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μ-Caputo type time-delay Langevin equations with two general fractional orders

Mustafa Aydin*[1,2] | Nazim I. Mahmudov[1]

[1]Department of Mathematics, Eastern Mediterranean University, Famagusta 99628 T.R. North Cyprus, Turkey
[2]Department of Mathematics, Istanbul Technical University, Sariyer, Istanbul 34469 , Turkey

Correspondence
*Mustafa AYDIN
Email: mustafaaydin@itu.edu.tr

Summary

In the present paper, a μ-delayed Mittag-Leffler type function is introduced as a fundamental function. By means of μ-delayed Mittag-Leffler type function, an exact analytical solution formula to non-homogeneous linear delayed Langevin equations involving two distinct μ-Caputo type fractional derivatives of general orders is given. Also, a global solution of nonlinear version of delayed Langevin equations is inferred from the findings on hand and is verified with the aid of the functional(substitutional) operator. In terms of exponential function, we estimate μ-delayed Mittag-Leffler type function. Existence uniqueness of solutions to nonlinear delayed Langevin fractional differential equations are obtained with regard to the weighted norm defined in accordance with exponential function. The notion of stability analysis in the sense of solutions to the described Langevin equations is discussed on the grounds of the fixed point approach. Numerical and simulated examples are shared to exemplify the theoretical findings. This paper provides novel results.

KEYWORDS:
μ-fractional Langevin equation, μ-delayed Mittag-Leffler function, an explicit solution, existence uniqueness, Ulam-Hyers stability

1 | INTRODUCTION

As to a solution of an arbitrary differential equation, the first notion to be explored is to determine a fundamental (matrix) function which forms the main structure(backbone) of the solution. The more complex a differential equation structure is, the more difficult and sometimes impossible it is to determine the fundamental function. Therefore, sometimes it is more important to identify the most convenient and usable fundamental function than to settle out the problem. Although the first foundation of the fractional derivative was laid with Leibniz's note to L'Hospital in 1695, fractional differential equations have become as important as ordinary differential equations, even more important today. This also makes it difficult to identify the fundamental functions. In this sense of finding the fundamental function for both ordinary and fractional differential equations, there are so many of original papers[1-9] which are about the fact that existence, uniqueness, distinct kinds of stabilities like Ulam-Hyers stability, Lapunov stability, different kinds of controllabilities like relative controllability, iterative learning control, etc related to solution with respect to a fundamental function are discussed.

The ancestor of the Langevin equations is the French physicist Paul Langevin. In the early 1900s, he gave a comprehensive description of Brownian motion which describes the progress or motion of physical phenomena in fluctuant media. In the progress of time, for a description of the more complex problems in complex fluctuant environments, the well-known ordinary Langevin equations remain incapable or fall short. So, general versions of Langevin equations are need to present more physical phenomena[18]. Among them, we can count fractional Langevin type equation consisting of fractional order derivative. As far as we know, although there are so many studies on the nonlinear Langevin differential equations consisting two distinct fractional orders derivatives which are presented and investigated, there are very few studies on delayed Langevin fractional differential equations. In the study[3], an explicit solution to fractional Langevin time-delay equations having two different classical Riemann-Liouville fractional derivatives by defining delayed Mittag-Leffler type function as a fundamental function. In the work[9], with the help of the same delayed Mittag-Leffler type function proposed in the reference[3], an exact analytical solution to nonlinear fractional Langevin time-delay equations having two distinct classical Caputo fractional orders derivatives.

In the recent times, the concept of a fractional derivative of a function with respect to another function[22] has attracted researchers' attention. While it is involved in many sorts of real-world problems, simultaneously it has been developed as in the studies[10]-[17].

Our aim is to study the nonlinear μ-Caputo type time-delay Langevin equations with two general fractional orders

$$\begin{cases} \left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t) - \kappa_2\left({}^{C}_{0^+}\beth^{\beta}_{\mu}x\right)(t) - \kappa_1 x(t-h) = \daleth\left(t, x(t)\right), \quad t \in (0, T], h > 0, \\ x(t) = \phi(t), \quad -h \le t \le 0, \end{cases} \tag{1}$$

where ${}^{C}_{0^+}\beth^{\alpha}_{\mu}$ and ${}^{C}_{0^+}\beth^{\beta}_{\mu}$ are μ-Caputo type fractional derivatives, $n-1 < \alpha \le n$, $n-2 < \beta \le n-1$, $n \ge 2$ with $\alpha - \beta \ge 1$ and $\daleth : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear disturbance. $\phi : [-h, 0] \to \mathbb{R}$ is an arbitrary $(n-1)$-times continuously differentiable, $\kappa_1, \kappa_2 \in \mathbb{R}$, $T = lh$, $l \in \mathbb{N}$ which is the set of all natural numbers.

Contribution of the current paper:

- we offer the μ- Caputo type fractional linear(or semi-linear) delayed Langevin differential equations involving two distinct fractional orders
- we propose novelly μ-delayed Mittag-Leffler type function as a fundamental function,
- we present an exact analytical solution to linear Langevin system and a global solution formula to the semi-linear Langevin system with respect to the μ-delayed Mittag-Leffler type function,
- we estimate μ-delayed Mittag-Leffler type function and an integral of it in terms of exponential function,
- we examined the existence and uniqueness of solutions of the μ-delayed Langevin differential equations' system,
- we make an analysis of stability in the sense of solutions to the described Langevin equations on the grounds of the fixed point approach
- we illustrate the theoretical findings with the help of numerical and simulated examples.

2 | PRELIMINARIES

Let $a, b \in \mathbb{R}$(or $\mathbb{R}^+$) which is the set of all real numbers(or all positive real numbers). For $-\infty < a < b < \infty$, $I = [a, b]$ is the interval of $\mathbb{R}$. $C([a,b], \mathbb{R}) = C(I)$ is the Banach space of continuous functions and also $AC([a,b], \mathbb{R}) = AC(I)$ is the space of absolutely continuous functions on $[a, b]$. For $n \in \{0, 1, 2, ...\}$, let $C^n([a,b], \mathbb{R}) = C^n(I)$ and $AC^n([a,b], \mathbb{R}) = AC^n(I)$ be the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ such that $f^{(n-1)} \in C([a,b], \mathbb{R})$ and $f^{(n-1)} \in AC([a,b], \mathbb{R})$, respectfully. Let $\mu \in C^1([a,b], \mathbb{R})$ with μ is increasing and $\mu'(t) \ne 0$ for every $t \in [a, b]$. The maximum norm $\|.\|_{\infty}$ and the μ-weighted maximum norm $\|.\|_{\mu,\omega}$ of the above spaces are

$$\|\daleth\|_{\infty} = \max_{t \in [a,b]} |\daleth(t)|,$$

and

$$\|\daleth\|_{\mu,\omega} = \max_{t \in [a,b]} \left| \frac{\daleth(t)}{e^{\omega\mu(t)}} \right|,$$

where $|.|$ is an arbitrary norm on $\mathbb{R}$.

Definition 1. [22] The gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0.$$

For this function, the reduction formula

$$\Gamma(\alpha+1) = \Gamma(\alpha),$$

holds.

Definition 2. [22] The beta function $B(\alpha, \beta)$ is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha > 0, \quad \beta > 0.$$

The Beta function is closely related to the Gamma function as noted below

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

holds.

Definition 3. [22] Let $\daleth$ be an integrable function defined on I and $\mu \in C^1(I)$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in I$. The μ-Riemann Liouville type fractional integral of order $\alpha > 0$ is defined as

$$\begin{aligned}\left({}^{RL}_{0^+}\gimel^{\alpha}_{\mu}\daleth\right)(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t \mu'(s) (\mu(t) - \mu(s))^{\alpha-1} \daleth(s) ds, \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (\mu(t) - \mu(s))^{\alpha-1} \daleth(s) d\mu(s)\end{aligned}$$

and the μ-Riemann Liouville type fractional derivative of order $\alpha > 0$ is defined as

$$\begin{aligned}\left({}^{RL}_{0^+}\beth^{\alpha}_{\mu}\daleth\right)(t) &:= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{\mu'(t)dt}\right)^n \int_0^t \mu'(s) (\mu(t) - \mu(s))^{n-\alpha-1} \daleth(s) ds, \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\mu(t)}\right)^n \int_0^t (\mu(t) - \mu(s))^{n-\alpha-1} \daleth(s) d\mu(s),\end{aligned}$$

where $n = [\alpha] + 1$, $[\alpha]$ means the integer part of α.

