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RESEARCH ARTICLE

Coordinate-free exponentials of general multivector in $Cl_{p,q}$ algebras for $p+q=3$

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Summary

Closed form expressions in real Clifford geometric algebras $Cl_{0,3}$, $Cl_{3,0}$, $Cl_{1,2}$, and $Cl_{2,1}$ are presented in a coordinate-free form for exponential function when the exponent is a general multivector. The main difficulty in solving the problem is connected with an entanglement (or mixing) of vector and bivector components a_i and a_{jk} in a form $(a_i - a_{jk})^2$, $i \neq j \neq k$. After disentanglement, the obtained formulas simplify to the well-known Moivre-type trigonometric/hyperbolic function for vector or bivector exponentials. The presented formulas may find wide application in solving GA differential equations, in signal processing, automatic control and robotics.

KEYWORDS:
Clifford (geometric) algebra, exponentials of Clifford numbers, computer-aided theory

1 | INTRODUCTION

In the complex number algebra, which is isomorphic to $Cl_{0,1}$ Clifford geometric algebra, the complex exponential may be expanded into trigonometric function sum (de Moivre's theorem). In 2D and 3D GA algebras similar formulas are known too under the name "polar decomposition". In particular, if the square of the blade is equal to ±1, then GA exponential can also be expanded in de Moivre-type sum, i.e., in either trigonometric or hyperbolic functions respectively [1,2,3]. However, expansion of GA exponential in case of 3D and higher algebras, when the exponent is a general multivector, as we shall see is much more complicated and as far as the authors know has not been analyzed fully as yet. The authors of articles [4,5] have considered general properties of functions of MV variable for Clifford algebras $n = p+q \leq 3$, including the exponential function. For this purpose they have made use of the property that in these algebras the pseudoscalar I commutes with all MV elements and $I^2 = \pm1$. This has allowed to introduce more general functions related to a polar decomposition of MVs. However, the analysis is not full enough. A different approach to resolve the problem is to factor, if possible, the exponential into product of simpler exponentials, for example, in the polar form [6,7,8,9]. A general bivector exponential in $Cl_{4,1}$ algebra was analyzed in [10] in connection with 3D conformal GA. In paper [11], exact and closed form expressions for coefficients at basis elements to calculate GA exponentials in coordinate form are presented for all 3D GAs. However, in this form the final MV formulas constructed in some orthogonal basis are rather complicated and inconvenient to carry a detailed analysis of the properties of GA exponential functions, although they may be useful in some practical cases, for example, for all-purpose computer programs to calculate GA exponentials with numerical coefficients.

In this paper the exact exponential formulas [11] are transformed to coordinate-free form what allows to carry a detailed analysis and gives a clear geometric interpretation to the problem. Also, special cases where various additional conditions and relations are imposed upon GA elements are considered what may be useful in applications of exponentials in practice. In Sec. 2 the notation is introduced. In Sec. 3 the exponential of the simplest, namely $Cl_{0,3}$ algebra is considered. Since algebras $Cl_{3,0}$ and

[0]**Abbreviations:** MV, multivector; GA, geometric (Clifford) algebra; 3D, three dimensional vector space

$Cl_{1,2}$ are isomorphic, in Sec. 4 the exponentials for both algebras are investigated simultaneously. In Sec. 5 the exponential of the most difficult $Cl_{2,1}$ algebra is presented. In Sec. 6 possible applications of exponentials to solve GA linear differential equations are presented. Finally, in Sec. 7 we discuss further development of the problem, including the inverse function, viz. the GA logarithm.

2 | NOTATION AND GENERAL PROPERTIES OF GA EXPONENTIAL

In calculations below we have intensively used our symbolic GA program written for *Mathematica* package[12]. In the program, in GA space endowed with orthonormal basis we expanded a general 3D MV in inverse degree lexicographic ordering: $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}$, where $\mathbf{e}_i$ are basis vectors, $\mathbf{e}_{ij}$ are the bivectors and I is the pseudoscalar.[1] The number of subscripts indicates the grade. The scalar is a grade-0 element, the vectors $\mathbf{e}_i$ are the grade-1 elements, etc. In the orthonormalized basis used here the geometric product of basis vectors satisfies the anti-commutation relation,

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}. \tag{1}$$

For $Cl_{3,0}$ and $Cl_{0,3}$ algebras the squares of basis vectors, correspondingly, are $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_i^2 = -1$, where $i = 1, 2, 3$. For mixed signature algebras such as $Cl_{2,1}$ and $Cl_{1,2}$ the squares are $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3^2 = -1$ and $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = -1$, respectively.

The general MV that belongs to real Clifford algebras $Cl_{p,q}$, when $n = p + q = 3$ can be expressed as

$$\begin{aligned} \mathsf{A} &= a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13} + a_{123}I \\ &\equiv a_0 + \mathbf{a} + \mathcal{A} + a_{123}I, \end{aligned} \tag{2}$$

where a_i, a_{ij} and a_{123} are the real coefficients, and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13}$ is, respectively, the vector and bivector. I is the pseudoscalar, $I = \mathbf{e}_{123}$. Similarly, the exponential of A is denoted as

$$\begin{aligned} \mathrm{e}^{\mathsf{A}} = \mathsf{B} &= b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{12} + b_{23}\mathbf{e}_{23} + b_{13}\mathbf{e}_{13} + b_{123}I \\ &\equiv b_0 + \mathbf{b} + \mathcal{B} + b_{123}I. \end{aligned} \tag{3}$$

