The Propagation of Hermite-Gauss wavepackets in Optics and Quantum Mechanics

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Abstract. The two-dimensional paraxial equation of optics and the two-dimensional time-dependent Schrödinger equation, derived as approximations of the three-dimensional Helmholtz equation and the three-dimensional time-independent Schrödinger equation respectively, are identical. Here the free propagation in space and time of Hermite-Gauss wavepackets (optics) or Harmonic Oscillator eigenfunctions (quantum mechanics) is examined in detail. The Gouy phase is shown to be a dynamic phase, appearing as the integral of the adiabatic eigenfrequency or eigenenergy. The wave packets propagate adiabatically in that at each space or time point they are solutions of the instantaneous harmonic problem. In both cases, it is shown that the form of the wave function is unchanged along the loci of the normals to wave fronts. This invariance along such trajectories is connected to the propagation of the invariant amplitude of the corresponding free wave number (optics) or momentum (quantum mechanics) wavepackets. It is shown that the van Vleck classical density of trajectories function appears in the wave function amplitude over the complete trajectory. A transformation to the co-moving frame along a trajectory gives a constant wave function multiplied by a simple energy or frequency phase factor. The Gouy phase becomes the proper time in this frame.

Keywords: Hermite-Gauss wave packets, Optics and quantum mechanics, Gouy phase, Bohm trajectory

1. Introduction

In optics texts, the propagation of Gaussian beams in one space direction is a standard problem [1, 2, 3]. Similarly it is a standard problem of wave function propagation in time in quantum mechanics (QM) and treated in very many text books, good examples are Cohen-Tannoudji et.al. in Ref. [4] or the book of Holland Ref. [5]. Nonetheless, in the following some novel features of this problem are illuminated.

Usually only a simple Gaussian function is considered, for example, in optics, as the output of a laser or, in QM as the harmonic oscillator ground state. Here the extension to the propagation of laser beams of general Hermite-Gauss form is treated. The terminology Hermite-Gauss is from optics, in QM these functions appear as the infinite number of energy eigenfunctions of the harmonic oscillator (HO).

It is often pointed out that there is a similarity of the paraxial equation (PE) of optics to the two-dimensional time-dependent Schrödinger equation (TDSE) of QM, although the propagation of the PE is in one space direction z, whereas the propagation in the TDSE is in the time coordinate. However, as shown in Ref. [6], the PE and TDSE are identical in that the approximation steps to derive the PE from the three-dimensional Helmholtz equation are *exactly* those leading from the three-dimensional time-independent Schrödinger equation to the two-dimensional TDSE.

Unfortunately, in the solution of the PE and TDSE describing the propagation of initial HG wave functions, the commonly-used nomenclature and mathematical notations differ between optics and QM. This clouds the insight into the equivalence of the two problems. One aim of this paper is to unite the concepts used in the two fields by providing simple formulae to transcribe between optics and QM notations.

The main aim, however, is to throw new light on the free propagation in space of Hermite-Gauss beams described by the PE, or equivalently, with the propagation in time of the TDSE of QM. In particular the emergence of the invariant wave function in wave number or momentum space along trajectories defined by the normals to the wave fronts is emphasised. The Gouy phase, well known in optics [1, 2, 3], is re-interpreted and shown to play a crucial role, especially in the transformation to a frame co-moving along the trajectory.

2. Equivalence of the PE and TDSE

The precise equivalence of the derivation of the 2-dimensional PE from the 3-dimensional Helmholtz equation and the derivation of the 2-dimensional TDSE from the 3-dimensional Schrödinger equation is given in Ref. [6].

The PE for classical light of wave number k propagating through a vacuum is of the form

$$\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)+2ik\,\frac{\partial\Psi}{\partial z}=0, \tag{1}$$

where $\Psi(x,y,z)$ is a scalar field variable for light of wave number k. In the paraxial approximation it is usually admissible to replace k by k_z, its component in the z direction of light propagation.

In two space dimensions (x, y) with force-free particle motion and hence a constant velocity v_z in the z direction, the TDSE for a particle of mass m is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)-i\hbar\frac{\partial\Psi}{\partial t}=0, \tag{2}$$

where $t = z/v_z$. Dividing this equation by $-\frac{\hbar^2}{2m}$ gives

$$\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)+2i\frac{m}{\hbar}\frac{\partial\Psi}{\partial t}=0, \tag{3}$$

The PE for light propagation is obtained from this equation by the standard replacement $m/\hbar \to k_z/c$ and putting $z = ct$ to obtain (1).

Alternatively, beginning with (1) one could define an effective mass of the optical field of energy E as $m \equiv E/c^2 = p/c = \hbar k_z/c$. Then substitution for k_z in (1) leads to the TDSE of (2). Hence, there is complete equivalence of PE and TDSE.

The mathematical equivalence of PE and TDSE is to be contrasted by their very different regions of physical application. For light, a characteristic distance of beam focussing is called the Rayleigh range z_R and is typically of the order of 100 millimetre. As defined below, the corresponding distance in QM is typically of the order of nanometres.

In the solution of the PE and TDSE for an initial HG wave, an important phase function arises, identical in the two equations and known as the Gouy phase. Here the origin and physical meaning of this phase is explained as arising from a quantised energy or frequency, and its equivalence in the QM and optics cases is demonstrated. It emerges as a manifestation of the property that the exact solution of the wave equation propagates *adiabatically* as space or time evolves. This adiabatic behaviour, showing that the wave packet adjusts as an HO eigenfunction, with corresponding adiabatic phase, at each step in the propagation is perhaps understandable in QM, where the velocity v_z can be varied. It is more remarkable in the optics case where the velocity can be c, the velocity of light.

In the QM of massive particles, beginning with Bohm [7], attempts have been made to introduce deterministic particle trajectories even in regions of the size of a wavelength. Such trajectories obey classical mechanics (CM) and therefore bring a new philosophical aspect into the QM of the Schrödinger equation [5, 8, 9, 10]. Recently, there have been suggestions to introduce similar trajectory concepts into the wave picture of the Helmholtz equation of optics [11, 12], also at distances less than the Rayleigh range, where normally wave fronts are shown. Here, if one accepts that such trajectories are merely the loci of the normals to wave fronts, their importance as conduits for the transport of the wave functions in k space is demonstrated. In particular, it is shown that the space wave function, evaluated at points along the trajectories, is form invariant. This is explained as due to the invariant form, under free propagation, of the corresponding wave number (optics) or momentum (quantum mechanics) function. In addition it is shown that, in a frame co-moving with the trajectory, the space wave function assumes this simple, invariant form where only the Gouy phase changes. By contrast, in the fixed laboratory frame the well-known analytic form of the space wave function is strongly time-dependent and ultimately spreads over all space.

