$$

Proof. Let $z(\cdot)$ be a complete and weak solution to equation (2.6). Then, by Lemma 3.1 and (5.1), we deduce

$$\begin{aligned}
\|z(t;\varsigma, z(\varsigma)\|^2 \leqslant & e^{-\sigma(t-\varsigma)}\|z(\varsigma)\|^2 + \frac{1}{\sigma}\int_\varsigma^t e^{-\sigma(t-s)}\|F(s)\|^2 \mathrm{d}s \\
\leqslant & e^{-\sigma(t-\varsigma)}\|z(\varsigma)\|^2 + \frac{1}{\sigma}\Big(\int_{t-\sigma^{-1}}^t + \int_{t-2\sigma^{-1}}^{t-\sigma^{-1}} + \int_{t-3\sigma^{-1}}^{t-2\sigma^{-1}} + \cdots\Big)e^{-\sigma(t-s)}\|F(s)\|^2\mathrm{d}s \\
\leqslant & e^{-\sigma(t-\varsigma)}\|z(\varsigma)\|^2 + \frac{\|F\|^2_{L^2(\sigma^{-1})}}{\sigma(1-e^{-1})},\ \ \forall t \geqslant \varsigma,\, \varsigma \in \mathbb{R}.
\end{aligned} \tag{5.4}$$

Now (3.1) and (5.4) imply that all elements $z(\cdot)$ within the pullback attractor $\hat{\mathcal{A}}$ are bounded uniformly with respect to $t \in \mathbb{R}$, with $\|z(t)\|^2 \leqslant c_7 := \frac{2\|F\|^2_{L^2(\sigma^{-1})}}{\sigma(1-e^{-1})}$. The fact that the pullback attractor $\hat{\mathcal{A}}$ consists of all complete and weak solutions of equation (2.6) has been proved in [11, Theorem 2.11]. Therefore, (5.2) is proved.

We next prove (5.3). Let $\{z(\cdot)\} \in \hat{\mathcal{A}}$. Then (3.3) and (5.2) imply

$$\|z(t)\|^2 + \sigma\int_\varsigma^t \|z(\theta)\|^2_{\mathbb{V}}\mathrm{d}\theta \leqslant \|z(\varsigma)\|^2 + \frac{1}{\sigma}\int_\varsigma^t \|F(\theta)\|^2\mathrm{d}\theta \leqslant c_7 + \frac{1}{\sigma}\int_\varsigma^t \|F(\theta)\|^2\mathrm{d}\theta,\ \ \forall\, t \geqslant \varsigma, \tag{5.5}$$

whence

$$\sigma\int_\varsigma^t \|z(\theta)\|^2_{\mathbb{V}}\mathrm{d}\theta \leqslant \|z(\varsigma)\|^2 + \frac{1}{\sigma}\int_\varsigma^t \|F(\theta)\|^2\mathrm{d}\theta \leqslant c_7 + \frac{1}{\sigma}\int_\varsigma^t \|F(\theta)\|^2\mathrm{d}\theta,\ \ \forall\, t \geqslant \varsigma. \tag{5.6}$$

Picking $t = \varsigma + \sigma^{-1}$, we obtain from (5.1) and (5.6) that

$$\int_\varsigma^{\varsigma+\sigma^{-1}} \|z(\theta)\|^2_{\mathbb{V}}\mathrm{d}\theta \leqslant \frac{c_7}{\sigma} + \frac{\|F\|^2_{L^2_{\mathbb{H}}(\sigma^{-1})}}{\sigma^3} := \frac{c_8}{\sigma}. \tag{5.7}$$

Inequality (5.7) implies that the measure of the set $\{\theta \in [\varsigma, \varsigma+\sigma^{-1}] : \|z(\theta)\|^2_{\mathbb{V}} \geqslant c_8 k\}$ satisfies

$$\text{mes}\,\{\theta \in [\varsigma, \varsigma+\sigma^{-1}] : \|z(\theta)\|^2_{\mathbb{V}} \geqslant c_8 k\} \leqslant \frac{1}{c_8 k}\frac{c_8}{\sigma} = \frac{1}{\sigma k},\ \ \ k = 1, 2, \cdots. \tag{5.8}$$

Obviously,

$$\text{mes}\,\{\theta\in[\varsigma,\varsigma+\sigma^{-1}]:\|z(\theta)\|_{\mathbb{V}}^2\leqslant c_8k\}=\sigma^{-1}-\text{mes}\{\theta\in[\varsigma,\varsigma+\sigma^{-1}]:\|z(\theta)\|_{\mathbb{V}}^2\geqslant c_8k\}\geqslant(1-\frac{1}{k})\sigma^{-1}.$$

This completes the proof. □

From (5.3) we can deduce that each weak solution $z(\cdot)$ within $\hat{\mathcal{A}}$ is "nearly" regular on $[\varsigma,\varsigma+\sigma^{-1}]$ for each $\varsigma\in\mathbb{R}$. But we cannot say that each $z(\cdot)$ within $\hat{\mathcal{A}}$ is a.e. regular on $[\varsigma,\varsigma+\sigma^{-1}]$. This is much like in the Lusin Theorem describing the relations between measurable functions and continuous functions: each measurable function on a interval I is "nearly" continuous on I. This "nearly" regular result of the elements within $\hat{\mathcal{A}}$ is stated as follows.