The main involutions, namely the reversion, grade inversion and Clifford conjugation are denoted, respectively, by tilde, circumflex and their combination,

$$\widetilde{\mathsf{A}} = a_0 + \mathbf{a} - \mathcal{A} - a_{123}I, \qquad \widehat{\mathsf{A}} = a_0 - \mathbf{a} + \mathcal{A} - a_{123}I, \qquad \widehat{\widetilde{\mathsf{A}}} = a_0 - \mathbf{a} - \mathcal{A} + a_{123}I. \tag{4}$$

2.1 | General properties of GA exponential

The exponential of MV is another MV that belongs to the same geometric algebra. Therefore, we shall assume that the defining equation for exponential is $\mathrm{e}^{\mathsf{A}} = \mathsf{B}$, where $\mathsf{A}, \mathsf{B} \in Cl_{p,q}$ and $p+q = 3$. The following general properties hold for MV exponential:

$$\begin{aligned} &\exp(\mathsf{A} + \mathsf{B}) = \exp(\mathsf{A})\exp(\mathsf{B}) \quad \text{if and only if } \mathsf{AB} = \mathsf{BA}, \\ &\widetilde{\mathrm{e}^{\mathsf{A}}} = \mathrm{e}^{\widetilde{\mathsf{A}}}, \quad \widehat{\mathrm{e}^{\mathsf{A}}} = \mathrm{e}^{\widehat{\mathsf{A}}}, \quad \widehat{\widetilde{\mathrm{e}^{\mathsf{A}}}} = \mathrm{e}^{\widehat{\widetilde{\mathsf{A}}}}, \\ &\mathsf{V}\exp(\mathsf{A})\mathsf{V}^{-1} = \exp(\mathsf{V}\mathsf{A}\mathsf{V}^{-1}). \end{aligned} \tag{5}$$

From the first formula the exponential identity follows $\exp\mathsf{A} = (\exp\mathsf{A}/\mathsf{m})^m$, $m \in \mathbb{N}$. In the numerical matrix function theory it is frequently used where it is called the inverse scaling and squaring method[13]. The middle line of (5) indicates that involution and exponentiation operations commute. In the last expression the transformation V, for example the rotor, has been lifted to exponent, i.e. similarity transformation commutes with exponentiation.

The GA exponential e^{A} can be expanded in a series that has exactly the same structure as a scalar exponential[2,14], from which GA trigonometric and hyperbolic GA functions as well as various other relations that are analogues of respective scalars

[1]An increasing order of digits in basis elements is used, i.e., we write $\mathbf{e}_{13}$ instead of $\mathbf{e}_{31} = -\mathbf{e}_{13}$. This convention is reflected in opposite signs when expressions are expanded in a coordinate basis.

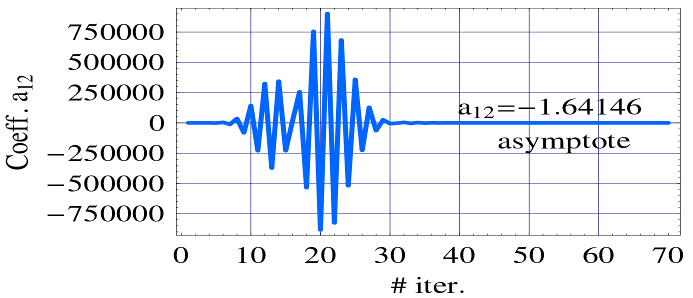

FIGURE 1 Change of value of coefficient $a_{12}=5$ of $Cl_{3,0}$ MV $\mathsf{A}=-8-6\mathbf{e}_2-9\mathbf{e}_3+5\mathbf{e}_{12}-5\mathbf{e}_{13}+6\mathbf{e}_{23}-4\mathbf{e}_{123}$ with increasing number of terms in the exponential series. The first significant figure for the MV coefficient is obtained after taking 64 series terms. Similar behaviour is characteristic to other MVs and coefficients.

functions follow[4,11]. For example,

$$\begin{aligned}
&\cos^2\mathsf{A}+\sin^2\mathsf{A}=1,\quad \cosh^2\mathsf{A}-\sinh^2\mathsf{A}=1,\\
&\sin(2\mathsf{A})=2\sin\mathsf{A}\cos\mathsf{A}=2\cos\mathsf{A}\sin\mathsf{A},\\
&\cos(2\mathsf{A})=\cos^2\mathsf{A}-\sin^2\mathsf{A}.
\end{aligned}\tag{6}$$

Also, it should be noted that GA functions of the same argument commute. Thus, the sine and cosine functions as well as hyperbolic GA sine and cosine functions satisfy: $\sin\mathsf{A}\cos\mathsf{A}=\cos\mathsf{A}\sin\mathsf{A}$ and $\sinh\mathsf{A}\cosh\mathsf{A}=\cosh\mathsf{A}\sinh\mathsf{A}$.

In sections 3-5 the exact (symbolic) formulas for GA exponentials in an expanded form but coordinate-free form are presented. If the MV is in a numerical form or one is interested in a checking of GA formula, for instance in a preliminary stage of calculation, a finite series expansion may be useful as well. It is known that GA exponential is convergent for all MVs[14], however convergence is not monotonous (see fig. 1). To minimize the number of multiplications it is convenient to represent the exponential in a nested form (aka Horner's rule)

$$\mathrm{e}^{\mathsf{A}}=1+\frac{\mathsf{A}}{1}(1+\frac{\mathsf{A}}{2}(1+\frac{\mathsf{A}}{3}(1+\frac{\mathsf{A}}{4}(1+\dots)))),\tag{7}$$

which requires a minimal number of MV products to calculate the truncated series than working out each power of A. If numerical coefficients in A are not too large the exponential e^{A} can be approximated to high precision by (7). The series may be programmed as a simple iterative procedure repeated k-times that begins from the end (dots) with the initial value at $\mathsf{A}/k=1$ and then iteratively moving to left.[2]

We start from the $Cl_{0,3}$ GA where the expanded exponential in the coordinate form has the simplest MV coefficients.