3. Freely-moving Hermite-Gaussian wave packets.

Since the x and y solutions are identical due to the separability of the PE and TDSE equations, it suffices to consider only the solutions in the x coordinate. Well-known solutions of the space propagation are summarised and the Gouy phase interpreted as an adiabatic energy or frequency phase.

3.1. Quantum mechanics

In quantum mechanics it is recognised that the HG functions are the eigenfunctions of the HO with eigenenergy equal to $(n + 1/2)\hbar\omega$ where the classical frequency of the oscillator of mass m is given by $\omega = \hbar/(m\sigma^2)$. The eigenstates of the oscillator, of the HG form are

$$\Psi_n(x,0) = \frac{1}{(\pi\sigma^2)^{1/4}} \left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{x}{\sigma}\right) \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{4}$$

where H_n is a Hermite polynomial and the Gauss function is of width σ. This is the form of the wave packet at arbitrary initial time $t = 0$.

The HG wave packet in momentum space at time $t = 0$ is obtained as the Fourier transform (FT) of (4) and reads

$$\begin{aligned} \tilde{\Psi}_n(p_x,0) &= (2\pi\hbar)^{-1/2} \int \Psi(x,0)\, e^{-ip_x x/\hbar}\, dx \\ &= i^{-n} \left(\frac{\sigma^2}{\hbar^2\pi}\right)^{1/4} \left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sigma p_x}{\hbar}\right)\, e^{-\sigma^2 p_x^2/(2\hbar^2)} \end{aligned} \tag{5}$$

with width given by $\hbar/\sigma$. For later comparison with the optics case, one can put $p_x/\hbar = k_x$ in the argument of the above wavefunction, where k_x is the wave number. The corresponding probability distributions $|\Psi_n(x,0)|^2$ and $|\tilde{\Psi}_n(p_x,0)|^2$ are both normalised to unity.

In accordance with conservation of momentum, the momentum wave packet, upon propagation to finite times, is unchanged in shape, acquiring simply an energy-time phase factor,

$$\tilde{\Psi}_n(p_x,t) = \tilde{\Psi}_n(p_x,0) \exp\left(-\frac{i}{\hbar}\frac{p_x^2}{2m}t\right). \tag{6}$$

Using (5) this wave function is written in the form

$$\tilde{\Psi}_n(p_x,t) = i^{-n}\left(\frac{\sigma^2}{\hbar^2\pi}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sigma p_x}{\hbar}\right) \times \exp\left(-\frac{\sigma^2\, p_x^2(1+i\tau)}{2\hbar^2}\right), \tag{7}$$

where the characteristic time scale $T \equiv m\sigma^2/\hbar$ is introduced and $\tau \equiv t/T$ is defined. Note that the physical parameters m and $\hbar$ appear in the fixed time T but τ is dimensionless. This important constant time T was identified by Heisenberg [13] at the dawn of quantum mechanics. It will be shown to determine the crossover between quantum and classical behaviour.

Propagation of the coordinate space wave packet to finite times is readily evaluated from the inverse Fourier transform of (7) which gives

$$\begin{aligned}\Psi_n(x,t) &= (2\pi\hbar)^{-1/2}\int \tilde{\Psi}_n(p_x,t)\, e^{ip_x x/h}\, dp_x \\ &= \frac{1}{[\pi\sigma^2(1+i\tau)^2]^{\frac{1}{4}}}\left(\frac{1-i\tau}{1+i\tau}\right)^{n/2}\left(\frac{1}{2^n n!}\right)^{1/2} \\ &\times H_n\left(\frac{x}{\sigma(1+\tau^2)^{1/2}}\right)\exp\left[-\frac{x^2}{2\sigma^2(1+i\tau)}\right],\end{aligned} \tag{8}$$

a normalised HG function but now with a time-dependent complex width $\sigma(1+i\tau)^{1/2}$.

The complex normalisation factor incorporates what is known in optics as the Gouy phase. This can be simplified using

$$\frac{1}{(1+i\tau)^{1/2}} = \frac{1}{(1+\tau^2)^{1/4}}\exp\left[-\frac{i}{2}\arctan\tau\right] \tag{9}$$

and

$$\left[\frac{(1-i\tau)}{(1+i\tau)}\right]^{n/2} = \exp\left[-i\, n\arctan\tau\right] \tag{10}$$

Then one can write the wave packet (8) in the form

$$\begin{aligned}\Psi_n(x,t) &= \frac{1}{[\pi\sigma^2(1+\tau^2)]^{\frac{1}{4}}}\exp\left[-i(n+\frac{1}{2})\arctan\tau\right] \\ &\times\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{x}{\sigma(1+\tau^2)^{1/2}}\right) \\ &\times\exp\left[-\frac{x^2}{2\sigma^2(1+\tau^2)}\right]\exp\left[i\frac{x^2\,\tau}{2\sigma^2(1+\tau^2)}\right].\end{aligned} \tag{11}$$

This result shows, as well-known for the ground state, that the HG function expands in time according to the factor $\sigma(1+\tau^2)^{1/2}$. Apart from the phase factor involving x, the last factor on the r.h.s. of (11), there is the purely time-dependent Gouy phase $\exp\left[-i(n+\frac{1}{2})\arctan\tau\right]$.

To a quantum practitioner the appearance of the factor $(n+\frac{1}{2})$ in the Gouy phase immediately "rings a bell". As explained above, the energy $(n+\frac{1}{2})\hbar\omega$ is precisely the eigenenergy of the instantaneous HG harmonic oscillator wave function. Indeed, since the oscillator frequency at time zero is $\omega_0 \equiv 1/T = \hbar/(m\sigma^2)$, then the time-dependent frequency

$$\omega(t) \equiv \frac{1}{T(1+\tau^2)} \tag{12}$$

is defined and the corresponding time-dependent eigenenergy $E_n(t) \equiv (n+\frac{1}{2})\hbar\omega(t)$. From the integral representation of the arctan function, one has

$$\begin{aligned} \exp\left[-i(n+\frac{1}{2})\arctan\tau\right] &= \exp\left[-\frac{i}{\hbar}\int^t (n+\frac{1}{2})\hbar\omega(t')\,dt'\right] \\ &= \exp\left[-\frac{i}{\hbar}\int^t E_n(t')\,dt'\right]. \end{aligned} \tag{13}$$

This generalises the result given in Ref. [6] for the ground state. It gives a new interpretation of the Gouy phase as the "adiabatic" energy phase arising from the continuous propagation of the harmonic oscillator HG wave function. At each fixed time t, the wavefunction becomes an eigenfunction of the harmonic oscillator with effective potential $\frac{1}{2}m\omega^2(t)x^2$. Irrespective of the z-direction velocity v_z which decides the time scale t, the exact confined oscillator wave function evolves adiabatically. Now it is shown that the same is true for the classical HG wave packet of optics.