Definition 4. [22] Let $\daleth \in AC^n(I)$ and $\mu \in C^1(I)$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in I$. The μ-Caputo type fractional derivative of order $\alpha > 0$ is defined as

$$\begin{aligned}\left({}^{RL}_{0^+}\beth^{\alpha}_{\mu}\daleth\right)(t) &:= \frac{1}{\Gamma(n-\alpha)} \int_0^t \mu'(s) (\mu(t) - \mu(s))^{n-\alpha-1} \left(\frac{d}{\mu'(s)ds}\right)^n \daleth(s) ds, \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (\mu(t) - \mu(s))^{n-\alpha-1} \left(\frac{d}{d\mu(s)}\right)^n \daleth(s) d\mu(s),\end{aligned}$$

where $n = [\alpha] + 1$.

Remark 1. If we consider $\mu(t) = t$, we obtain the classical Riemann Liouville fractional integral $\left({}^{RL}_{0^+}\gimel^{\alpha}_{t}\daleth\right)(t)$, the classical Riemann Liouville fractional derivative $\left({}^{RL}_{0^+}\beth^{\alpha}_{t}\daleth\right)(t)$, and the classical Caputo fractional derivative $\left({}^{C}_{0^+}\beth^{\alpha}_{t}\daleth\right)(t)$. Furthermore, we can obtain other fractional derivatives like Hadamard fractional derivative [25,26], Erdélyi–Kober fractional derivative [27], etc.

Theorem 1. [23,24] If $\daleth \in AC^n(I)$ and $\alpha > 0$, then

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}\daleth\right)(t) = \left({}^{RL}_{0^+}\beth^{\alpha}_{\mu}\daleth\right)\left[\daleth(t) - \sum_{k=0}^{n-1} \frac{(\mu(t) - \mu(0))^k}{k!} \left(\frac{1}{\mu'(t)}\right)^k \daleth^{(k)}(0)\right].$$

Lemma 1. [22] Given $\beta \in \mathbb{R}$. For $\alpha > 0$,

$$
{}_{0^+}^{C}\beth_{\mu}^{\alpha}\left(\mu(t)-\mu(0)\right)^{\beta-1} = \begin{cases} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mu(t)-\mu(0)\right)^{\beta-\alpha-1}, & \beta > [\alpha], \\ 0, & \beta = 0,1,2,...,[\alpha], \\ \text{undefined}, & \text{otherwise}. \end{cases}
$$

The well-known binomial identity is

$$
\binom{m+p}{p} = \binom{m+p-1}{p} + \binom{m+p-1}{p-1}, \quad m,p \geq 0.
$$

3 | MAIN RESULTS

3.1 | Exact analytical solutions of the linear homogeneous μ-Caputo type fractional Langevin type time-delay differential equations with general fractional orders

First of all, we shall define the following function which is a solution of the linear homogeneous μ-Caputo type fractional Langevin type time-delay differential equations. This function can be regarded as a fundamental function for the delayed Langevin equations with two fractional orders.

Definition 5. Let $\mu \in C^1(I)$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in I$. The μ-Delayed version of Mittag-Leffler function generated by $\lambda_1, \lambda_2 \in \mathbb{R}$ of three parameters with respect to another function μ with two variables t, s; $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}$: $\mathbb{R} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is defined by

$$
\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\lambda_1,\lambda_2;t,s\right) = \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\lambda_1^m\lambda_2^p}{\Gamma(m\alpha_1+p\alpha_2+\alpha_3)}\left(\mu(t)-\mu(s+mh)\right)^{m\alpha_1+p\alpha_2+\alpha_3-1}\mathcal{H}\left(\mu(t)-\mu(s+mh)\right),
$$

where $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$, $h > 0$, and $\mathcal{H}(.) : \mathbb{R} \to \mathbb{R}$ is the Heaviside function as described below

$$
\mathcal{H}(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}
$$

For simplicity, we use the symbol ${}_{\mu}\mathcal{H}_s^t := \mathcal{H}(\mu(t)-\mu(s))$. This is a fundamental matrix for system (1).

Remark 2. The graphs of the μ-delayed M-L type function $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\lambda_1,\lambda_2;.,.\right)$, the delayed analogue of M-L type function generated by $\kappa_1, \kappa_2 \in \mathbb{R}$ of three parameters $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h}\left(\kappa_1,\kappa_2;t\right)$ and the two-parameter M-L function $\mathrm{E}_{\alpha_1,\alpha_2}(t)$ with certain values or functions of the parameters can be found in Figure 1 . Depending on choosing μ under specific values of the rest of the parameters, the graphs of different μ-delayed M-L type functions are compared with each other in Figure 2 .

Remark 3. The μ-Delayed Mittag-Leffler(M-L) type function $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\lambda_1,\lambda_2;.,.\right)$ in Definition 5 is the more general function because it includes many of known special function in the literature. Here are some of them.

1. $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\kappa_1,\kappa_2;t,s\right) = \mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h}\left(\kappa_1,\kappa_2;t\right)$, where $\mu(t) = t$, $s = 0$, the delayed analogue of M-L type function generated by $\kappa_1, \kappa_2 \in \mathbb{R}$ of three parameters $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h}\left(\kappa_1,\kappa_2;t\right)$ is studied in the references[3 9].
2. $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\kappa_1,\kappa_2;t,s\right) = \mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{\delta}\left(\kappa_1,\kappa_2\right)$, where $\mu(t) = t$, $s = 0$, $h = 0$, $\delta = 1$, the bivariate M-L function $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{\delta}\left(\kappa_1,\kappa_2\right)$ is considered in the reference[19].
3. $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\kappa_1,\kappa_2;t,s\right) = \mathrm{E}_{\alpha_1,\alpha_3}^{h}\left(\kappa_1,t\right)$ where $\mu(t) = t$, $s = 0$, $\kappa_2 = 0$, the delayed analogue of M-L type function of two parameters $\mathrm{E}_{\alpha_1,\alpha_3}^{h}\left(\kappa_1,t\right)$ is researched in the references[8 20 2].
4. $\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\kappa_1,\kappa_2;t,s\right) = t^{\alpha_3-1}\mathrm{E}_{\alpha_2,\alpha_3}\left(\kappa_2 t^{\alpha_2}\right)$ where κ_1, the two-parameter M-L function is investigated in the reference[22].

Lemma 2. Let $\mu \in C^1(I)$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in I$. If $\lambda_1, \lambda_2, \alpha, \alpha_1, \alpha_2, \alpha_3, h \in \mathbb{R}$ satisfying $\alpha_1 > 0$, $h > 0$, and $\alpha, \alpha_3 - 1 > [\alpha]$, then

$$
{}_{0^+}^{C}\beth_{\mu}^{\alpha}\mathrm{E}_{\alpha_1,\alpha_2,\alpha_3}^{h,\mu}\left(\lambda_1,\lambda_2;t,s\right) = \mathrm{E}_{\alpha_1,\alpha_2,\alpha_3-\alpha}^{h,\mu}\left(\lambda_1,\lambda_2;t,s\right).
$$

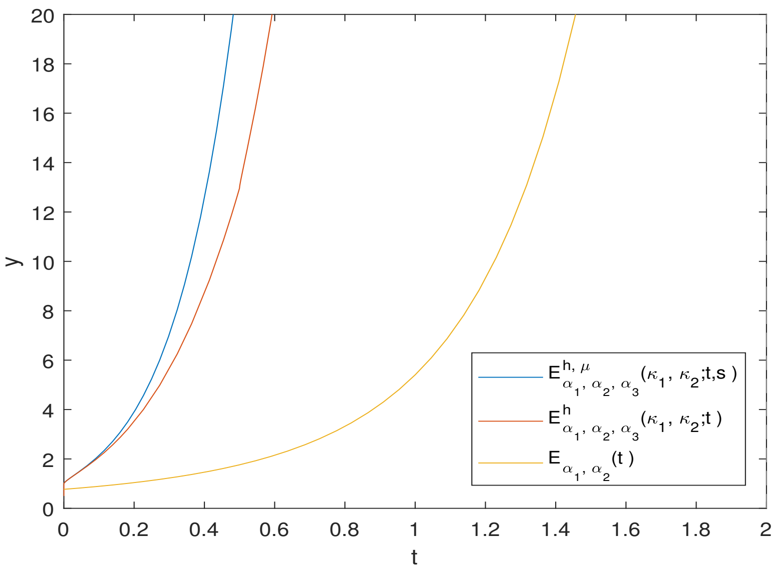

FIGURE 1 Comparison of functions $E^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3}(\kappa_1,\kappa_2;t,s)$, $E^{h}_{\alpha_1,\alpha_2,\alpha_3}(\kappa_1,\kappa_2;t)$, and $E_{\alpha_1,\alpha_2}(t)$ for $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 1$, $\lambda_1 = 2$, $\lambda_2 = 3$, $h = 0.5$, $s = 0$, $\mu(t) = e^t$.