3 | MV EXPONENTIALS IN $Cl_{0,3}$ ALGEBRA

3.1 | Exponential in coordinate-free form

In GA the symbolic formulas may be written in coordinate and coordinate-free forms. The latter presentation is compact and carries clear geometrical interpretation and therefore is preferred. However, the formulas written in the coordinates sometimes may be useful too, in particular, in GA numerical calculations by non-symbolic programmes. In[11] we have found a general MV exponentials in coordinate form for all 3D GAs. Although the expressions are rather involved, however, they acquire a simple form if coordinate-free notation (the second lines in Eqs. (2) and (3)) are used. Moreover, geometrical analysis of GA formulas becomes simpler and more evident when formulas are rewritten in a coordinate-free form.

[2]In *Mathematica* the algorithm reads expHorner[A_, n_] := Module[{B = 1, $s=n+1$}, While[($s=s-1$) > 0, B = 1. + GP[B, A/s]]; B], where n is the number of iterations and GP is the geometric product. If MV coefficients are large ($a_J\geq 3$), in addition, the formula $(\exp(\mathsf{A}/m))^m$, where m is the integer, may be applied at first to accelerate the convergence and then to raise the result to the mth power.

In a case of $Cl_{0,3}$ algebra, after multiplication of coordinates by respective basis elements and then collection to vector, bivector, trivector and their products one can transform the exponential components [11] to a generic coordinate-free form,

$$\exp(\mathsf{A}) = \frac{1}{2}\mathrm{e}^{a_0}\Big(\mathrm{e}^{a_{123}}(1+I)\big(\cos a_+ + \frac{\sin a_+}{a_+}(\mathbf{a}+\mathcal{A})\big) + \mathrm{e}^{-a_{123}}(1-I)\big(\cos a_- + \frac{\sin a_-}{a_-}(\mathbf{a}+\mathcal{A})\big)\Big). \tag{8}$$

where a_- and a_+ are the scalars,

$$a_- = \sqrt{-(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})+2I\mathbf{a}\wedge\mathcal{A}} = \sqrt{(a_3+a_{12})^2+(a_2-a_{13})^2+(a_1+a_{23})^2}, \tag{9}$$

$$a_+ = \sqrt{-(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})-2I\mathbf{a}\wedge\mathcal{A}} = \sqrt{(a_3-a_{12})^2+(a_2+a_{13})^2+(a_1-a_{23})^2}\,. \tag{10}$$

The scalars show how the vector and bivector components are entangled (mixed up). As we shall see the appearance of trigonometric functions in Eq. (8) indicates that the exponential in $Cl_{0,3}$ (and in all remaining 3D algebras) has an oscillatory character as a function of the coefficients, similarly as it is in the Moivte formula case. When the denominator in the formula (8), either a_+ or a_-, reduces to zero we will have a special case. The generic formula (8) then should be modified by replacing the corresponding ratios by their limits, $\lim_{a_\pm\to 0}\frac{\sin a_\pm}{a_\pm}=1$.

If either vector $\mathbf{a}$ or bivector $\mathcal{A}$ in (8)-(10) is absent then $a_+ = a_- \equiv a$, where a is a magnitude of the vector $a = |\,\mathbf{a}\,| = \big(\mathbf{a}\widetilde{\widehat{\mathbf{a}}}\big)^{\frac{1}{2}} = \sqrt{a_1^2+a_2^2+a_3^2}$, or of the bivector $a = |\,\mathcal{A}\,| = \big(\mathcal{A}\widetilde{\widehat{\mathcal{A}}}\big)^{\frac{1}{2}} = \sqrt{a_{12}^2+a_{13}^2+a_{23}^2}$. If, in addition, the scalar and pseudoscalar are absent, $a_0 = a_{123} = 0$, the formula (8) reduces to the well-known trigonometric expressions for exponential of vector and bivector in a polar form [2], namely,

$$\mathrm{e}^{\mathbf{a}} = \cos|\,\mathbf{a}\,| + \frac{\mathbf{a}}{|\,\mathbf{a}\,|}\sin|\,\mathbf{a}\,|, \quad \mathrm{e}^{\mathcal{A}} = \cos|\,\mathcal{A}\,| + \frac{\mathcal{A}}{|\,\mathcal{A}\,|}\sin|\,\mathcal{A}\,|\,. \tag{11}$$

Note that the appearance of trigonometric functions in (11) is due to vector and bivector properties, $\mathbf{a}^2 < 0$ and $\mathcal{A}^2 < 0$ in $Cl_{0,3}$. If exponential consists of scalar and pseudoscalar only then $a_+ = a_- = 0$ and the Eq. (8) simplifies to hyperbolic sine and cosine functions,

$$\mathrm{e}^{a_0+Ia_{123}} = \mathrm{e}^{a_0}(\cosh a_{123} + I\sinh a_{123}), \quad I^2 = 1. \tag{12}$$

In the following we shall distinguish two kinds of coordinate-free formulas for exponential functions, namely, generic and special. The formula (8) is an example of generic formula since it is valid for almost all real coefficient a_J values, where J is a compound index: $J = i, ij$, or ijk. The expression (12) represents the special formula, since in the case $a_+ = 0$ and/or $a_- = 0$ we have division by zero in (8) and therefore should use a modified formula (which, in this case can be obtained by computing limit of (8) when $a_+ \to 0$, and/or $a_- \to 0$). On the other hand the Eq. (11) represents an important in practice case of generic solution (obtained by simply equating the coefficients at scalar and pseudoscalar and, respectively, at bivector and vector, to zero). For completeness, it would be interesting to remark that in a case of logarithmic functions one may add an additional free MV to the generic or special symbolic solution [15]. Such free terms do not appear for GA exponential functions.