3.2. Paraxial optics

From the identity of the paraxial equation and the TDSE, all that is necessary to derive the propagation of a classical wave is to make the substitution $m/\hbar \to k_z/c$ in the quantum equations. However, since it is customary in optics to use the z direction, rather than time, as the propagation coordinate then the substitution $t = z/c$ is made. Furthermore, it is usual to replace the constant $\sqrt{2}\sigma$ of the quantum gaussian by the notation W_0 which is the width or "beam waist" at $z = 0$. With these changes, (11) becomes

$$\begin{aligned} \Psi_n(x,t) = &\frac{1}{[\pi W_0^2(1+\tau^2)/2]^{\frac{1}{4}}} \exp\left[-i(n+\frac{1}{2})\arctan\tau\right] \\ &\times \left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sqrt{2}x}{W_0(1+\tau^2)^{1/2}}\right) \\ &\times \exp\left[-\frac{x^2}{W_0^2(1+\tau^2)}\right] \exp\left[i\frac{x^2\,\tau}{W_0^2(1+\tau^2)}\right], \end{aligned} \tag{14}$$

where now $\tau \equiv z/z_R$ is a dimensionless "distance" and z_R is the Rayleigh length, which corresponds to the fixed time T in the quantum case via the correspondence $z_R/c \to T$. As T gives the demarcation of QM and CM characteristics, so z_R gives the demarcation between light wave and light beam descriptions.

The size of the beam waist is related to the Rayleigh length as $z_R = k_z W_0^2/2$. With the definitions $W(z) \equiv W_0(1+\tau^2)$ and the phase front curvature $R(z) \equiv z_R(1+\tau^2)/\tau$ the wave function can be expressed in the compact form

$$\begin{aligned}\Psi_n(x,t) = & \frac{1}{[\pi W_z^2/2]^{\frac{1}{4}}} \exp\left[-i(n+\frac{1}{2})\arctan\tau\right]\left(\frac{1}{2^n n!}\right)^{1/2} \\ & \times H_n\left(\frac{\sqrt{2}x}{W_z}\right)\exp\left[-\frac{x^2}{W_z^2}\right]\exp\left[i\frac{k_z x^2}{2R(z)}\right].\end{aligned} \tag{15}$$

This is the standard form of HG beams in optics, usually given in two transverse dimensions (x, y), see ch.3 of Saleh and Teich [1] for example.

Exactly as in quantum mechanics, it is shown that the Gouy phase has its origin in the "quantisation" of the transverse wave to fit into the confinement of the finite space occupied by the wave at each fixed z value. An examination of the harmonic oscillator equation giving the eigenenergies for $\Psi_n(x,t)$ in the quantum case shows that the $n = 0$ ground state corresponds in the classical case to the replacement of the quantum frequency $\omega = 1/T$ by the mode frequency $\omega = c/z_R$ i.e again $T \to z_R/c$. For finite distances z, as in (12) of the quantum case, an instantaneous frequency

$$\omega(z) \equiv \frac{1}{(z_R/c)(1+\tau^2)} = \frac{c}{z_R\left(1+(\frac{z}{z_R})^2\right)} \tag{16}$$

is defined.

The frequencies of all higher HG states n are given by $(n+1/2)\omega(z)$. Then, as in (13), the Gouy phase is explained as the accumulated adiabatic phase of the instantaneous eigenfrequencies

$$\begin{aligned}\exp\left[-i(n+\frac{1}{2})\arctan\tau\right] &= \exp\left[-i(n+\frac{1}{2})\arctan\left(\frac{z}{z_R}\right)\right] \\ &= \exp\left[-\frac{i}{c}\int^z (n+\frac{1}{2})\,\omega(z')\,dz'\right].\end{aligned} \tag{17}$$

This result is identical to the quantum case and shows again that the Gouy phase has its origin in the confinement of the harmonic transverse wave in a finite area at each fixed z value. This proof supports the same idea contained in the work of Feng and Winful [14], who showed that the finite average value of the effective transverse wave number k_x^2/k_z also leads to the correct Gouy phase. This paper contains an interesting discussion of the history of different interpretations of this phase. The proof given here demonstrates more directly the connection to the quantum energy phase through the appearance of the factor $(n+1/2)\omega$.

The Gouy phase in the form of (13) is known as a dynamic phase in quantum mechanics. In wave packets generated by laser ionisation [15], it has been identified with the Maslov phases of semi-classical dynamics. It is certainly to be distinguished

from a geometric phase, to which in the past the Gouy phase has been attributed [16, 17]
.

Again the exact propagating wave function (15) with the adiabatic phase (17) and instantaneous oscillator frequency (16) indicates that the wave function propagates fully adiabatically in z. This establishment of the full transverse distribution at each fixed z, out to macroscopic distances, is perhaps remarkable considering that the velocity of propagation c is the maximum allowed physically.

4. The far-field asymptotes: light beams and particle trajectories.

It is very well-documented that in the far-field asymptotic limit $z \gg z_R$ the paraxial wave function corresponds to straight-line beams of light. Also, that the space wave function assumes the functional form of the wave number function, apart from phase and normalisation factors (Fraunhofer limit) is found in many text books. Strangely, the corresponding result in quantum mechanics with asymptote $t \gg T$, is much less known and its implications are not treated in most quantum text books. In particular the fact that the coordinates x, t are connected by classical mechanics asymptotically has not been accorded the importance it deserves. Recently, attention has been called increasingly to the fact that the asymptotic form of the wave function represents the quantum to classical transition in a very general way [18, 6]. Of course this is the direct equivalent of the wave optics to ray optics transition.

Again, with suitable notation change, as shown next, the two asymptotic forms in optics and quantum mechanics are identical.

4.1. The far-field optical wave

Since much better known, it is appropriate to begin with the optics case. The asymptote to be taken in (14) is $z \gg z_R$ or $\tau \to \infty$. This gives

$$\begin{aligned}\Psi_n(x,t) \approx \frac{W_0^{1/2}}{(2\pi)^{\frac{1}{4}}} \left[\frac{k_z}{z}\right]^{1/2} \exp\left[-i(n+\frac{1}{2})\pi/2\right] \left(\frac{1}{2^n n!}\right)^{1/2} \\ \times H_n\left(\frac{k_z W_0}{\sqrt{2}z}x\right) \exp\left[-\frac{k_z^2 W_0^2}{4z^2}x^2\right] \exp\left[i\frac{k_z}{2z}x^2\right],\end{aligned} \tag{18}$$