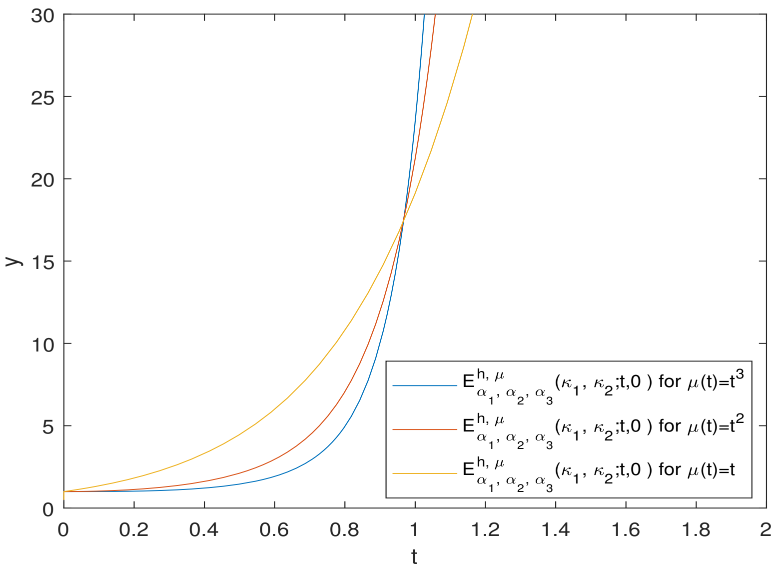

FIGURE 2 Comparison of functions $E^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3}(\kappa_1,\kappa_2;t,s)$, for a few of different functions $\mu(t) = t^3$, $\mu(t) = t^2$, $\mu(t) = t$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\lambda_1 = 2$, $\lambda_2 = 3$, $h = 0.5$.

Proof.

$$
\begin{aligned}
&{}^{C}_{0^+}\beth^{\alpha}_{\mu}\mathrm{E}^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3}\left(\lambda_1,\lambda_2;t,s\right)\\
&={}^{C}_{0^+}\beth^{\alpha}_{\mu}\left(\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\lambda_1^m\lambda_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha_1+p\alpha_2+\alpha_3-1}}{\Gamma(m\alpha_1+p\alpha_2+\alpha_3)}\mathcal{H}\left(\mu(t)-\mu(s+mh)\right)\right)
\end{aligned}
$$

$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\lambda_1^m\lambda_2^p\mathcal{H}\left(\mu(t)-\mu(s+mh)\right)}{\Gamma(m\alpha_1+p\alpha_2+\alpha_3)}\,{}_{0^+}^{C}\beth_{\mu}^{\alpha}\left((\mu(t)-\mu(s+mh))^{m\alpha_1+p\alpha_2+\alpha_3-1}\right)$$
$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\lambda_1^m\lambda_2^p(\mu(t)-\mu(s+mh))^{m\alpha_1+p\alpha_2+\alpha_3-\alpha-1}}{\Gamma(m\alpha_1+p\alpha_2+\alpha_3-\alpha)}\mathcal{H}\left(\mu(t)-\mu(s+mh)\right)$$
$$= \mathrm{E}^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3-\alpha}\left(\lambda_1,\lambda_2;t,s\right).$$

□

Theorem 2. With x belonging to the Banach space of absolutely continuously differentiable functions up to order n on $[-h,T]$,

$$x(t) = \sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\right]\phi_0^{(i)}+\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)}$$
$$+\kappa_1\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s) \tag{2}$$

where $\phi_0^{(i)}=\phi^{(i)}(0)$ with $i\in\{0,1,...,n-1\}$, is an analytical solution of the Cauchy type problem (1) with $\daleth\equiv 0$.

Proof. To make this proof understandable and clearer, we separately calculate the fractional derivative of each of terms in (2). With the help of the binomial identity and Lemma 1 , we acquire

$${}_{0^+}^{C}\beth_{\mu}^{\alpha}\left(\sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\right]\phi_0^{(i)}\right)$$
$$=\sum_{i=0}^{n-2}\kappa_1\left({}_{0^+}^{C}\beth_{\mu}^{\alpha}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\right)\phi_0^{(i)}$$
$$=\sum_{i=0}^{n-2}\kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,i+1}\left(\kappa_1,\kappa_2;t,h\right)\phi_0^{(i)}$$
$$=\sum_{i=0}^{n-2}\kappa_1\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\sum_{m=1}^{\infty}\sum_{p=0}^{\infty}\binom{m+p-1}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+(m+1)h)\right)^{m\alpha+p(\alpha-\beta)+i}}{\Gamma(m\alpha+p(\alpha-\beta)+i+1)}\,{}_{\mu}\mathcal{H}^{t}_{s+(m+1)h}\right.$$
$$\left.+\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\binom{m+p-1}{p-1}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+(m+1)h)\right)^{m\alpha+p(\alpha-\beta)+i}}{\Gamma(m\alpha+p(\alpha-\beta)+i+1)}\,{}_{\mu}\mathcal{H}^{t}_{s+(m+1)h}\right]\phi_0^{(i)}$$
$$=\kappa_1\sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\kappa_1\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+(m+2)h)\right)^{(m+1)\alpha+p(\alpha-\beta)+i}}{\Gamma((m+1)\alpha+p(\alpha-\beta)+i+1)}\,{}_{\mu}\mathcal{H}^{t}_{s+(m+2)h}\right]\phi_0^{(i)}$$
$$+\kappa_1\kappa_2\sum_{i=0}^{n-2}\left[\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+(m+1)h)\right)^{m\alpha+(p+1)(\alpha-\beta)+i}}{\Gamma(m\alpha+(p+1)(\alpha-\beta)+i+1)}\,{}_{\mu}\mathcal{H}^{t}_{s+(m+1)h}\right]\phi_0^{(i)}$$
$$=\kappa_1\sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,2h\right)\right]\phi_0^{(i)}+\kappa_1\kappa_2\sum_{i=0}^{n-2}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\phi_0^{(i)}.$$

Now, we will compute the fractional derivative of the second one in (2). Again, by using the binomial identity and Lemma 1 we get

$$
\begin{aligned}
&= {}^{C}_{0^+}\daleth^{\alpha}_{\mu} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)} \\
&= {}^{C}_{0^+}\daleth^{\alpha}_{\mu}\left(\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha+p(\alpha-\beta)+n-1}}{\Gamma(m\alpha+p(\alpha-\beta)+n)}{}_{\mu}\mathcal{H}^t_{s+mh}\right)\phi_0^{(n-1)} \\
&+ {}^{C}_{0^+}\daleth^{\alpha}_{\mu}\left[\frac{\mu^{m-1}(t)}{\Gamma(m)}+\sum_{m=1}^{\infty}\sum_{p=0}^{\infty}\binom{m+p-1}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha+p(\alpha-\beta)+n-1}}{\Gamma(m\alpha+p(\alpha-\beta)+n)}{}_{\mu}\mathcal{H}^t_{s+mh}\right. \\
&\left.+\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\binom{m+p-1}{p-1}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha+p(\alpha-\beta)+n-1}}{\Gamma(m\alpha+p(\alpha-\beta)+n)}{}_{\mu}\mathcal{H}^t_{s+mh}\right]\phi_0^{(n-1)} \\
&= \sum_{m=1}^{\infty}\sum_{p=0}^{\infty}\binom{m+p-1}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha+p(\alpha-\beta)+n-\alpha-1}}{\Gamma(m\alpha+p(\alpha-\beta)+n-\alpha)}{}_{\mu}\mathcal{H}^t_{s+mh}\phi_0^{(n-1)} \\
&+ \sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\binom{m+p-1}{p-1}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha+p(\alpha-\beta)+n-\alpha-1}}{\Gamma(m\alpha+p(\alpha-\beta)+n-\alpha)}{}_{\mu}\mathcal{H}^t_{s+mh}\phi_0^{(n-1)} \\
&= \kappa_1\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+(m+1)h)\right)^{(m+1)\alpha+p(\alpha-\beta)+n-\alpha-1}}{\Gamma((m+1)\alpha+p(\alpha-\beta)+n-\alpha)}{}_{\mu}\mathcal{H}^t_{s+(m+1)h}\phi_0^{(n-1)} \\
&+ \kappa_2\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\kappa_1^m\kappa_2^p\left(\mu(t)-\mu(s+mh)\right)^{m\alpha+(p+1)(\alpha-\beta)+n-\alpha-1}}{\Gamma(m\alpha+(p+1)(\alpha-\beta)+n-\alpha)}{}_{\mu}\mathcal{H}^t_{s+mh}\phi_0^{(n-1)} \\
&= \kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,n}\left(\kappa_1,\kappa_2;t,h\right)\phi_0^{(n-1)}+\kappa_2\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,n-\beta}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)}
\end{aligned}
$$