Example 1. *Exponential of MV in $Cl_{0,3}$.* Let's compute the exponential of $\mathsf{A} = -8 - 6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23} - 4\mathbf{e}_{123}$ using the coordinate-free expression (8). We find $a_- = \sqrt{53}$ and $a_+ = \sqrt{353}$. The exact numerical answer then is

$$\exp(\mathsf{A}) = \tfrac{\mathrm{e}^{-8}}{2}\Big(\mathrm{e}^{4}(1-\mathbf{e}_{123})\Big(\cos\sqrt{53} + \tfrac{\sin\sqrt{53}}{\sqrt{53}}(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23})\Big)$$
$$+\,\mathrm{e}^{-4}(1+\mathbf{e}_{123})\Big(\cos\sqrt{353} + \tfrac{\sin\sqrt{353}}{\sqrt{353}}(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23})\Big)\Big).$$

For comparison, the calculation of the exponential series (7) by Mathematica v.12 in a floating point regime gives six significant figures of the exact solution after summation of 70 series terms (iterations). However, it should be stressed that the convergence is alternating and not monotonic (see fig. 1), and to get the first significant figure for all basis MV elements no less than 64 series terms (iterations) are needed in this particular case. The iteration number may be decreased if the inverse scaling and squaring method [13] is applied (see subsection 2.1) at a price of computing MV powers.

3.2 | Vector-bivector entanglement in $Cl_{0,3}$

In equations (9) and (10), the outer product $\mathbf{a}\wedge\mathcal{A}$ in general case is a trivector. It entangles or mixes up the components of vector and bivectorin in the exponential (8). This is easy to see if we equate to zero either $\mathbf{a}$ or $\mathcal{A}$. Then, $a_+ = a_- = |\,\mathbf{a}\,|$ if $\mathcal{A} = 0$ and $a_+ = a_- = |\,\mathcal{A}\,|$ if $\mathbf{a} = 0$, where $|\,\mathbf{a}\,| = \big(\mathbf{a}\widetilde{\widehat{\mathbf{a}}}\big)^{\frac{1}{2}}$ and $|\,\mathcal{A}\,| = \big(\mathcal{A}\widetilde{\widehat{\mathcal{A}}}\big)^{\frac{1}{2}}$. The trivector also vanishes if $\mathbf{a}$ and $\mathcal{A}$ are unequal to zero but the vector $\mathbf{a}$ lies in the plane $\mathcal{A}$. In this case[3] the components satisfy the condition $I\mathbf{a}\wedge\mathcal{A} = a_1a_{23} - a_2a_{13} + a_3a_{12} = 0$. Then the entanglement (mixing) coefficients become $a_+ = a_- \to a_m = \sqrt{|\,\mathbf{a}\,|^2 + |\,\mathcal{A}\,|^2}$ and the exponential (8) reduces to

$$\mathrm{e}^{\mathsf{A}} = \mathrm{e}^{a_0}\mathrm{e}^{Ia_{123}}\big(\cos a_m + \frac{\mathbf{a}+\mathcal{A}}{a_m}\sin a_m\big), \quad \mathbf{a}\|\mathcal{A}. \tag{13}$$

Thus, the exponential e^{A} in this case can be factorized. It is interesting that the multiplier in round brackets now represents a disentangled Moivre-type formula for a sum of vector and bivector, where the magnitude of $(\mathbf{a}+\mathcal{A})$ is

$$a_m = |\,\mathbf{a}+\mathcal{A}\,| = \sqrt{(\mathbf{a}+\mathcal{A})\widetilde{\widetilde{(\mathbf{a}+\mathcal{A})}}} = \sqrt{|\,\mathbf{a}\,|^2 + |\,\mathcal{A}\,|^2}. \tag{14}$$

We shall remind that in Eq. (13) the vector $\mathbf{a}$ lies in the plane $\mathcal{A}$. A similar formula can be obtained in opposite case if we assumes that, for example, the vector $\mathbf{a}\|\mathbf{e}_3$ is perpendicular to bivector $\mathcal{B}\|\mathbf{e}_{12}$ and $a_3a_{12}\neq 0$, i.e., the vector and bivector are characterized by a single scalar term in the entanglement formula.[4] Then the expression (8) gives

$$\mathrm{e}^{\mathsf{A}} = -\big(\cos(a_{12}-a_3) + \frac{a_{12}\mathbf{e}_{12}+a_3\mathbf{e}_3}{a_{12}-a_3}\sin(a_{12}-a_3)\big). \tag{15}$$

In conclusion, apart from Moivre-type expressions (see Eqs. (11) and (12)), the generic GA exponential (8) also contains entangled MVs which under additional conditions may be disentangled as seen from Eqs (13) and (15).