Lemma 5.2. *Let ς be any real number. Then, for any $\epsilon>0$ and $\{z(\cdot)\}\in\hat{\mathcal{A}}$ there corresponds a set $E(\epsilon,z,\varsigma)\subset[\varsigma,\varsigma+\sigma^{-1}]$ and a positive constant c_ϵ such that the measure of the set $E(\epsilon,z,\varsigma)$ satisfies $\text{mes}(E(\epsilon,z,\varsigma))<\epsilon$ and $\|z(\theta)\|_{\mathbb{V}}^2\leqslant c_\epsilon$, $\forall\theta\in[\varsigma,\varsigma+\sigma^{-1}]\setminus E(\epsilon,z,\varsigma)$.*

Proof. For any $\epsilon>0$ there exists a positive integer k_ϵ such that $\frac{1}{\sigma k_\epsilon}<\epsilon$. Let ς be any real number. For each $\{z(\cdot)\}\in\hat{\mathcal{A}}$ and above ϵ we denote, $E(\epsilon,z,\varsigma)=\{\theta\in[\varsigma,\varsigma+\sigma^{-1}]:\|z(\theta)\|_{\mathbb{V}}^2\geqslant c_8k_\epsilon\}$. Then from (5.3) we deduce

$$\begin{aligned}\text{mes}\,(E(\epsilon,z,\varsigma))=&\sigma^{-1}-\text{mes}(\{\theta\in[\varsigma,\varsigma+\sigma^{-1}]:\|z(\theta)\|_{\mathbb{V}}^2\leqslant c_8k_\epsilon\})\\ \leqslant&\sigma^{-1}-(1-\frac{1}{k_\epsilon})\sigma^{-1}=\frac{1}{\sigma k_\epsilon}<\epsilon.\end{aligned}\tag{5.9}$$

Clearly, $\|z(\theta)\|_{\mathbb{V}}^2\leqslant c_8k_\epsilon:=c_\epsilon$, $\forall\theta\in[\varsigma,\varsigma+\sigma^{-1}]\setminus E(\epsilon,z,\varsigma)$. This ends the proof. □

Lemma 5.2 shows that the set $E(\epsilon,z,\varsigma)$ depends on ϵ, $\{z(\cdot)\}\in\hat{\mathcal{A}}$ and ς. Thus, for fixed ϵ and ς, different elements $z(\cdot)$ within $\hat{\mathcal{A}}$ will correspond to different sets $E(\epsilon,z,\varsigma)$. We next prove that if the (generalized) Grashof number $\langle\|F\|^2\rangle_{\leqslant t}$ defined in (1.5) is small enough, then the pullback attractor $\hat{\mathcal{A}}$ degenerates to a single complete trajectory.

Lemma 5.3. *Let assumptions* (H1)-(H2) *hold and let $F(\cdot)\in L_b^2(\mathbb{H})$. Then the pullback attractor obtained in Theorem 3.1 degenerates to a single trajectory,*

$$\hat{\mathcal{A}}=\{\tilde{z}(t):\tilde{z}(t)\ \textit{is the unique complete and bounded weak solution to equation}\ (2.6)\}\tag{5.10}$$

provided that the generalized Grashof number is small enough, namely

$$\langle\|F\|^2\rangle_{\leqslant t}<c_2^{-2}\lambda_1^3\delta_1^4.\tag{5.11}$$

Proof. Let $z_1(t,\tau;z_{1\tau})=(u_1(t,\tau;u_{1\tau}),\omega_1(t,\tau;\omega_{1\tau})$ and $z_2(t,\tau;z_{2\tau})=(u_2(t,\tau;u_{2\tau}),\omega_2(t,\tau;\omega_{2\tau})$ be two weak solutions of problem (2.6)-(2.7) corresponding the initial values $z_{1\tau}=(u_{1\tau},\omega_{1\tau})$ and $z_{2\tau}=(u_{2\tau},\omega_{2\tau})$, respectively. Set $\tilde{z}=\tilde{z}(t)=z_1(t)-z_2(t)=(u_1(t)-u_2(t),\omega_1(t)-\omega_2(t))$. Then $\tilde{z}$ is a solution of the problem

$$\frac{\mathrm{d}\tilde{z}(t)}{\mathrm{d}t}+A\tilde{z}(t)+Bz_1(t)-Bz_2(t)+R\tilde{z}(t)+Nz_1(t)-Nz_2(t)=0,\ \text{in}\ \mathcal{D}'((\tau,+\infty),\mathbb{V}^*),\tag{5.12}$$

$$\tilde{z}(\tau)=z_{1\tau}-z_{2\tau}=(u_{1\tau}-u_{2\tau},\omega_{1\tau}-\omega_{2\tau}).\tag{5.13}$$