In a similar manner, one easily calculate the fractional derivative of the last one in (2) by applying the binomial identity and Lemma 1 as noted below:

$$
\begin{aligned}
&{}^{C}_{0^+}\daleth^{\alpha}_{\mu}\left(\kappa_1\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s)\right) \\
&= \kappa_1^2\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+2h\right)\phi(s)d\mu(s) \\
&+ \kappa_1\kappa_2\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha-\beta}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s).
\end{aligned}
$$

To sum up, the fractional derivative of solution (2) is

$$
\begin{aligned}
\left({}^{C}_{0^+}\daleth^{\alpha}_{\mu}x\right)(t) &= \kappa_1\sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,2h\right)\right]\phi_0^{(i)}+\kappa_1\kappa_2\sum_{i=0}^{n-2}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\phi_0^{(i)} \\
&+ \kappa_1^2\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+2h\right)\phi(s)d\mu(s)+\kappa_1\kappa_2\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha-\beta}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s) \\
&+ \kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,n}\left(\kappa_1,\kappa_2;t,h\right)\phi_0^{(n-1)}+\kappa_2\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,n-\beta}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)}.
\end{aligned}
\tag{3}
$$

Now, we will take the rest of expressions in (1) into consideration:

$$\kappa_2 \left({}^{C}_{0^+}\beth^{\beta}_{\mu} x\right)(t) = {}^{C}_{0^+}\beth^{\beta}_{\mu}\left[\sum_{i=0}^{n-2}\left(\frac{\mu^i(t)}{\Gamma(i+1)} + \kappa_1 \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\right)\phi_0^{(i)} + \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)}\right.$$
$$\left. + \kappa_1 \int_{-h}^{\min\{t-h,0\}} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s)\right]$$
$$= \kappa_1\kappa_2 \sum_{i=0}^{n-2} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha-\beta+i+1}\left(\kappa_1,\kappa_2;t,h\right)\phi_0^{(i)} + \kappa_2 \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m-\beta}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)}$$
$$+ \kappa_1\kappa_2 \int_{-h}^{\min\{t-h,0\}} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha-\beta}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s), \tag{4}$$

and

$$\kappa_1 x(t-h) = \kappa_1 \sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)} + \kappa_1 \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,2h\right)\right]\phi_0^{(i)} + \kappa_1 \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,2\right)\phi_0^{(n-1)}$$
$$+ \kappa_1^2 \int_{-h}^{\min\{t-h,0\}} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+2h\right)\phi(s)d\mu(s). \tag{5}$$

Taking a linear combination of (3), (4), and (5) provides the desired result. □

3.2 | Integral representation of solutions to the linear inhomogeneous μ-Caputo type fractional Langevin type DDEs with general fractional orders

Before starting main theorem, we share a simple lemma as follows:

Lemma 3. For $n-1 < \alpha \le n$, $n-2 < \beta \le n-1$ with $n \ge 2$ and $\alpha - \beta \ge 1$, the following equation holds true:

$$\int_{r+mh}^{t} \left(\mu(t)-\mu(s)\right)^{n-\alpha-1}\left(\mu(s)-\mu(r+mh)\right)^{m\alpha+p(\alpha-\beta)+\alpha-1} d\mu(s) = \left(\mu(s)-\mu(r+mh)\right)^{m\alpha+p(\alpha-\beta)+n-1} B(n-\alpha, m\alpha+p(\alpha-\beta)+\alpha).$$

Since the proof is based on the only substitution $(\mu(t)-\mu(r+mh))\,y = \mu(t)-\mu(s)$, it is enough easy to omit it.

Theorem 3. With x belonging to the Banach space of absolutely continuously differentiable functions up to order n on $[0,T]$,

$$x(t) = \int_0^t \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)\daleth(s)d\mu(s), \quad h>0, \tag{6}$$

is a solution of the Cauchy type problem (1) with zero initial conditions $x(t)=0$ for $t\in[-h,0)$, and $x^{(k)}(0)$, $0\le k\le n-1$.

Proof. We will exploit the variation of constants' approach to prove this theorem. Assume that x is a solution of non-homogenous system. Then it must be in the below form:

$$x(t) = \int_0^t \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)\eta(s)d\mu(s), \quad h>0,$$

where $\eta(s)$, $0\le s\le t$ is an unknown differentiable function and $x(0)=0$. Having Theorem 1 and the given conditions in the statement of this theorem, we acquire

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu} x\right)(t) = \left({}^{RL}_{0^+}\beth^{\alpha}_{\mu} x\right)(t).$$

Let's continue to finish proving in the this step or calculation.

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}(\mu(t)-\mu(s))^{n-\alpha-1}\int\limits_{0}^{s}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;s,r\right)\eta(r)d\mu(r)\,d\mu(s)$$

$$=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}\int\limits_{r}^{t}(\mu(t)-\mu(s))^{n-\alpha-1}\,\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;s,r\right)\eta(r)d\mu(s)\,d\mu(r)$$

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\frac{\kappa_1^m\kappa_2^p}{\Gamma(m\alpha+p(\alpha-\beta)+\alpha)}\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}\eta(r)$$

$$\times\left[\int\limits_{r}^{t}(\mu(t)-\mu(s))^{n-\alpha-1}(\mu(s)-\mu(r+mh))^{m\alpha+p(\alpha-\beta)+\alpha-1}\,{}_{\mu}\mathcal{H}^{s}_{r+mh}d\mu(s)\right]d\mu(r).$$

By applying Lemma 3 to the inner integral in the last equation above, we get

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t)=\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{m+p}{p}\kappa_1^m\kappa_2^p\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}\frac{(\mu(t)-\mu(r+mh))^{m\alpha+p(\alpha-\beta)+n-1}\,{}_{\mu}\mathcal{H}^{s}_{r+mh}}{\Gamma(m\alpha+p(\alpha-\beta)+n)}\eta(r)d\mu(r)$$

by utilizing the binomial identity, we obtain

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t)=\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}\frac{(\mu(t)-\mu(r))^{n-1}\,{}_{\mu}\mathcal{H}^{s}_{r}}{\Gamma(n)}\eta(r)d\mu(r)$$

$$+\sum_{m=1}^{\infty}\sum_{p=0}^{\infty}\binom{m+p-1}{p}\kappa_1^m\kappa_2^p\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}\frac{(\mu(t)-\mu(r+mh))^{m\alpha+p(\alpha-\beta)+n-1}\,{}_{\mu}\mathcal{H}^{s}_{r+mh}}{\Gamma(m\alpha+p(\alpha-\beta)+n)}\eta(r)d\mu(r)$$

$$+\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\binom{m+p-1}{p-1}\kappa_1^m\kappa_2^p\left(\frac{d}{d\mu(t)}\right)^{n}\int\limits_{0}^{t}\frac{(\mu(t)-\mu(r+mh))^{m\alpha+p(\alpha-\beta)+n-1}\,{}_{\mu}\mathcal{H}^{s}_{r+mh}}{\Gamma(m\alpha+p(\alpha-\beta)+n)}\eta(r)d\mu(r)$$

applying Leibniz rule for higher-order derivatives, we acquire

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t)=\eta(t)+\sum_{m=1}^{\infty}\sum_{p=0}^{\infty}\binom{m+p-1}{p}\kappa_1^m\kappa_2^p\int\limits_{0}^{t}\frac{(\mu(t)-\mu(r+mh))^{m\alpha+p(\alpha-\beta)-1}\,{}_{\mu}\mathcal{H}^{s}_{r+mh}}{\Gamma(m\alpha+p(\alpha-\beta))}\eta(r)d\mu(r)$$

$$+\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\binom{m+p-1}{p-1}\kappa_1^m\kappa_2^p\int\limits_{0}^{t}\frac{(\mu(t)-\mu(r+mh))^{m\alpha+p(\alpha-\beta)-1}\,{}_{\mu}\mathcal{H}^{s}_{r+mh}}{\Gamma(m\alpha+p(\alpha-\beta))}\eta(r)d\mu(r).$$