4 | MV EXPONENTIALS IN $Cl_{3,0}$ AND $Cl_{1,2}$ ALGEBRAS

After multiplication of the scalar coefficients given in[11] by respective basis elements and collection into a sum, and finally combining the resulting expression into a coordinate-free form we find the generic exponential of MV A,

$$\begin{aligned}\exp(\mathsf{A}) =& \mathrm{e}^{a_0}(\cos a_{123} + I\sin a_{123})\Big(\cos a_-\cosh a_+ + I\sin a_-\sinh a_+\\ &+ \frac{1}{a_+^2+a_-^2}\big(\cosh a_+\sin a_- - I\cos a_-\sinh a_+\big)\big(a_-(\mathbf{a}+\mathcal{A}) + a_+I(\mathbf{a}+\mathcal{A})\big)\Big),\end{aligned} \tag{16}$$

where scalar coefficients $a_\pm$ are

$$\begin{aligned}
&a_- = \frac{-2I\mathbf{a}\wedge\mathcal{A}}{\sqrt{2}\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}+\sqrt{(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})^2-4(\mathbf{a}\wedge\mathcal{A})^2}}},\\
&a_+ = \frac{\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}+\sqrt{(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})^2-4(\mathbf{a}\wedge\mathcal{A})^2}}}{\sqrt{2}}, \qquad \text{when}\quad \mathbf{a}\wedge\mathcal{A}\neq 0, \quad\text{and}\\
&\begin{cases} a_+ = \sqrt{\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}}, \quad a_- = 0, \quad \mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A} > 0\\ a_+ = 0, \quad a_- = \sqrt{-(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})}, \quad \mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A} < 0,\end{cases} \qquad \text{when}\quad \mathbf{a}\wedge\mathcal{A} = 0.
\end{aligned} \tag{17}$$

Since Eq. (16) is in a coordinate-free form the above formulas are valid for both mutually isomorphic $Cl_{3,0}$ and $Cl_{1,2}$ algebras. If formulas are expanded into coordinates, of course, the resulting expressions will differ by signs at some terms. Note that determinant[5] of a vector and bivector part $\mathbf{a}+\mathcal{A}$ is $\mathrm{Det}(\mathbf{a}+\mathcal{A}) = (a_+^2+a_-^2)^2$. When $\mathrm{Det}(\mathbf{a}+\mathcal{A}) = 0$ we have special case $\exp(\mathsf{A}) = \mathrm{e}^{a_0}(\cos a_{123} + I\sin a_{123})$ which again can be straightforwardly obtained by computing limit of (16), when both $a_+\to 0$ and $a_-\to 0$. Simultaneous vanishing of a_+ and a_- means vanishing of both the inner $\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}$ and outer $\mathbf{a}\wedge\mathcal{A}$ products. Then the well-known Moivre-type formulas from Eq. (16) follow

$$\exp\mathsf{A} = \mathrm{e}^{a_0}(\cos a_{123} + \sin a_{123}I) \qquad \mathbf{a} = \mathcal{A} = 0, \tag{18}$$

$$\exp\mathcal{A} = \cos|\,\mathcal{A}\,| + \frac{\mathcal{A}}{|\,\mathcal{A}\,|}\sin|\,\mathcal{A}\,| \qquad a_0 = a_{123} = \mathbf{a} = 0, \tag{19}$$

$$\exp\mathbf{a} = \cosh|\,\mathbf{a}\,| + \frac{\mathbf{a}}{|\,\mathbf{a}\,|}\sinh|\,\mathbf{a}\,| \qquad a_0 = a_{123} = \mathcal{A} = 0. \tag{20}$$

[3] Such a situation is encountered in classical electrodynamics where magnetic field bivector and electric field vector of a free wave lie in a the same plane.

[4] This approach reminds a popular method in physics where a judicious choice of mutual orientation of the fields and coordinate vectors allows to simplify the problem substantially.

[5] In 3D algebras the determinant of MV A is defined by $\mathrm{Det}(\mathsf{A}) = \mathsf{A}\widehat{\mathsf{A}}\widetilde{\widehat{\mathsf{A}}}\widetilde{\mathsf{A}}$ [16,17]. This determinant should not be confused with a MV transformation determinant [18].

The equation (18) represents a special case when $a_+ = a_- = 0$.

Similarly to $Cl_{0,3}$ algebra (see subsec. 3.2), in the exponential (16) the vector and bivector may be disentangled if we assume that $\mathbf{a}\|\mathcal{A}$, i.e., the vector $\mathbf{a}$ lies in the plane $\mathcal{A}$. Then, the vector-bivector sum is expressed by trigonometric and hyperbolic functions in both $Cl_{3,0}$ and $Cl_{1,2}$ algebras,

$$\exp \mathsf{A} = \begin{cases} \cos\sqrt{|\mathcal{A}|^2 - |\mathbf{a}|^2} + \frac{\mathbf{a}+\mathcal{A}}{\sqrt{|\mathcal{A}|^2-|\mathbf{a}|^2}} \sin\sqrt{|\mathcal{A}|^2 - |\mathbf{a}|^2} & \text{if } \mathbf{a}^2 < \mathcal{A}^2, \\ \cosh\sqrt{|\mathbf{a}|^2 - |\mathcal{A}|^2} + \frac{\mathbf{a}+\mathcal{A}}{\sqrt{|\mathbf{a}|^2-|\mathcal{A}|^2}} \sinh\sqrt{|\mathbf{a}|^2 - |\mathcal{A}|^2} & \text{if } \mathbf{a}^2 > \mathcal{A}^2. \end{cases} \tag{21}$$