In this limit the phase fronts are approximately planar and the normals to these fronts are the straight-line beams defined in terms of the transverse wave number k_x by $x = (k_x/k_z)\, z$. With this substitution, the x-space wave function can be written in terms of k_x and becomes

$$\begin{aligned}\Psi_n(x,t) \approx \frac{W_0^{1/2}}{(2\pi)^{\frac{1}{4}}} \left[\frac{k_z}{z}\right]^{1/2} \exp\left[-i(n+\frac{1}{2})\pi/2\right] \left(\frac{1}{2^n n!}\right)^{1/2} \\ \times H_n\left(\frac{W_0 k_x}{\sqrt{2}}\right) \exp\left[-\frac{W_0^2}{4}\, k_x^2\right] \exp\left[i\frac{k_x^2}{2k_z}z\right],\end{aligned} \tag{19}$$

which is recognised as the propagating *wave number* function, proportional to the FT of the initial HG wave function. Of course this is just the well-known Fraunhofer result, valid for any wave packet, that the far-field function is proportional to the FT of the initial wave function. In the wave function the x and z coordinates are connected giving light beams propagating along the straight lines $x = (k_x/k_z)\, z$. The corresponding result in quantum mechanics describes a transition from quantum waves to classical straight line trajectories with $x = p_x t/m = \hbar k_x t/m$, to connect the x and t coordinates. Again one sees the identity of this condition with the beam optics. Replacing $m/\hbar$ with k_z/c and, defining $t = z/c$ in the expression $x = \hbar k_x t/m$, gives the optical beam "trajectory" $x = (k_x/k_z)\, z$.

4.2. The far-field quantum wave

In the quantum case, the far-field limit $\tau \gg 1$ or $t \gg T$ is taken in the exact wave function (11) and then x is replaced by the classical condition $x = pt/m$. That the asymptotic space wave function is proportional to the initial momentum wave function for free propagation has been derived over the years in scattering theory [19, 20], where it is now known as the "imaging theorem" [21]. The result appears also in many treatments of Bohm trajectories [5, 8, 9, 10]. More recently it has been shown [18] that it is valid for any wave packet and also for extraction by electromagnetic fields. It contains the essence of the quantum to classical transition.

One can use the stationary phase evaluation of (11) or alternatively one can take the optics result (18) and substitute $z = ct$, $\sqrt{2}\sigma$ for W_0 and put $k_z/c = m/\hbar$. In either case the result again is that the asymptotic space quantum wave function in proportional to the initial *momentum* wave function. From the $t \gg T$ asymptote of (11) one has

$$\begin{aligned}\Psi_n(x,t) \approx \frac{\sigma^{1/2}}{\pi^{\frac{1}{4}}}\left[\frac{m}{\hbar t}\right]^{1/2} \exp\left[-i(n+\frac{1}{2})\pi/2\right]\left(\frac{1}{2^n n!}\right)^{1/2} \\ \times H_n\left(\frac{m\sigma x}{\hbar t}\right)\exp\left[-\frac{1}{2}\left(\frac{m\sigma}{\hbar t}\right)^2 x^2\right]\exp\left[i\frac{m\sigma}{2\hbar t}x^2\right].\end{aligned} \tag{20}$$

With the classical replacement $mx/t = p_x$ and, recognising that the asymptotic Gouy phase $\exp\left[-i(n+\frac{1}{2})\pi/2\right] = i^{-(n+1/2)}$, the asymptotic space wave function becomes

$$\begin{aligned}\Psi_n(x,t) \approx \frac{\sigma^{1/2}}{\pi^{\frac{1}{4}}}\left[\frac{m}{i\hbar t}\right]^{1/2} i^{-n}\left(\frac{1}{2^n n!}\right)^{1/2} \\ \times H_n\left(\frac{\sigma p_x}{\hbar}\right)\exp\left[-\frac{1}{2}\left(\frac{\sigma^2 p_x^2}{\hbar^2}\right)\right]\exp\left[\frac{i}{\hbar}\frac{p_x^2}{2m}t\right].\end{aligned} \tag{21}$$

Hence, the asymptotic space wave function is proportional to the invariant momentum wave function of Eq. (7). Defining the quantum wave number as $k_x = p_x/\hbar$ and replacing, as usual $m/\hbar$ with k_z/c, this result corresponds exactly to (19) of the optics case.

Thus there is complete agreement in the transition from wave to beam optics and the transition from quantum to classical mechanics in the wave function.

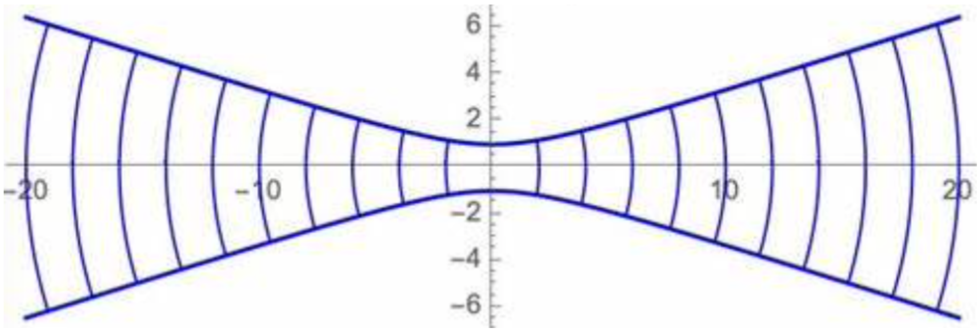

Figure 1. The propagation of an HG wave packet showing the beam waist trajectory in the z direction (abscissa) and the asymptotic straight line beams. The ordinate is the x direction. The curved wave fronts are shown at successive z values. For light the length scales are typically in μm.

One further point is important for what follows in section 6. The time-dependent normalisation factor $[m/t]^{1/2}$, occurring in (21), is recognised in QM as a purely classical quantity. The square of this function, since $p_x = mx/t$, is simply dp_x/dx, the density of classical trajectories. Then $[m/t]^{1/2} = (dp_x/dx)^{1/2}$ is called the van Vleck factor [22] and is much used in semi-classical QM [23]. The term $[m/(\hbar t)]^{1/2}$ of (21) can be written as $[dk_x/dx]^{1/2}$. The identical factor occurs in the optics case of (19) in the form $[k_z/z] = [dk_x/dx]$ for the asymptotic light beam. Again this factor can be viewed as giving the density of ray trajectories or equivalently, the local light beam density.

5. The near field: trajectories and beams.

In optics, the passage of a HG beam into and out of its focus is usually depicted by showing the curved wave fronts at each z value, as shown in figure 1. The wave fronts are perpendicular to the x direction at $z = 0$. The beam width function is shown as two bold lines described by the hyperbolae $W(z) = \pm W_0(1 + (z/z_R)^2)$, with W_0 the beam waist at $z = 0$.