As a result, we get

$$\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t)=\eta(t)+\kappa_1\int\limits_{0}^{t}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)\eta(s)d\mu(s)+\kappa_2\int\limits_{0}^{t}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha-\beta}\left(\kappa_1,\kappa_2;t,s\right)\eta(s)d\mu(s)$$

$$=\eta(t)+\kappa_2\left({}^{C}_{0^+}\beth^{\beta}_{\mu}x\right)(t)+\kappa_1x(t-h),$$

which ensures that $\daleth(t)=\eta(t)$ for $\in[0,T]$. This is the end of the proof. □

Theorem 4. An explicit solution $x \in AC^n(I)$ of the Cauchy type problem (1) has the below form

$$
\begin{aligned}
x(t) = & \sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)} + \kappa_1 \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\right]\phi_0^{(i)} + \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)} \\
& + \kappa_1 \int_{-h}^{\min\{t-h,0\}} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s) \\
& + \int_0^t \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)\daleth(s)d\mu(s).
\end{aligned} \tag{7}
$$

Proof. When theorems 2 and 3 are considered simultaneously, the proof of this theorem is clear. □

Remark 4. Just as the μ-Caputo type time-delay fractional Langevin differential equations (1) contain a few of fractional differential equations, the solutions (7) contain some different ones, too.

1. For $h = 0$, the system (1) turns into Langevin type linear inhomogeneous fractional system. Solutions (7) overlaps with the explicit solutions of Langevin fractional system with general fractional orders by means of the bivariate M-L type function in the reference [21], as well as coinciding with the solutions with the aid of the Fox-Wright functions given in the reference [22].
2. Solutions (7) with κ_1 match up with the solutions of Caputo fractional differential equations with the help of two-parameter M-L or Wiman's functions which is considered in the reference [22].
3. Solutions (7) with κ_2 are the general case of the solutions of fractional delayed differential system with constant coefficient which are examined for $\alpha \in (0,1]$ in the references [20,2].

3.3 | Existence and uniqueness problem for nonlinear time-delay FLE

In this section, we introduce the nonlinear μ-Caputo type time-delay fractional Langevin differential equations. To be able to discuss the existence and uniqueness, we have to obtain its solutions. After that, we debate the existence and uniqueness of the solutions.

In order to prove the main theorem, which will be expressed below, with a different approach, we need to talk about the known operational process. As mentioned in the references [22,29], the generalised fractional operators like ${}^{C}_{0^+}\beth^{\alpha}_{\mu}$, ${}^{RL}_{0^+}\beth^{\alpha}_{\mu}$, and ${}^{RL}_{0^+}\gimel^{\alpha}_{\mu}$ can be expressed as the conjugation of the classical fractional operators with the help of the operation of composition with μ or μ^{-1} :

$$
{}^{RL}_{0^+}\gimel^{\alpha}_{\mu} = \mathcal{Q}_\mu \circ \left({}^{RL}_{\mu(0)^+}\gimel^{\alpha}_{t}\right) \circ \mathcal{Q}_\mu^{-1}, \quad {}^{RL}_{0^+}\beth^{\alpha}_{\mu} = \mathcal{Q}_\mu \circ \left({}^{RL}_{\mu(0)^+}\beth^{\alpha}_{t}\right) \circ \mathcal{Q}_\mu^{-1}, \quad {}^{C}_{0^+}\beth^{\alpha}_{\mu} = \mathcal{Q}_\mu \circ \left({}^{C}_{\mu(0)^+}\beth^{\alpha}_{t}\right) \circ \mathcal{Q}_\mu^{-1},
$$

where $\mathcal{Q}_\mu$ is the substitution operator

$$
\mathcal{Q}_\mu(x) = x(\mu),
$$

and $\mathcal{Q}_\mu^{-1}$ is its inverse operator and $\left({}^{RL}_{\mu(0)^+}\gimel^{\alpha}_{t}\right)$, $\left({}^{RL}_{\mu(0)^+}\beth^{\alpha}_{t}\right)$, $\left({}^{C}_{\mu(0)^+}\beth^{\alpha}_{t}\right)$ are the classical Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, respectively.

It is time to state the main theorem.

Theorem 5. The solution $x \in AC^n(I)$ of the nonlinear μ-Caputo type time-delay fractional Langevin differential equations

$$
\begin{cases}
\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t) - \kappa_2\left({}^{C}_{0^+}\beth^{\beta}_{\mu}x\right)(t) - \kappa_1 x(t-h) = \daleth(t, x(t)), & t \in (0,T], h > 0, \\
x(t) = \phi(t), & -h \le t \le 0,
\end{cases} \tag{8}
$$

where ${}^{C}_{0^+}\beth^{\alpha}_{\mu}$ and ${}^{C}_{0^+}\beth^{\beta}_{\mu}$ are μ-Caputo type fractional derivatives, $n-1 < \alpha \le n$, $n-2 < \beta \le n-1$, $n \ge 2$ with $\alpha - \beta \ge 1$ and $\daleth : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear disturbance. $\phi : [-h,0] \to \mathbb{R}$ is an arbitrary $(n-1)$-times continuously differentiable,

$\kappa_1, \kappa_2 \in \mathbb{R}$, $T = lh$, $n \in \mathbb{N}$ which is the set of all natural numbers, has the following form:

$$x(t) = \sum_{i=0}^{n-2} \left[\frac{\mu^i(t)}{\Gamma(i+1)} + \kappa_1 \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1} \left(\kappa_1, \kappa_2; t, h\right) \right] \phi_0^{(i)} + \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m} \left(\kappa_1, \kappa_2; t, 0\right) \phi_0^{(n-1)}$$
$$+ \kappa_1 \int_{-h}^{\min\{t-h,0\}} \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha} \left(\kappa_1, \kappa_2; t, s+h\right) \phi(s) d\mu(s) + \int_0^t \mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha} \left(\kappa_1, \kappa_2; t, s\right) \daleth(s, x(s)) d\mu(s). \tag{9}$$

Proof. According to Theorem 4, it is easy to see that the solution of the nonlinear μ-Caputo type time-delay fractional Langevin differential equations is as given in (9). Now, we try to obtain the same solution with a different approach.

By exploiting the identity ${}^{C}_{0^+}\beth^{\alpha}_{\mu} = Q_\mu \circ \left({}^{C}_{\mu(0)^+}\beth^{\alpha}_{t}\right) \circ Q_\mu^{-1}$, the nonlinear μ-Caputo type time-delay fractional Langevin differential equations in (8) could be transformed into the nonlinear Caputo type time-delay fractional Langevin differential equations in the reference[9] as follows:

$$\begin{cases} \left({}^{C}_{\mu(0)^+}\beth^{\alpha}_{t} y\right)(t) - \kappa_2 \left({}^{C}_{\mu(0)^+}\beth^{\beta}_{t} y\right)(t) - \kappa_1 y(t-h) = \digamma(t, y(t)), \\ y(t) = \varphi(t), \end{cases} \tag{10}$$

where $y = Q_\mu^{-1} x$, $\digamma = Q_\mu^{-1} \daleth$, and $\varphi = Q_\mu^{-1} \phi$. Again, in the same reference[9], the solution of system (10) is given by

$$x(t) = \sum_{i=0}^{n-2} \left[\frac{t^i}{\Gamma(i+1)} + \kappa_1 \mathrm{E}^{h}_{\alpha,\alpha-\beta,\alpha+i+1} \left(\kappa_1, \kappa_2; t-h\right) \right] \phi_0^{(i)} + \mathrm{E}^{h}_{\alpha,\alpha-\beta,m} \left(\kappa_1, \kappa_2; t\right) \phi_0^{(n-1)}$$
$$+ \kappa_1 \int_{-h}^{\min\{t-h,0\}} \mathrm{E}^{h}_{\alpha,\alpha-\beta,\alpha} \left(\kappa_1, \kappa_2; t-s-h\right) \phi(s) ds + \int_0^t \mathrm{E}^{h}_{\alpha,\alpha-\beta,\alpha} \left(\kappa_1, \kappa_2; t-s\right) \digamma(s, x(s)) ds, \tag{11}$$

where $\mathrm{E}^h_{\alpha_1,\alpha_2,\alpha_3}\left(\kappa_1, \kappa_2; t\right)$ is the delayed Mittag-Leffler type matrix function as in the reference[28]. By applying Q_μ^{-1} to both sides of (11), we get the solution (9) as requested. This completes the proof. □