Example 2. *Exponential of MV in $Cl_{3,0}$.* Let's take the same MV $\mathsf{A} = -8-6\mathbf{e}_2-9\mathbf{e}_3+5\mathbf{e}_{12}-5\mathbf{e}_{13}+6\mathbf{e}_{23}-4\mathbf{e}_{123}$ as in Example 1 and calculate the exponential using the coordinate-free expression (16). We find $\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A} = 31$, $-2I\mathbf{a}\wedge\mathcal{A} = -150$. Then $a_- = -75\sqrt{\frac{2}{31+\sqrt{23461}}}$ and $a_+ = \sqrt{\frac{31+\sqrt{23461}}{2}}$. Finally, the exact numerical answer is

$$\begin{aligned} \exp(\mathsf{A}) =& \frac{1}{\mathrm{e}^8}\big(\cos(4) - \sin(4)I\big)\Bigg(\cos\Big(75\sqrt{\tfrac{2}{31+\sqrt{23461}}}\Big)\cosh\Big(\sqrt{\tfrac{31+\sqrt{23461}}{2}}\Big) - \sin\Big(75\sqrt{\tfrac{2}{31+\sqrt{23461}}}\Big)\sinh\Big(\sqrt{\tfrac{31+\sqrt{23461}}{2}}\Big)I \\ &+ \frac{1}{\sqrt{23461}}\Bigg(\Big(-75\sqrt{\tfrac{2}{31+\sqrt{23461}}}(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23}) + \sqrt{\tfrac{31+\sqrt{23461}}{2}}(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23})I\Big) \\ &\times\Big(-\sin\Big(75\sqrt{\tfrac{2}{31+\sqrt{23461}}}\Big)\cosh\Big(\sqrt{\tfrac{31+\sqrt{23461}}{2}}\Big) - \cos\Big(75\sqrt{\tfrac{2}{31+\sqrt{23461}}}\Big)\sinh\Big(\sqrt{\tfrac{31+\sqrt{23461}}{2}}\Big)I\Big)\Bigg)\Bigg). \end{aligned} \tag{22}$$

Example 3. *Exponential in $Cl_{1,2}$ with disentanglement included.* Let's take a simple MV, $\mathsf{A} = 3 - \mathbf{e}_1 + 2\mathbf{e}_{12}$, which represents disentangled case because $\mathbf{a}\wedge\mathcal{A} = 0$. Then we have $a_+ = \sqrt{5}$ and $a_- = 0$. The answer is expressed in hyperbolic functions: $\exp(\mathsf{A}) = \mathrm{e}^3\Big(\cosh\sqrt{5} + \big(-\mathbf{e}_1 + 2\mathbf{e}_{12}\big)\frac{\sinh\sqrt{5}}{\sqrt{5}}\Big)$.

5 | MV EXPONENTIAL IN $Cl_{2,1}$

After assembling coefficients given in[11] into MV and then regrouping them to coordinate-free form we find the following exponential formula in $Cl_{2,1}$

$$\exp(\mathsf{A}) = \frac{1}{2}\mathrm{e}^{a_0}\Big(\mathrm{e}^{a_{123}}(1+I)\big(\mathrm{co}(a_+^2) + \mathrm{si}(a_+^2)(\mathbf{a}+\mathcal{A})\big) + \mathrm{e}^{-a_{123}}(1-I)\big(\mathrm{co}(a_-^2) + \mathrm{si}(a_-^2)(\mathbf{a}+\mathcal{A})\big)\Big), \tag{23}$$

where the scalar coefficients $a_\pm$ are

$$\begin{aligned} a_-^2 &= -(\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A}) + 2I\mathbf{a}\wedge\mathcal{A}, \qquad a_-^2 \gtrless 0, \\ a_+^2 &= -(\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A}) - 2I\mathbf{a}\wedge\mathcal{A}, \qquad a_+^2 \gtrless 0. \end{aligned} \tag{24}$$

To simplify notation in Eq. (23) we have introduced si and co functions that depending on sign under square root go over to either trigonometric or hyperbolic functions. All in all, this gives four cases for both si and co functions,

$$\mathrm{si}(a_+^2) = \begin{cases} \frac{\sin\sqrt{a_+^2}}{\sqrt{a_+^2}}, & a_+^2 > 0 \\ \frac{\sinh\sqrt{-a_+^2}}{\sqrt{-a_+^2}}, & a_+^2 < 0 \end{cases}; \qquad \mathrm{co}(a_+^2) = \begin{cases} \cos\sqrt{a_+^2}, & a_+^2 > 0 \\ \cosh\sqrt{-a_+^2}, & a_+^2 < 0 \end{cases}; \tag{25}$$

$$\mathrm{si}(a_-^2) = \begin{cases} \frac{\sin\sqrt{a_-^2}}{\sqrt{a_-^2}}, & a_-^2 > 0 \\ \frac{\sinh\sqrt{-a_-^2}}{\sqrt{-a_-^2}}, & a_-^2 < 0 \end{cases}; \qquad \mathrm{co}(a_-^2) = \begin{cases} \cos\sqrt{a_-^2}, & a_-^2 > 0 \\ \cosh\sqrt{-a_-^2}, & a_-^2 < 0 \end{cases}. \tag{26}$$

When $a_- = 0$ and/or $a_+ = 0$ we have special cases, which again can be easily included taking already mentioned limits, i.e., by putting $\mathrm{co}(0) = 1$ and $\mathrm{si}(0) = 1$.

5.1 | Special cases

If both the vector $\mathbf{a}$ and the bivector $\mathcal{A}$ are equal to zero the exponential (23) simplifies to $\exp \mathsf{A} = \exp(a_0 + I a_{123}) = \mathrm{e}^{a_0}(\cosh a_{123} + I \sinh a_{123})$.