Taking the dual pairing of $\tilde{z}(t)$ with (5.12), then using (2.11), (2.20) and (2.23), we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}(t)\|^2 + \delta_1\|\tilde{z}(t)\|_{\mathbb{V}}^2 \leqslant |b_1(u_1-u_2,u_2,u_1-u_2)| + |b_2(u_1-u_2,\omega_2,\omega_1-\omega_2)|. \tag{5.14}$$

Now, by a classical estimate (see [12, (A.46d)]) we have

$$\begin{cases} |b_1(u_1-u_2,u_2,u_1-u_2)| \leqslant c_2\|u_1-u_2\|\|u_1-u_2\|_V\|u_2\|_V \leqslant \frac{c_2^2}{4\delta_1}\|\tilde{z}\|^2\|z_2\|_{\mathbb{V}}^2 + \frac{\delta_1}{4}\|\tilde{z}\|_{\mathbb{V}}^2, \\ |b_2(u_1-u_2,\omega_2,\omega_1-\omega_2)| \leqslant c_2\|\tilde{z}\|\|\tilde{z}\|_{\mathbb{V}}\|\omega_2\|_{1,2} \leqslant \frac{c_2^2}{4\delta_1}\|\tilde{z}\|^2\|z_2\|_{\mathbb{V}}^2 + \frac{\delta_1}{4}\|\tilde{z}\|_{\mathbb{V}}^2. \end{cases} \tag{5.15}$$

Inserting (5.15) into (5.14) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}(t)\|^2 + \delta_1\|\tilde{z}(t)\|_{\mathbb{V}}^2 \leqslant \frac{c_2^2}{\delta_1}\|\tilde{z}(t)\|^2\|z_2(t)\|_{\mathbb{V}}^2. \tag{5.16}$$

Integrating (5.16) over $[\tau,t]$ and then using Poincaré's inequality, we obtain

$$\|\tilde{z}(t)\|^2 \leqslant \|\tilde{z}(\tau)\|^2 + \int_\tau^t (\frac{c_2^2}{\delta_1}\|z_2(\theta)\|_{\mathbb{V}}^2 - \sigma)\|\tilde{z}(\theta)\|^2\mathrm{d}\theta. \tag{5.17}$$

Note that (5.5) implies

$$\|z_2(\theta)\|_{\mathbb{V}}^2 \leqslant \limsup_{\gamma\to-\infty}\frac{1}{t-\gamma}\int_\gamma^t \|z_2(s)\|_{\mathbb{V}}^2\mathrm{d}s \leqslant \frac{\langle\|F\|^2\rangle_{\leqslant t}}{\sigma^2},\ \ \forall\theta\leqslant t,$$

and if (5.11) holds true, then

$$\frac{c_2^2}{\delta_1}\|z_2(\theta)\|_{\mathbb{V}}^2 - \sigma \leqslant \frac{c_2^2\langle\|F\|^2\rangle_{\leqslant t}}{\sigma^2\delta_1} - \sigma := c_9 < 0,\ \ \forall\,\theta\leqslant t, \tag{5.18}$$

Taking into account (5.17)-(5.18) and using Gronwall's inequality, we arrive at

$$\|\tilde{z}(t)\|^2 \leqslant \|\tilde{z}(\tau)\|e^{-c_9(t-\tau)},\ \ \forall\,t\geqslant\tau,$$

which implies that $\tilde{z}(\theta)=0$ for $\theta\in(-\infty,t]$, that is, $z_1(\theta)=z_2(\theta)$ for $\theta\in(-\infty,t]$. The proof of Lemma 5.3 if complete. □

Recall that the (partial) regularity of the statistical solution refers to that it is supported on a set in which all weak solutions are (partially) strong solutions. Combining Lemma 5.2 and Lemma 5.3, we conclude that the statistical solution $\{m\}_{t\in\mathbb{R}}$ is "nearly" regular (but not a.e regular) on each interval $[\varsigma,\varsigma+\sigma^{-1}]$, $\varsigma\in\mathbb{R}$. This Lusin type degenerate regular result of the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ is stated as follows.

Theorem 5.1. *Let assumptions* (H1)-(H2) *and* (5.11) *hold, and let* $F(\cdot)\in L_b^2(\mathbb{H})$*. Then the statistical solution* $\{m_t\}_{t\in\mathbb{R}}$ *obtained in Theorem 4.2 possesses the following degenerate regularity: for each* $\varsigma\in\mathbb{R}$ *and for any* $\epsilon>0$*, there exists a subset* $E(\epsilon,\varsigma)\subset[\varsigma,\varsigma+\sigma^{-1}]$ *such that* mes $(E(\epsilon,\varsigma))<\epsilon$ *and* $\{m_t\}_{t\in\mathbb{R}}$ *is regular for* $t\in[\varsigma,\varsigma+\sigma^{-1}]\setminus E(\epsilon,\varsigma)$.

Proof. Theorem 4.1 shows that the support of m_t is contained in $\mathcal{A}(t)$ for each $t\in\mathbb{R}$. Then the result follows directly from lemmas 5.2 and 5.3. □

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