Although not done usually in optics books, one can draw any number of such curves, the loci of wave front normals, intersecting the x axis at different points. The hyperbolic curves are the projections of the straight beam lines back into the near field zone. Such Bohmian "trajectories" have been defined in QM where they are ascribed to particle trajectories. A similar construction in optics has been suggested by Orefice and co-workers [11, 12], who calculated the curves numerically for all z. In section 5.2, for HG beams, it is shown that a simple analytical form of such trajectories can be derived. The beam waist curve is just one such beam trajectory. The "beams" pass through the near zone and connect smoothly to the asymptotic straight-line beams. In addition an interesting connection of these beams to the wave function in wave-number space is given in section 6.

Since the definition of trajectories in the near field stems from QM, it is appropriate in this section to treat the QM case first.

5.1. The near-field quantum wave and Bohm trajectories.

The derivation of Bohm trajectories involves defining a velocity from the phase of the wave function [7, 5]. This classical velocity can then be integrated over time to give a trajectory $x(t)$ *for all time*, not just asymptotically. However, their association with "real" particle trajectories is still disputed. The definition of trajectories starts with the exact wave function of (11) expressed in the form

$$\Psi_n(x,t) = R_n(x,t)\, e^{\frac{i}{\hbar}S_n(x,t)}. \tag{22}$$

The action function of this exact wave function is

$$S_n = \hbar \frac{x^2}{(2\sigma^2)} \frac{\tau}{(1+\tau^2)} - (n+\frac{1}{2})\hbar \arctan\tau. \tag{23}$$

The practitioners of Bohmian mechanics now define, in complete analogy to classical mechanics, a velocity,

$$\dot{x} \equiv \frac{1}{m}\frac{\partial S_n}{\partial x} = \frac{x}{T}\frac{\tau}{(1+\tau^2)} \tag{24}$$

which can be written also as

$$\dot{x} = \frac{x}{t}\frac{\tau^2}{(1+\tau^2)}. \tag{25}$$

Note that, since the second term in S_n of (23) does not depend upon x, then the velocity function is independent of the eigenstate n. Hence, the trajectories are identical for all HG functions. One recognises that asymptotically, when $\tau \gg 1$, then $\dot{x} = x/t$ is the constant classical speed v_x.

From the velocity equation one can integrate over time to give a trajectory

$$x(t) = x_0(1+\tau^2)^{1/2} = x_0\left(1+\left(\frac{t}{T}\right)^2\right)^{1/2}, \tag{26}$$

where $x_0 \equiv x(0)$ is the intersection of the trajectory with the x axis at $t = 0$. In the large time limit, one has the classical trajectory $x(t) \approx v_x t$. To satisfy this limit one then has $x_0 = \pm v_x T$. Hence, for a fixed final velocity v_x, the trajectory has a fixed position $v_x T$ at $t = 0$. Each trajectory is then characterised solely by its final velocity v_x and the constant T.

That the trajectory is of hyperbolic form is made clear by putting $z = v_z t$ to write (26) as

$$\left(\frac{x}{v_x}\right)^2 - \left(\frac{z}{v_z}\right)^2 = T^2 \tag{27}$$

Using wave number rather than classical velocity, this equation assumes the form

$$\left(\frac{x}{k_x}\right)^2 - \left(\frac{z}{k_z}\right)^2 = \sigma^4 \tag{28}$$

with straight-line asymptote $x = (k_x/k_z)\, z$ exactly as in the optics case.

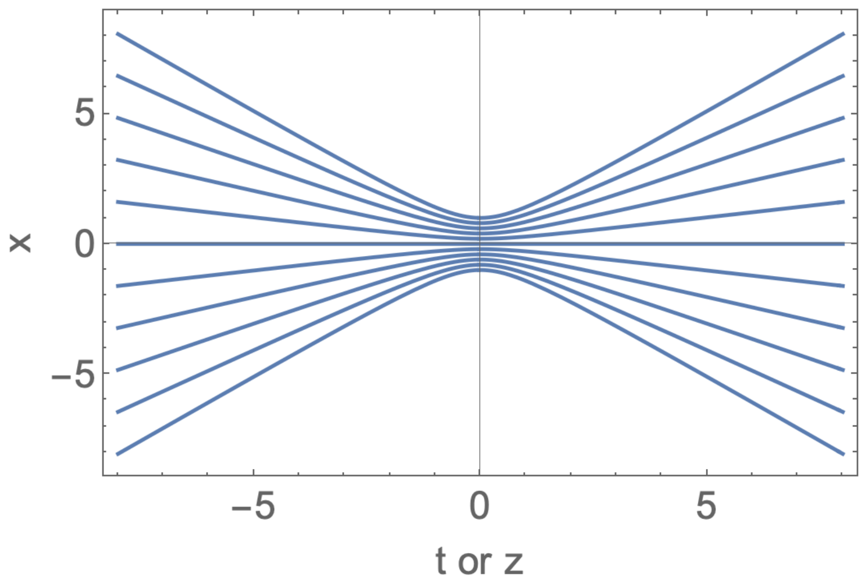

Figure 2. The hyperbolic trajectories $x(t)$ (QM) or $x(z)$ (optics) representing the focussing in one dimension (vertical axis) of an HO or HG wave function. The constant T (QM) or z_R (optics) has the value unity. The units are arbitrary.

The hyperbolic trajectories are shown in figure 2. In Bohmian mechanics these trajectories are viewed as the deterministic trajectories of real particles. Whilst the asymptotic reality of the trajectory $x = v_x T$ is mathematically and experimentally justified, the extension as a particle path into the near zone is an additional *postulate* in QM. However, here, to connect to the optics case, it is only necessary to accept the trajectories as the loci of the normals to the phase fronts of the propagating wave packets.

5.2. The near-field optical wave and Bohm trajectories.

With the understanding that one is referring to the locus of wave front normals, it is straightforward to transform the QM trajectories into those of the paraxial equation. The equation (26) is preserved but in terms of z rather than t, i.e.

$$x(z) = x_0(1+\tau^2)^{1/2} = x_0\left(1+\left(\frac{z}{z_R}\right)^2\right)^{1/2} \tag{29}$$

and, to satisfy the asymptotic beam trajectory $x = (k_x/k)z$, now with $x_0 = (k_x/k_z)z_R = k_x W_0^2/2$. Then, fixing the initial point on the x axis for $z = 0$, fixes the value of k_x. For example, the beam waist trajectory occurs when $x_0 = W_0$ to give $k_x = 2/W_0$.

Again, the beam trajectories are hyperbolae following the equation

$$\left(\frac{x}{k_x}\right)^2 - \left(\frac{z}{k_z}\right)^2 = \frac{W_0^4}{4} \tag{30}$$

which is the same as the QM case (28), with the same straight-line beam asymptote $x = (k_x/k_z)\,z$.