Lemma 4. The below inequalities hold true:

- $\left|\mathrm{E}^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3+k}\left(\lambda_1, \lambda_2; t, s\right)\right| \le [\mu(t) - \mu(s)]^{\alpha_3+k-1} e^{|\lambda_1|\mu^{\alpha_1}(t) + |\lambda_2|\mu^{\alpha_2}(t)}$, for $k = 0, 1, 2, ..., n-1$,
- $\int_0^t (\mu(t) - \mu(s))^{\alpha-1} e^{w\mu(s)} d\mu(s) \le e^{w\mu(t)} \frac{\Gamma(\alpha)}{w^\alpha}$.

Lemma 5. Let's start with the first item,

$$\begin{aligned} \left|\mathrm{E}^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3+k}\left(\lambda_1, \lambda_2; t, s\right)\right| &\le \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \binom{m+p}{p} \frac{|\lambda_1|^m |\lambda_2|^p \left(\mu(t) - \mu(s+mh)\right)^{m\alpha_1 + p\alpha_2 + \alpha_3 + k - 1}}{\Gamma(m\alpha_1 + p\alpha_2 + \alpha_3 + k)} {}_{\mu}\mathcal{H}^t_{s+mh} \\ &\le \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \binom{m+p}{p} \frac{|\lambda_1|^m |\lambda_2|^p \mu(t)^{m\alpha_1 + p\alpha_2 + \alpha_3 + k - 1}}{\Gamma(m\alpha_1 + p\alpha_2 + \alpha_3 + k)} \\ &= [\mu(t) - \mu(s)]^{\alpha_3 + k - 1} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(m+p)!}{m!p!} \frac{|\lambda_1|^m |\lambda_2|^p [\mu(t)]^{m\alpha_1 + p\alpha_2}}{\Gamma(m\alpha_1 + p\alpha_2 + \alpha_3 + k)}, \end{aligned}$$

Since $n-1 < \alpha \le n$, $n-2 < \beta \le n-1$ with $n \ge 2$ and $\alpha - \beta \ge 1$, we have

$$\Gamma(m\alpha_1 + p\alpha_2 + \alpha_3 + k) > \Gamma(m + p + 1) = (m+p)!.$$

With this relation, we get

$$\begin{aligned}\left|\mathrm{E}^{h,\mu}_{\alpha_1,\alpha_2,\alpha_3+k}\left(\lambda_1,\lambda_2;t,s\right)\right| &\le [\mu(t)-\mu(s)]^{\alpha_3+k-1}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(m+p)!}{m!p!}\frac{|\lambda_1|^m|\lambda_2|^p[\mu(t)]^{m\alpha_1+p\alpha_2}}{\Gamma(m\alpha_1+p\alpha_2+\alpha_3+k)}\\ &= [\mu(t)-\mu(s)]^{\alpha_3+k-1}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(m+p)!}{m!p!}\frac{|\lambda_1|^m|\lambda_2|^p[\mu(t)]^{m\alpha_1+p\alpha_2}}{(m+p)!}\\ &= [\mu(t)-\mu(s)]^{\alpha_3+k-1}\sum_{m=0}^{\infty}\frac{|\lambda_1|^m[\mu(t)]^{m\alpha_1}}{m!}\sum_{p=0}^{\infty}\frac{|\lambda_2|^p[\mu(t)]^{p\alpha_2}}{p!}\\ &= [\mu(t)-\mu(s)]^{\alpha_3+k-1}e^{|\lambda_1|\mu^{\alpha_1}(t)+|\lambda_2|\mu^{\alpha_2}(t)}.\end{aligned}$$

With the aid of the substitution $u=\mu(t)-\mu(s)$, we get

$$\int_0^t(\mu(t)-\mu(s))^{\alpha-1}e^{w\mu(s)}d\mu(s)=\int_0^{\mu(t)}u^{\alpha-1}e^{w(\mu(t)-u)}du=e^{w\mu(t)}\int_0^{\mu(t)}u^{\alpha-1}e^{-wu}du.$$

Now, again with the aid of another substitution $v=wu$, we get

$$\int_0^t(\mu(t)-\mu(s))^{\alpha-1}e^{w\mu(s)}d\mu(s)=e^{w\mu(t)}\int_0^{w\mu(t)}\frac{v^{\alpha-1}}{w^\alpha}e^{-v}dv\le e^{w\mu(t)}\int_0^{\infty}\frac{v^{\alpha-1}}{w^\alpha}e^{-v}dv=e^{w\mu(t)}\frac{\Gamma(\alpha)}{w^\alpha}.$$

In order to guarantee that solution of system (8) is existent and unique, we must put two main conditions over the disturbance $\daleth(t,x(t))$ as described below:

O_1 : The function $\daleth:[0,T]\times\mathbb{R}\to\mathbb{R}^n$ is continuous;

O_2 : $|\daleth(t,x)-\daleth(t,y)|\le L_\daleth|x-y|,\quad t\in[0,T],\quad x,y\in\mathbb{R},\quad L_f>0.$

Theorem 6. If O_1 and O_2 are satisfied, then system (8) has a unique global continuous solution on $[0,T]$

Proof. We will define a ball and an integral operator on it. Let $\mathfrak{D}_r=\{x\in C([0,T],\mathbb{R}):\|y\|_w\le r,\ w>0\}$ with

$$r=\frac{w^\alpha\left(R_1+R_2\right)+\left(|\kappa_1|R_3\|\phi\|_w+R_3R_4\right)\Gamma(\alpha)}{w^\alpha-L_\daleth R_3\Gamma(\alpha)} \tag{12}$$

where

$$R_1=\sum_{j=0}^{n-2}\frac{\mu^j(T)}{\Gamma(j+1)}|\phi_0^{(j)}|,\quad R_3=e^{|\kappa_1|\mu^\alpha(T)+|\kappa_2|\mu^{\alpha-\beta}(T)},\quad R_4=\max\left\{\frac{|\daleth(t,0)|}{e^{w\mu(t)}},0\le t\le T\right\},$$

$$R_2=\sum_{j=0}^{n-2}|\kappa_1|\mu^{\alpha+j}(T)|\phi_0^{(j)}|R_3+\mu^{(m-1)}(T)|\phi_0^{(m-1)}|R_3.$$

It is time to define an integral operator $\mathfrak{G}$ on $\mathfrak{D}_r$ as noted below:

$$\mathfrak{G}:\mathfrak{D}_r\subset C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$$

by the following formula

$$\begin{aligned}\mathfrak{G}x(t)=&\sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+\kappa_1\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)\right]\phi_0^{(i)}+\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,0\right)\phi_0^{(n-1)}\\ &+\kappa_1\int_{-h}^{\min\{t-h,0\}}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)\phi(s)d\mu(s)+\int_0^t\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)\daleth(s,x(s))d\mu(s).\end{aligned}$$

Because of O_1, $\mathfrak{G}$ is well-defined. So, the existence and uniqueness of solution of the Cauchy type problem (8) in $AC^n([0,T],\mathbb{R})$ is equivalent to the existence and uniqueness of a fixed point of the integral operator $\mathfrak{G}$ on $\mathfrak{D}_r\subset C([0,T],\mathbb{R})$. To achieve this, we use contraction mapping principle.