The exponential of vector, when $a_0 = a_{123} = \mathcal{A} = 0$, is

$$\exp(\mathbf{a}) = \begin{cases} \cos\sqrt{-\mathbf{a}^2} + \frac{\mathbf{a}}{\sqrt{-\mathbf{a}^2}} \sin\sqrt{-\mathbf{a}^2} & \text{if } \mathbf{a}^2 < 0, \\ \cosh\sqrt{\mathbf{a}^2} + \frac{\mathbf{a}}{\sqrt{\mathbf{a}^2}} \sinh\sqrt{\mathbf{a}^2} & \text{if } \mathbf{a}^2 > 0. \end{cases} \tag{27}$$

The exponential of bivector, when $a_0 = a_{123} = \mathbf{a} = 0$, is

$$\exp(\mathcal{A}) = \begin{cases} \cos\sqrt{-\mathcal{A}^2} + \frac{\mathcal{A}}{\sqrt{-\mathcal{A}^2}} \sin\sqrt{-\mathcal{A}^2} & \text{if } \mathcal{A}^2 < 0, \\ \cosh\sqrt{\mathcal{A}^2} + \frac{\mathcal{A}}{\sqrt{\mathcal{A}^2}} \sinh\sqrt{\mathcal{A}^2} & \text{if } \mathcal{A}^2 > 0. \end{cases} \tag{28}$$

If a_0 and a_{123} are not equal to zero then $\exp(\mathbf{a})$ and $\exp(\mathcal{A})$ should be multiplied by $\mathrm{e}^{a_0}(\cosh a_{123} + I \sinh a_{123})$.

Example 4. *Exponential of MV in $Cl_{2,1}$. Case $a_-^2 < 0,\ a_+^2 > 0$.*
Using the same MV $\mathsf{A} = -8 - 6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23} - 4\mathbf{e}_{123}$ for $Cl_{2,1}$ now we have $a_-^2 = -141$, $a_+^2 = 159$. The answer then is

$$\begin{aligned} \exp(\mathsf{A}) =& \frac{1}{2\mathrm{e}^8}\bigg(\frac{1}{\mathrm{e}^4}(1+I)\Big(\tfrac{\sin\sqrt{159}}{\sqrt{159}}\big(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23}\big) + \cos\sqrt{159}\Big) \\ &+ \mathrm{e}^4(1-I)\Big(\tfrac{\sinh\sqrt{141}}{\sqrt{141}}\big(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23}\big) + \cosh\sqrt{141}\Big)\bigg). \end{aligned}$$

Example 5. *Exponential in $Cl_{2,1}$. Case $a_-^2 < 0,\ a_+^2 < 0$.* Multivector $\mathsf{A} = -6\mathbf{e}_2 + 5\mathbf{e}_{12} + \mathbf{e}_{123}$ of $Cl_{2,1}$. $a_-^2 = -11$, $a_+^2 = -11$. The exponential is

$$\exp(\mathsf{A}) = \frac{1}{2}\bigg(\big(\mathrm{e}(1+I) + \mathrm{e}^{-1}(1-I)\big)\Big(\tfrac{\sinh\sqrt{11}}{\sqrt{11}}\big(-6\mathbf{e}_2 + 5\mathbf{e}_{12}\big) + \cosh\sqrt{11}\Big)\bigg).$$

Example 6. *Exponential in $Cl_{2,1}$. Case $a_-^2 > 0,\ a_+^2 > 0$.* Exponential of $\mathsf{A} = 2 + \mathbf{e}_3 + 6\mathbf{e}_{12} + 3\mathbf{e}_{123}$ of $Cl_{2,1}$. We have $a_-^2 = 49$, $a_+^2 = 25$. The answer then is

$$\exp(\mathsf{A}) = \frac{\mathrm{e}^2}{2}\bigg(\mathrm{e}^3(1+I)\Big(\tfrac{\sin 5}{5}\big(\mathbf{e}_3 + 6\mathbf{e}_{12}\big) + \cos 5\Big) + \mathrm{e}^{-3}(1-I)\Big(\tfrac{\sin 7}{7}\big(\mathbf{e}_3 + 6\mathbf{e}_{12}\big) + \cos 7\Big)\bigg).$$

Example 7. *Exponential in $Cl_{2,1}$. Case $a_-^2 > 0,\ a_+^2 < 0$.* $\mathsf{A} = 2 - 10\mathbf{e}_2 - 10\mathbf{e}_3 + 2\mathbf{e}_{13} + \mathbf{e}_{23} + \mathbf{e}_{123}$, $a_-^2 = 35$, $a_+^2 = -45$. The answer is

$$\begin{aligned} \exp(\mathsf{A}) =& \frac{\mathrm{e}^2}{2}\bigg(\mathrm{e}(1+I)\Big(\tfrac{\sinh 3\sqrt{5}}{3\sqrt{5}}\big(-10\mathbf{e}_2 - 10\mathbf{e}_3 + 2\mathbf{e}_{13} + \mathbf{e}_{23}\big) + \cosh(3\sqrt{5})\Big) \\ &+ \mathrm{e}^{-1}(1-I)\Big(\tfrac{\sin\sqrt{35}}{\sqrt{35}}\big(-10\mathbf{e}_2 - 10\mathbf{e}_3 + 2\mathbf{e}_{13} + \mathbf{e}_{23}\big) + \cos\sqrt{35}\Big)\bigg). \end{aligned}$$