6. The wave packet form on the trajectory

6.1. The Bohm trajectory quantum wave

Here it is shown that the space wave function along the Bohm trajectory assumes the form of the momentum wave function at *all* times, not just asymptotically. To emphasise that the wave function is to be evaluated along $x(t)$, the notation $\Psi_n(x(t))$ is used rather than $\Psi_n(x,t)$. Substituting the trajectory coordinate dependence $x(t) = x_0(1+\tau^2)^{1/2}$ from (26) into (11) gives the space wave function at points along the trajectory,

$$\begin{aligned}\Psi_n(x(t)) = {} & \frac{1}{[\pi\sigma^2(1+\tau^2)]^{\frac{1}{4}}}\exp\left[-i(n+\frac{1}{2})\arctan\tau\right] \\ & \times\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{x_0}{\sigma}\right)\exp\left[-\frac{x_0^2}{2\sigma^2}\right]\exp\left[i\frac{x_0^2\,\tau}{2\sigma^2}\right].\end{aligned} \tag{31}$$

Thus one sees that the modulus squared of this wave function, the probability density, is constant along the trajectory and retains its initial $t = 0$ form. Only the normalisation and phase factors change with time.

Recognising that $x_0 = v_x T$ and that $x_0/\sigma = p_x\,\sigma/\hbar$, with $p_x = mv_x$ being the variable appearing in the unchanging amplitude of the momentum wave function of (6), one can write the space wave function evaluated along the trajectory as

$$\Psi_n(x(t)) = \frac{1}{[\pi\sigma^2(1+\tau^2)]^{\frac{1}{4}}} \exp\left[-i(n+\frac{1}{2})\arctan\tau - in\pi/2\right] \\ \times i^{-n}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sigma p_x}{\hbar}\right)\exp\left[-\frac{\sigma^2 p_x^2}{2\hbar^2}\right]\exp\left[\frac{i}{\hbar}\frac{p_x^2 t}{2m}\right]. \tag{32}$$

Also, along the trajectory, given by $p_x = mx_0/T = mx/(T(1+\tau^2)^{1/2})$, one has

$$\left(\frac{dp_x}{dx}\right)^{1/2} = \frac{\hbar^{1/2}}{\sigma}(1+\tau^2)^{-1/4}. \tag{33}$$

Hence, the space wave function (32), using (5) and (6), can be written in the simple form

$$\Psi_n(x(t)) = \left(\frac{dp_x}{dx}\right)^{1/2}\tilde{\Psi}_n(p_x,t)\exp\left[-i(n+\frac{1}{2})\arctan\tau - in\pi/2\right], \tag{34}$$

valid *for all points* t along the trajectory, where $\tilde{\Psi}_n(p_x,t)$ is the momentum wave function.

Forming the probability density along a tube $dx(t)$ one has the simple result

$$|\Psi_n(x(t))|^2 = \left(\frac{dp_x}{dx(t)}\right)|\tilde{\Psi}_n(p_x,0)|^2, \tag{35}$$

or

$$|\Psi_n(x(t))|^2 dx(t) = |\tilde{\Psi}_n(p_x,0)|^2 dp_x. \tag{36}$$

This result generalises its asymptotic validity [18], to all t on the trajectory $x(t) = x_0(1+\tau^2)^{1/2}$. That is, the trajectory provides a conduit along which the probability of the initial free momentum wave function propagates form unchanged.

6.2. The Bohm trajectory optical wave

The foregoing analysis is applied readily to the optical case.

Along the "trajectory" specified by a given value of x_0, i.e. $x(z) = x_0(1+\tau^2)^{1/2} = x_0\left(1+\left(\frac{z}{z_R}\right)^2\right)^{1/2}$ the space wave function corresponding to (31) becomes

$$\Psi_n(x(z)) = \frac{1}{[\pi W_0^2(1+\tau^2)/2]^{\frac{1}{4}}}\exp\left[-i(n+\frac{1}{2})\arctan\tau - in\pi/2\right] \\ \times (i)^{-n}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sqrt{2}x_0}{W_0}\right)\exp\left[-\frac{x_0^2}{W_0^2}\right]\exp\left[i\frac{x_0^2\,\tau}{W_0^2}\right]. \tag{37}$$

For a particular trajectory one has $x_0 = k_x W_0^2/2$, to give,

$$\Psi_n(x(z)) = \frac{1}{[\pi w_0^2(1+\tau^2)/2]^{\frac{1}{4}}}\exp\left[-i(n+\frac{1}{2})\arctan\tau - in\pi/2\right] \\ \times (i)^{-n}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{W_0 k_x}{\sqrt{2}}\right)\exp\left[-\frac{W_0^2 k_x^2}{4}\right]\exp\left[i\frac{k_x^2}{2k}z\right], \tag{38}$$

which, as expected, is the wave number function whose asymptotic form is (19).

Along any trajectory, given in the optical case by $k_x = 2x_0/W_0^2 = 2x/(W_0^2(1+\tau^2)^{1/2})$, the square root of the trajectory density (van Vleck factor) is

$$\left(\frac{dk_x}{dx}\right)^{1/2} = \frac{\sqrt{2}}{W_0}\,(1+\tau^2)^{-1/4}. \tag{39}$$

Hence, the space wave function (38) can be written in the simple form

$$\Psi_n(x(z)) = \left(\frac{dk_x}{dx}\right)^{1/2} \tilde{\Psi}_n(k_x, z) \exp\left[-i(n+\frac{1}{2}) \arctan\tau - in\pi/2\right], \tag{40}$$

valid for all z along the trajectory, where $\tilde{\Psi}_n(k_x, z)$ is the wave number function. This is the optics version of the quantum equation (34).

As in the quantum case of Eq.(36), along a tube of trajectories $dx(z)$ the intensity follows the conservation rule

$$|\Psi_n(x(z))|^2 dx(z) = |\tilde{\Psi}_n(k_x, 0)|^2 dk_x. \tag{41}$$

7. Scaling of space coordinates.

7.1. The quantum case: time scaling

The TDSE for free motion (2) in one space dimension is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - i\hbar\,\frac{\partial\Psi}{\partial t} = 0. \tag{42}$$

In [6] it is shown, following Ref. [24] and Ref. [25], that this equation can be transformed by a time-scaling of space. That is, new space and time coordinates are defined as

$$q \equiv \frac{x}{a(t)} \qquad \bar{t} \equiv \int^t \frac{dt'}{a(t')^2}, \tag{43}$$

where $a(t)$ is a dimensionless function. In (42) this transformation gives a new wave function

$$\Phi(q, \bar{t}\,) = a^{1/2} \exp\left(-\frac{i}{\hbar}\frac{m}{2} a\dot{a} q^2\right) \Psi(x, t), \tag{44}$$

where the dot signifies differentiation w.r.t. the original time t. Although one is considering free motion, rather remarkably the function $\Phi(q, \bar{t}\,)$ satisfies the TDSE of a harmonic oscillator (HO) with a *time-dependent* frequency, i.e.