With a simple calculation, one can easily show that the norms $\|.\|_\infty$ and $\|.\|_w$ are equivalent. Eventually, since $\|.\|_\infty$ and $\|.\|_w$ are equivalent, and also $\left(C([0,T],\mathbb{R}),\|.\|_\infty\right)$ is a complete metric space, $\left(C([0,T],\mathbb{R}),\|.\|_w\right)$ must be a complete metric space. First of all, we will show that $\mathfrak{G}\left(\mathfrak{D}_r\right)\subset\mathfrak{D}_r$, that is, $\mathfrak{G}x\in\mathfrak{D}_r$ for all $x\in\mathfrak{D}_r$. Take an arbitrary $x\in\mathfrak{D}_r$.

$$
\begin{aligned}
|\left(\mathfrak{G}x\right)(t)| \le{}& \sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+|\kappa_1||\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha+i+1}\left(\kappa_1,\kappa_2;t,h\right)|\right]|\phi_0^{(i)}|+|\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,m}\left(\kappa_1,\kappa_2;t,0\right)||\phi_0^{(n-1)}|\\
&+|\kappa_1|\int_{-h}^{\min\{t-h,0\}}|\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s+h\right)||\phi(s)|d\mu(s)\\
&+\int_0^t|\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)||\daleth(s,x(s))-\daleth(s,0)+\daleth(s,0)|d\mu(s)
\end{aligned}
$$

By applying O_2 and Lemma 4 and dividing both sides by $e^{w\mu(t)}$ we acquire

$$
\begin{aligned}
\frac{|\left(\mathfrak{G}x\right)(t)|}{e^{w\mu(t)}} \le{}& \sum_{i=0}^{n-2}\left[\frac{\mu^i(t)}{\Gamma(i+1)}+|\kappa_1||\mu^{\alpha+i}(t)|e^{|\kappa_1|\mu^{\alpha}(t)+|\kappa_2|\mu^{\alpha-\beta}(t)}\right]|\phi_0^{(i)}|+|\mu^{m-1}(t)|e^{|\kappa_1|\mu^{\alpha}(t)+|\kappa_2|\mu^{\alpha-\beta}(t)}|\phi_0^{(n-1)}|\\
&+\frac{|\kappa_1|}{e^{w\mu(t)}}\int_{-h}^{0}\left(\mu(t)-\mu(s)\right)^{\alpha-1}|\phi(s)|d\mu(s)e^{|\kappa_1|\mu^{\alpha}(t)+|\kappa_2|\mu^{\alpha-\beta}(t)}\\
&+\frac{L_\daleth}{e^{w\mu(t)}}\int_0^t\left(\mu(t)-\mu(s)\right)^{\alpha-1}|x(s)|d\mu(s)e^{|\kappa_1|\mu^{\alpha}(t)+|\kappa_2|\mu^{\alpha-\beta}(t)}\\
&+\frac{1}{e^{w\mu(t)}}\int_0^t\left(\mu(t)-\mu(s)\right)^{\alpha-1}|\daleth(s,0)|d\mu(s)e^{|\kappa_1|\mu^{\alpha}(t)+|\kappa_2|\mu^{\alpha-\beta}(t)}.
\end{aligned}
$$

By taking the μ-weighted maximum norm on the right hand side, we get

$$
\begin{aligned}
\frac{|\left(\mathfrak{G}x\right)(t)|}{e^{w\mu(t)}} \le{}& R_1+R_2+\frac{|\kappa_1|R_3\|\phi\|_w}{e^{w\mu(t)}}\int_0^h\left(\mu(t)-\mu(s)\right)^{\alpha-1}e^{w\mu(s)}d\mu(s)\\
&+\frac{L_\daleth R_3\|x\|_w}{e^{w\mu(t)}}\int_0^t\left(\mu(t)-\mu(s)\right)^{\alpha-1}e^{w\mu(s)}d\mu(s)\\
&+\frac{R_3R_4}{e^{w\mu(t)}}\int_0^t\left(\mu(t)-\mu(s)\right)^{\alpha-1}e^{w\mu(s)}d\mu(s)
\end{aligned}
$$

From the second item of Lemma 4, we obtain

$$
\frac{|\left(\mathfrak{G}x\right)(t)|}{e^{w\mu(t)}} \le R_1+R_2+\left(|\kappa_1|R_3\|\phi\|_w+L_\daleth R_3\|x\|_w+R_3R_4\right)\frac{\Gamma(\alpha)}{w^\alpha}
$$

Taking the μ-weighted maximum norm on the left hand side and exploiting equality (12), we acquire $\|\mathfrak{G}x\|_w\le r$, and so $\mathfrak{G}:\mathfrak{D}_r\to\mathfrak{D}_r$, which means $\mathfrak{G}$ is well-defined on $\mathfrak{D}_r$. Now, we prove that $\mathfrak{G}$ is a contraction on $[0,T]$. For $x,y\in[0,T]$, consider

$$\frac{|\left(\mathfrak{G}x\right)(t)-\left(\mathfrak{G}y\right)(t)|}{e^{w\mu(t)}}\leq\frac{1}{e^{w\mu(t)}}\int\limits_{0}^{t}|\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)||\daleth(s,x(s))-\daleth(s,y(s))|d\mu(s)$$

$$\leq\frac{L_{\daleth}R_3}{e^{w\mu(t)}}\int\limits_{0}^{t}\left(\mu(t)-\mu(s)\right)^{\alpha-1}e^{w\mu(s)}d\mu(s)\|x-y\|_w$$

$$\leq\frac{L_{\daleth}R_3\Gamma(\alpha)}{w^{\alpha}}\|x-y\|_w.$$

If we choose $w>\left(L_{\daleth}R_3\Gamma(\alpha)\right)^{\frac{1}{\alpha}}$, the integral operator $\mathfrak{G}$ is a contraction. By using contraction mapping principle, there is a unique fixed point of $\mathfrak{G}$ which is the unique global continuous solution of system (8). This completes the proof. □

Remark 5. Since the existence interval $[0,T]$ is independent of choices of the parameters, the interval can be extended to the interval $[0,\infty)$ provided that O_1 and O_2 are satisfied by $\daleth$ for each $t\in[0,\infty)$.

3.4 | Ulam-Hyers stability analysis on nonlinear time-delay FLEs

Definition 6. Let $\varepsilon>0$. The system (8) is said to be Ulam-Hyers stable if for every solution $y\in C\left([0,T],\mathbb{R}\right)$ of inequality,

$$\left\|\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}y\right)(t)-\kappa_2\left({}^{C}_{0^+}\beth^{\beta}_{\mu}y\right)(t)-\kappa_1 y(t-h)-\daleth\left(t,y(t)\right)\right\|\leq\varepsilon, \tag{13}$$

there is a solution $x\in C\left([0,T],\mathbb{R}\right)$ of the system (8), and $\sigma>0$ such that

$$\|y-x\|_w\leq\varepsilon\sigma\quad t\in[0,T]. \tag{14}$$

Remark 6. A function $y\in C^1\left([0,T],\mathbb{R}\right)$ is a solution of the inequality equation (13) if and only if there exists a function $f\in C\left([0,T],\mathbb{R}\right)$, such that

- $\|f(t)\|<\varepsilon$,
- $\left({}^{C}_{0^+}\beth^{\alpha}_{\mu}y\right)(t)-\kappa_2\left({}^{C}_{0^+}\beth^{\beta}_{\mu}y\right)(t)-\kappa_1 y(t-h)=\daleth\left(t,y(t)\right)+f(t)$.

Theorem 7. Suppose that O_1 and O_2 are satisfied by $\daleth$. The system (8) is Ulam-Hyers stable on $[0,T]$.