6 | EXAMPLES OF APPLICATION: DIFFERENTIAL GA EQUATION-SOLVING

The exponential function plays an important role in solution of linear differential equations [19]. For example, the solution of a homogeneous equation

$$\frac{\mathrm{d}\mathsf{X}}{\mathrm{d}t} = \mathsf{A}\mathsf{X}, \quad \mathsf{X} = \mathsf{X}_0 \quad \text{at } t = 0, \tag{29}$$

with respect to MV X gives GA exponential function $\mathsf{X}(t) = \mathrm{e}^{t\mathsf{A}}\mathsf{X}_0$, where t is the parameter, for instance, the time. Treating $t\mathsf{A}$ as a new MV, after expansion of $\mathrm{e}^{t\mathsf{A}}$ we will get the evolution of X in time. More generally, with suitable assumptions upon smoothness of $\mathsf{X}(t)$, the solution of the inhomogeneous system

$$\frac{\mathrm{d}\mathsf{X}}{\mathrm{d}t} = \mathsf{A}\mathsf{X} + f(t), \quad \mathsf{X} = \mathsf{X}_0 \quad \text{at } t = 0, \tag{30}$$

may be expressed by

$$\mathsf{X}(t) = \mathrm{e}^{t\mathsf{A}}\mathsf{X}_0 + \int_0^t \mathrm{e}^{(t-s)\mathsf{A}} f(s)\mathrm{d}s. \tag{31}$$

Some of MV differential equations, for example,

$$\frac{\mathrm{d}\mathsf{X}}{\mathrm{d}t} = \mathsf{A}\mathsf{X} \pm \mathsf{X}\mathsf{B}, \quad \mathsf{X} = \mathsf{X}_0 \quad \text{at } t = 0, \tag{32}$$

have solution that consists of a product of GA exponentials: $\mathsf{X}(t) = \mathrm{e}^{t\mathsf{A}}\, X_0 \mathrm{e}^{\pm t\mathsf{B}}$. The answer can be easily checked by direct substitution of $\mathsf{X}(t)$ into Eq. (32) and application of Leibniz's differentiation theorem[18]. If $\mathsf{B} = \mathsf{A}$ and the sign is negative we will get the rotor equation.

Trigonometric GA functions, as well as GA roots, arise in the solution of second order differential equations. For example, the GA equation[13]

$$\frac{\mathrm{d}^2\mathsf{X}}{\mathrm{d}t^2} + \mathsf{A}\mathsf{X} = 0, \quad \text{at} \quad t = 0,\ \mathsf{X} = \mathsf{X}_0 \text{ and } (\mathrm{d}\mathsf{X}/\mathrm{d}t)_{t=0} = \mathsf{X}_0', \tag{33}$$

has the solution

$$\mathsf{X}(t) = \cos\big(\sqrt{\mathsf{A}}\, t\big)\mathsf{X}_0 + \big(\sqrt{\mathsf{A}}\big)^{-1} \sin\big(\sqrt{\mathsf{A}}\, t\big)\mathsf{X}_0', \tag{34}$$

where $\sqrt{\mathsf{A}}$ is the square root of A. The trigonometric functions of MV argument can be expressed by exponentials[4,11]. Closed form expression for square root of MV when $p+q = 3$ are presented in[20]. A concrete example of application of GA exponentials in physics can be found in paper[11].

7 | CONCLUSIONS AND DISCUSSION

The main results of this paper are the formulas (8), (16) and (23), where real GA exponentials are presented in an expanded coordinate-free form. Since in 3D algebras the scalar and pseudoscalar belong to GA center, the related coefficients a_0 and a_{123} appear in scalar exponentials only. In all algebras the entanglement (mixing) of vector and bivector components takes place. The mixing is characterized by scalar coefficients a_+ and a_-, where the terms of the form $(a_i - a_{jk})^2$, $i \neq j \neq k$ appear. The entanglement can be eliminated by equating to zero either vector or bivector, as a result the well-known trigonometric and hyperbolic Moivre-type formulas for vector and bivector exponentials are recovered. However, more interesting case is the disentanglement when the vector is parallel to bivector. In this orientation we obtain the exponential which consists of a sum of scalar, vector and bivector, Eqs. (13) and (21). Finally, we shall note that for $n = p+q = 3$ algebras the characteristic polynomial of matrix representation is of degree 4. Since the algebraic equations of degree 4 are solvable in radicals, our results, the Eqs (8), (16) and (23) are consistent with this theorem. Moreover, characteristic polynomials of GAs with $n = 4$ are also of degree 4. It follows that explicit expressions for exponents can in principle be derived for $n = 4$ and even for $n = 5$, due to block diagonal form of matrix representations of $n = 5$ algebras.

The GA exponentials with complex coefficients also need deeper analysis, more so, since the relativity theory may be embedded into complexified 3D algebras[21].

The logarithm function is closely related to exponential. Our attempts to find real GA logarithm from general GA exponentials revealed that the GA logarithm problem is more difficult[15], albeit symbolically trackable. Similarly to complex logarithm, the GA logarithm is a not a single valued function and therefore one must proceed with caution not to mix different branches in reducing the logarithm to a principal value. In addition, it appeared that to GA logarithm one may add a free MV that vanishes after exponentiation. Knowledge of explicit forms of both exponents and logarithms opens a way to compute exact symbolic expressions for trigonometric/hyperbolic functions and their inverses of single MV at least for 3D GAs.

Conflict of interest

The authors declare no potential conflict of interests.

SUPPORTING INFORMATION

The following supporting information is available as part of the online article: https://github.com/ArturasAcus/GeometricAlgebra.

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