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2} m\, a^3 \ddot{a}\, q^2\right)\Phi = i\hbar\,\frac{\partial\Phi}{\partial\bar{t}}. \tag{45}$$

Since the frequency is not constant, this equation does not appear particularly useful. However, if one chooses the time dependence of the Bohm trajectory of the HG wave functions (26), that is choose $a(t) = (1+\tau^2)^{1/2}$, where $\tau = t/T$, then one obtains the TDSE with a time-independent fixed frequency $\omega_0 = 1/T$,

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2} m\,\omega_0^2\, q^2\right)\Phi = i\hbar\,\frac{\partial \Phi}{\partial \bar{t}}. \tag{46}$$

The solutions to (46) are the HO wave functions with eigenvalues $E_n = (n+1/2)\,\hbar\omega_0$, i.e.

$$\Phi_n(q,\bar{t}) = \frac{1}{(\pi\sigma^2)^{1/4}}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{q}{\sigma}\right)\exp\left(-\frac{q^2}{2\sigma^2}\right)\exp\left(-\frac{i}{\hbar}E_n\bar{t}\right), \tag{47}$$

For the chosen scaling of time along the parabolic trajectories, the phase factors are $E_n\bar{t}/\hbar = (n+\frac{1}{2})\omega_0\bar{t}$. Remembering that $\omega_0 = 1/T$ and defining the new dimensionless time parameter $\bar{\tau} \equiv \bar{t}/T$, from the definition of $\bar{t}$ in (43), one has the result that $\bar{\tau} = \arctan\tau$. Hence, the new "time" is simply the Gouy phase itself. The phase factors become $E_n\bar{t}/\hbar = (n+\frac{1}{2})\bar{\tau}$.

Furthermore with the scaling function $a(t) = (1+\tau^2)^{1/2}$, one has $q = x/a = x_0 = v_x T$. Hence, in the co-moving frame the functions are fixed with value of q given by the intersection of the trajectory with the $t=0$ (or $z=0$) axis. The phase factors change linearly with $\bar{\tau}$. In analogy to a Lorentz frame transformation, one can view the time $\bar{\tau}$ expressed in units of the time T, as a "proper" time in the co-moving frame. This proper time is simply the Gouy phase.

7.2. The optics paraxial case: space scaling

In the optics case the same procedure is followed, except that the space coordinate z along the beam is scaled. That is the space is warped by the transformations,

$$q \equiv \frac{x}{a(z)} \qquad\qquad \bar{z} \equiv \int^z \frac{dz'}{a(z')^2}, \tag{48}$$

where $a(z)$ is dimensionless.

Choosing $a(z)$ to describe the ray trajectory $a(z) = (1+\tau^2)^{1/2}$, where $\tau = z/z_R$, and following the same transformation steps as in (42) to (45) one arrives at the analogue to (46),

$$\left(\frac{\partial^2}{\partial q^2} - \left(\frac{2}{W_0^2}\right)^2 q^2\right)\Phi = i2k_z\frac{\partial \Phi}{\partial \bar{z}}. \tag{49}$$

The solutions to this equation in the co-moving frame again are of the form of constant HG functions multiplied by the Gouy phase factor, i.e the analogue of (47),

$$\Phi_n(q,\bar{z}) = \frac{1}{(\pi W_0^2/2)^{1/4}}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sqrt{2}q}{W_0}\right)\exp\left(-\frac{q^2}{W_0^2}\right)\exp\left(-i(n+1/2)\frac{\bar{z}}{z_R}\right), \tag{50}$$

where now the fixed value of q is given by $q = x_0 = k_x W_0^2/2$. Again this gives the intersection point of the wave front normal trajectory with the x axis.

In exactly the same way as for the quantum case, in the co-moving frame the wave function is constant. Only the Gouy phase changes linearly in $\bar{\tau} \equiv \bar{z}/z_R$. This dimensionless parameter $\bar{\tau}$ can be viewed as the "proper" time $c\,\bar{z}$ in units of the constant time $c\,z_R$.

7.3. An arbitrary wave packet

A normalisable wave packet at time zero can be expanded in a complete set of HO (HG) wave functions

$$\chi(x,0) = \sum_n a_n \, \Psi_n(x,0) \tag{51}$$

where the $\Psi_n(x,0)$ are defined in (4). The coefficients a_n can be calculated if $\chi(x,0)$ is known. In the laboratory frame, the time development is then given by the time development of each HO wavefunction $\Psi_n(x,t)$ given in Eq.(11). These functions expand in time.

The time development is much simpler when transformed to the co-moving frame of the Bohm trajectories, along which the functions are constant. If the transformed wave packet of $\chi(x,t)$ is denoted by $\bar{\chi}(q,\bar{t}\,)$, then the time development is given by

$$\bar{\chi}(q,\bar{t}\,) = \sum_n a_n \, \Phi_n(q,\bar{t}\,) \tag{52}$$

where, from (47), $q = x_0$ is fixed and only the energy phase factors change with $\bar{t}$. Then the sum becomes simply a Fourier series. For a light wave of general form, the only change is to replace $\bar{t}$ with $\bar{z}$ and use the $\Phi_n(q,\bar{z}\,)$ from (50).

8. Adiabatic and diabatic trajectories

It has been noted already that the exact wave packet propagates adiabatically with the Gouy phase as an adiabatic energy phase. In QM this is the form of (11) and (13). In optics it is the form of (15) and (17). This adiabatic behaviour is seen in the trajectories of figure 3 which show a clear "avoided crossing" at $t = 0$. Normally in QM, it is the eigenenergies as a function of the time parameter which are shown as an avoided crossing. Here the analogy is extended to the behaviour of the trajectories $x(t)$.

8.1. The QM avoided crossing

In figure 3 is shown a typical pair of such adiabatic trajectories. Also plotted are the straight-line classical asymptotes $x_\pm = \pm\, v_x t$. These diabatic trajectories cross at $t = 0$. The avoidance of crossing is given simply by $2T$ and results in the switched adiabatic asymptotic connection $x_\pm = \mp\, v_x t$. The change in sign is due to the change of π in the

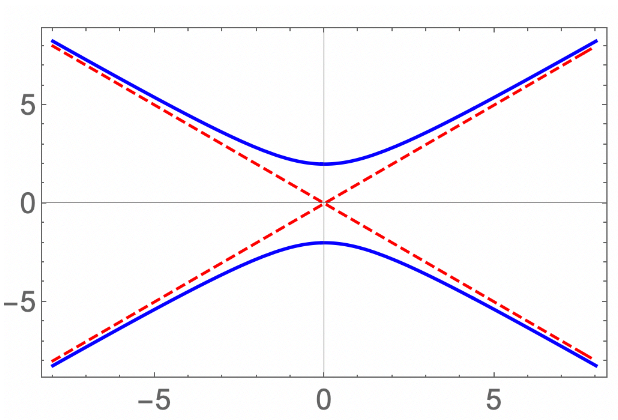

Figure 3. Two parabolic adiabatic trajectories for a Hermite-Gauss wave function (continuous lines, blue). The units are arbitrary. The abscissa is the time t in QM or the distance z in optics. The ordinate is the distance $x(t)$ in QM or $x(z)$ for optics. The two straight dashed lines (red) are the diabatic classical trajectories for QM or the light beam trajectories in optics.