Proof. Let $y\in C\left([0,T],\mathbb{R}\right)$ which satisfies the inequality (13), and let $x\in C\left([0,T],\mathbb{R}\right)$ which is the unique solution to system (8) with the initial condition $x(t)=y(t)$ for $t\in[0,T]$. By keeping the definition of $\mathfrak{G}$ and Remark 6 in mind, we can obtain

$$\|f(t)\|<\varepsilon,\quad y(t)=\left(\mathfrak{G}y\right)(t)+\int\limits_{0}^{t}\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)f(s)d\mu(s),$$

and also $x=\left(\mathfrak{G}x\right)(t)$ for $t\in[0,T]$. One can easily obtain

$$|y(t)-\left(\mathfrak{G}y\right)(t)|\leq\int\limits_{0}^{t}|\mathrm{E}^{h,\mu}_{\alpha,\alpha-\beta,\alpha}\left(\kappa_1,\kappa_2;t,s\right)||f(s)|d\mu(s)\leq\mu^{\alpha-1}(T)R_3\varepsilon.$$

We can choose $w>\left(L_{\daleth}R_3\Gamma(\alpha)\right)^{\frac{1}{\alpha}}$. We are ready to estimate $\|y-x\|_w$:

$$\|y-x\|_w\leq\|\left(\mathfrak{G}x\right)-\left(\mathfrak{G}y\right)\|_w+\|\left(\mathfrak{G}y\right)-y\|_w$$

$$\leq\frac{L_{\daleth}R_3\Gamma(\alpha)}{w^{\alpha}}\|x-y\|_w+\mu^{\alpha-1}(T)R_3\varepsilon,$$

which yields

$$\|y-x\|_w\leq\varepsilon\sigma,$$

where

$$\sigma=\frac{w^{\alpha}\mu^{\alpha-1}(T)R_3}{w^{\alpha}-L_{\daleth}R_3\Gamma(\alpha)}>0,$$

which is the desired result. □

Remark 7. The result stability is valid for $[0, \infty)$ because of the same explanations as Remark 5.

3.5 | Numerical and simulated examples

Example 8. In order to exemplify the linear inhomogeneous μ-Caputo type time-delay fractional Langevin type differential equations, we firstly consider the following system

$$\begin{cases} \left({}^{C}_{0^+}\beth^{\alpha}_{\mu}x\right)(t) - 10\left({}^{C}_{0^+}\beth^{\beta}_{\mu}x\right)(t) - 15x\,(t-0.2) = t^3, \quad t \in (0, 0.4]\,, \\ x\,(t) = t^2 - 1, \quad -0.2 \le t \le 0, \end{cases} \tag{15}$$

where $\alpha = 2$, $\beta = 1$, $\mu(t) = t + 1$. The exact analytical solution of system (15) is given by

$$x(t) = 1 - 15\mathrm{E}^{0.2,t+1}_{2,1,3}(15, 10; t, 0.2) + 15\int_{-0.2}^{\min\{t-0.2,0\}} \mathrm{E}^{0.2,t+1}_{2,1,2}(15, 10; t, s+0.2)\,(s^2-1)ds + \int_{0}^{t} \mathrm{E}^{0.2,t+1}_{2,1,2}(15, 10; t, s)\,s^3 ds,$$

which can be obtained from Theorem 4, and easily verified whether it satisfies system (15). The graph of the solution $x(t)$ can be found in Figure 3

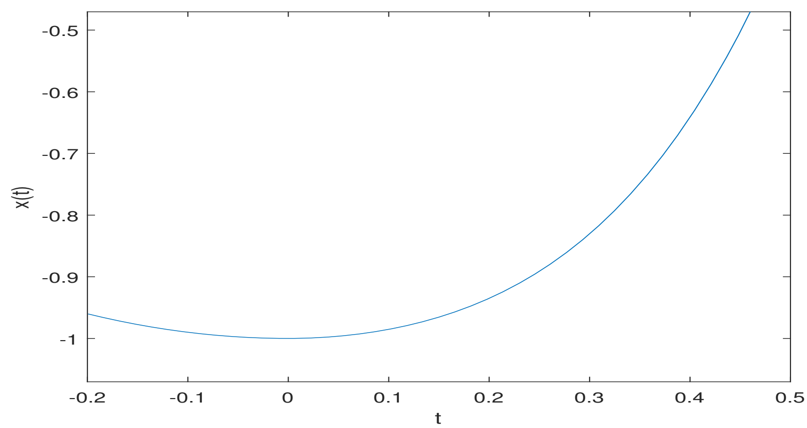

FIGURE 3 Graph of solution $x(t)$ to system (15).

Example 9. To illustrate the theoretical result of Theorem 6, we will examine the following non-linear inhomogeneous $\sin t$-Caputo type time-delay fractional Langevin type differential equations

$$\begin{cases} \left({}^{C}_{0^+}\beth^{8.7}_{\sin t}x\right)(t) - 40\left({}^{C}_{0^+}\beth^{7.2}_{\sin t}x\right)(t) - 10x\,(t-0.6) = \frac{e^{2t}}{25\left(1+e^{2t}\right)}\sin\left(x(t)\right), \quad t \in (0, 6]\,, \\ x\,(t) = 25t^2, \quad -0.2 \le t \le 0. \end{cases} \tag{16}$$

To ensure that the solution of system (16) is existent and unique, two conditions of Theorem 6 are confirmed for this system. We know that composition of two continuous functions is continuous. $\daleth(t, x(t)) = \frac{e^{2t}}{25\left(1+e^{2t}\right)}\sin\left(x(t)\right)$ since $x(t), \sin t, e^{2t} \in C([-0.2, 5], \mathbb{R})$. Also, $\daleth$ satisfies the Lipschitz condition wit $L_{\daleth} = 0.4$. According to Theorem 6, solutions of system (16) is existent and unique on $C([-0.2, 5], \mathbb{R}) \cong \mathfrak{D}_r$ with $r \cong \infty$.

Example 10. To show the validity of Theorem 7, we will take into consideration the following non-linear inhomogeneous $\log t$-Caputo type time-delay fractional Langevin type differential equations

$$\begin{cases} \left({}^{C}_{0^+}\beth^{11.1}_{\log t}x\right)(t) - \frac{1}{5}\left({}^{C}_{0^+}\beth^{10}_{\log t}x\right)(t) - \frac{\sqrt{2}}{2}x\,(t-0.4) = \frac{\arctan x(t)}{\pi}, \quad t \in (0, 2]\,, \\ x\,(t) = \phi\,(t)\,, \quad -h \le t \le 0. \end{cases} \tag{17}$$

It is obvious that $\daleth(t, x(t)) = \frac{\arctan x(t)}{\pi}$ is continuous. For an arbitrary $x, y \in C([-0.4, 2], \mathbb{R})$, we have

$$\left\| \frac{\arctan x}{\pi} - \frac{\arctan y}{\pi} \right\|_w \leq \frac{1}{\pi} \|x - y\|_w$$

which provides that $\daleth(t, x(t)) = \frac{\arctan x(t)}{\pi}$ satisfies Lipschitz condition with $L_\daleth = 0.5$. Since two conditions of Theorem 7 are satisfied by $\daleth(t, x(t)) = \frac{\arctan x(t)}{\pi}$, in accordance with the result of Theorem 7, system (17) is stable in the sense of Ulam-Hyers approach.

Remark 8. With some special cases of μ, we obtain the Caputo fractional derivative[29], Hadamard fractional derivative[22], the Caputo–Hadamard fractional derivative[30 31] and the Caputo–Erdélyi–Kober fractional derivative[32]. So, if we properly change types of fractional derivatives as noted above, then all of the findings and results are valid for them. This also makes the present paper wealthier and more valuable.

4 | CONCLUSION

In this work, we firstly present the nonlinear μ-Caputo type fractional delayed Langevin differential equations involving two distinct fractional orders. To acquire an exact analytical solution to linear Langevin system and a global solution formula to the semilinear Langevin system, we novelly propose the μ-delayed Mittag-Leffler type function which is quite different from ones in the references[3 9]. We estimate μ-delayed Mittag-Leffler type function on the grounds of exponential function. Existence uniqueness and Ulam-Hyers stability of solutions to nonlinear delayed Langevin fractional differential equations are obtained with regard to the weighted norm defined in accordance with exponential function and on the grounds of the fixed point approach.

From the viewpoint of fractional(ordinary) Langevin equations and some of their qualitative properties, this paper includes various sorts of comprehensive works as stated in Remarks 4 and 8.

To extend this paper, there are many possibilities. For example, the solution of nonlinear delayed Langevin equations can be found with the aid of the Laplace transform which is the most powerful tool for differential equations. Also, the next further work can be devoted to study exponential stability, finite time stability, asymptotic stability, and also Lyapunov type stability of the μ- Caputo type fractional delayed Langevin equations. Another direction for additional studies is to investigate approximate controllability, P-type and D-type learning laws concepts for controlled Langevin type delayed systems with (general) fractional orders. For $\mathbb{R}^n$, the proposed system and everything recommended on in can be reconsidered.

CONFLICT OF INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could haveappeared to influence the work reported in this paper. This work does not have any conflicts of interest.

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