Gouy phase as the avoided crossing is traversed. The diabatic (asymptotic) wave has a fixed Gouy phase for all time.

Let us consider any pair of the trajectories shown in figure 3, beginning, for large negative time, as $x_-(t) = v_x t$ and $x_+(t) = -v_x t$. The Bohmian adiabatic trajectories swap their classical character through the avoided crossing to become $x_-(t) = -v_x t$ and $x_+(t) = v_x t$ for large positive time. In the classical diabatic basis, the time development is given by $x^{cl}_{\pm}(t) = \pm v_x t$ for all times t, or

$$\begin{pmatrix} x^{cl}_{+}(t) \\ x^{cl}_{-}(t) \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \begin{pmatrix} v_x \\ v_x \end{pmatrix} \tag{53}$$

As is shown explicitly in Ref. [6] the change from classical to quantum propagation is characterised by time becoming complex i.e. $t \to t - iT$. This is manifest as a coupling between the classical trajectories due to the finite time iT, mirrored in the change of the Gouy phase. This gives a new coupling matrix, i.e.

$$\begin{pmatrix} x_+(t) \\ x_-(t) \end{pmatrix} = \begin{pmatrix} t & -iT \\ iT & -t \end{pmatrix} \begin{pmatrix} v_x \\ v_x \end{pmatrix} \tag{54}$$

To make the vectors both of dimensions length, this equation can be written

$$\begin{pmatrix} x_+(t) \\ x_-(t) \end{pmatrix} = \begin{pmatrix} \tau & -i \\ i & -\tau \end{pmatrix} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \tag{55}$$

where $\tau = t/T$ and $x_0 = v_x T = p_x \sigma^2/\hbar = k_x \sigma^2$.

Diagonalisation of the matrix gives the eigenvalues $\pm(1+\tau^2)^{1/2}$ and the trajectories

$$\begin{pmatrix} x_+(t) \\ x_-(t) \end{pmatrix} = \begin{pmatrix} (1+\tau^2)^{1/2} & 0 \\ 0 & -(1+\tau^2)^{1/2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \tag{56}$$

These trajectories, the normals to the wave fronts, are precisely the Bohmian trajectories of (26). Hence, the quantum Bohmian trajectories are derived from the $\pm v_x t$ classical trajectories by the repulsion due to the finite time T. This avoidance of the crossing results in an adiabatic swapping of the asymptotic velocity from $\pm v_x$ to $\mp v_x$. Extrapolation of the classical trajectories to $t = 0$ implies a singularity. The avoidance of a point singularity of the crossing *classical* trajectories is due to the finite extent σ of the quantum *wave* and the corresponding quantisation leading to a finite zero-point energy of the HO wave functions.

8.2. The optics avoided crossing

The only change for the optics case is that the diabatic light beam connection is given by

$$\begin{pmatrix} x_+^{ray}(t) \\ x_-^{ray}(t) \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \begin{pmatrix} k_x/k_z \\ k_x/k_z \end{pmatrix} \tag{57}$$

which are the straight-line ray asymptotes. The adiabatic connection of the normals to the wave fronts is again (56)

$$\begin{pmatrix} x_+(t) \\ x_-(t) \end{pmatrix} = \begin{pmatrix} (1+\tau^2)^{1/2} & 0 \\ 0 & -(1+\tau^2)^{1/2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \tag{58}$$

but where now, instead of the quantum $\tau = t/T$, one has $\tau = z/z_R$, see (29). The avoidance of trajectory crossing has the value $2z_R$ at $z = 0$. In optics this is called the spot size. Instead of the quantum $x_0 = k_x\sigma^2$ one has $x_0 = (k_x/k_z)z_R = k_x W_0^2/2$. Again, the trajectories of the wave front normals change their character adiabatically through the avoided crossings of figure 3. The avoidance of a point focus singularity of the light *rays* is due to the finite extent W_0 of the optical *wave* giving a focus of finite size.

9. Conclusions

Although treated in many standard texts of optics and QM, this study of free wavepacket propagation unifies the presentation of the results in the two fields. Furthermore, the results given here are in some important aspects new. The new perspectives are:

1) The identity of the two-dimensional paraxial equation and the two-dimensional Schroedinger equation describing the propagation of a Hermite-Gauss beam has been shown in some detail.

2) The Gouy phase arises from the quantisation of the wave form in an effective harmonic potential. The phase is the integral over time of the instantaneous oscillator frequency. In the classical limit of QM or the ray picture of optics, the phase is a constant and corresponds to a delta function singularity of the wave function of vanishing extent.

3) The locus of the wave front normals defines a "trajectory" both in QM, where it is known as the Bohm trajectory, and in optics. This trajectory has a simple hyperbolic form which is the same for all HG waves.

4) A transition to a frame moving with the trajectory gives a constant HG function, with a phase changing linearly with a "proper time" which is the Gouy phase itself.

5) The adiabatic propagation of the HG wave packet is mirrored in the adiabatic propagation of the wave front trajectories. The singularity at $t = 0$ in the diabatic classical particle trajectories, or at $z = 0$ for light beams, is avoided due to the finite extent of the HG wave, see figure 3. This is analysed as a typical "avoided crossing" phenomenon.

6) The momentum (QM) or wave number (optics) function propagates unchanged except for a phase factor. The space wave function spreads in time (QM) or in z-direction (optics). It has been shown that, along the continuous hyperbolic trajectories, the *space* wave function is simply proportional to the fixed wave number or momentum wave function.

These points give more credence to the assignment of physical meaning to the "trajectories", the loci of constant probability (QM) or light intensity (optics). In QM, many researchers interpret them as deterministic particle trajectories, in view of their smooth transformation to the asymptotic classical trajectories which are confirmed experimentally. The usefulness of the same trajectories in optics is suggested. However, clearly, in any experiment detecting particles, the paths of maximum probability are where particles are detected optimally. Such paths of constant probability do trace out the Bohm trajectories. In a similar way, the paths of constant high light intensity are the trajectories along which photons are detected optimally. Since typical Rayleigh ranges for light are much larger than the corresponding range for QM, it is possible to measure light in the near field below the Rayleigh range. Indeed, it has been claimed [26] that the photon Bohm trajectories have been observed. In the near field, they can be associated with the structures known as "Talbot carpets", as has been shown in Ref. [27].

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