Speed determinacy of the traveling waves for a three species time-periodic Lotka-Volterra competition system

Qiong Wu* Chaohong Pan† and Hongyong Wang‡

Abstract

In this paper, speed selection of the time periodic traveling waves for a three species time-periodic Lotka-Volterra competition system is studied via the upper-lower solution method as well as the comparison principle. Through constructing specific types of upper and lower solutions to the system, the speed selection of the minimal wave speed can be determined under some sets of sufficient conditions composed of the parameters in the system.

Keywords and Phrases: Time-periodic Lotka-Volterra system; speed determinacy; periodic traveling waves

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1 Introduction

The competition system of Lotka-Volterra type

$$\begin{cases} u_t = u_{xx} + u(r_1(t) - b_{11}(t)u - b_{12}(t)v - b_{13}(t)w), \\ v_t = d_1(t)v_{xx} + v(r_2(t) - b_{21}(t)u - b_{22}(t)v), \\ w_t = d_2(t)w_{xx} + w(r_3(t) - b_{31}(t)u - b_{33}(t)w),\ x \in R,\ t \in R^+, \end{cases} \tag{1.1}$$

models the population dynamics in a time periodic environment. In (1.1), the unknown functions $u(t,x)$, $v(t,x)$ and $w(t,x)$ account for respectively the population densities of three species; $d_i(t),\ i = 1,2$ are the diffusive coefficients; $b_{ii}(t), i = 1,2,3$ are the interspecific competition coefficients, which are used to quantify a numerical indicator of the degree of competition between the same species; $b_{1j}(t), b_{j1}(t)$, $j = 2,3$ are the interspecific competition coefficients; the coefficients $r_i(t), i = 1,2,3$ represent the growth

*School of Mathematics and Physics, University of South China, Hengyang 421001, China. Email: m18939662786@163.com

†School of Mathematics and Physics, University of South China, Hengyang 421001, China. Email: pan.ch@mail.scut.edu.cn. This work is supported by the National Natural Science Foundation of China grant (11626129 and 11801263), the Natural Science Foundation of Hunan Province grant (2018JJ3418).

‡Corresponding author. School of Mathematics and Physics, University of South China, Hengyang 421001, China. Email: sysuwhylj@aliyun.com. This work is partially supported by the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 20B512).

rates. Moreover, all of the coefficients in (1.1) are assumed to be T-periodic, positive and continuous functions.

In recent years, two-dimensional competition models have been extensively studied. Specially, the existence, uniqueness and stability of the corresponding traveling wave solutions as well as the persistence issues, etc were addressed perfectly or in part in the literatures, for examples, [12–19, 35, 36] and more references therein. For three-dimensional competition models like (1.1), the study of the aforementioned problems becomes more challenging due to the higher phase space dimension. This aspect of research also has attracted much attention from scientist community. Among them, we refer the reader to some references for showing the permanence [21–23], extinction [24], existence [25, 27], coexistence [28] and more references therein.

Particularly, we remark that the speed determinacy keeps its activity among the groups of mathematician and biologist for a long time. For such a topic, we refer the reader to [2–4, 13] for a diffusive Lotka-Volterra model with constant coefficients, [5] for a integro-difference model, [6] for cooperative models, [7, 8] for lattice system, [9, 36] for a spatial and time periodic Lotka-Volterra competition model and so on.

Yet, much less is known for the speed selection for a three-species Lotka-Volterra competition system except for the work [10]. Since the coefficients of the model considered in [10] are constant, it is a autonomous system. However, the model (1.1) is non-autonomous, which makes the analysis more delicate. Moreover, we shall investigate the nonlinear selection which seems to be not addressed in the existing publications (to the best of our knowledge). In this paper, we make an effort in this direction. Speed selection of the system (1.1) is closely related to the time-periodic traveling wave solutions which are assumed to have the following form of

$$\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} = \begin{pmatrix} X(t,x-ct) \\ Y(t,x-ct) \\ Z(t,x-ct) \end{pmatrix} := \begin{pmatrix} X(t,z) \\ Y(t,z) \\ Z(t,z) \end{pmatrix},$$

obeying the condition

$$\begin{pmatrix} X(t+T,z) \\ Y(t+T,z) \\ Z(t+T,z) \end{pmatrix} = \begin{pmatrix} X(t,z) \\ Y(t,z) \\ Z(t,z) \end{pmatrix},$$

where $z = x - ct$, the constant c is the wave speed. The dynamic behaviors of the kinetic system of (1.1) is very complicate, and it has at least five equilibrium points as below

$$e_0 = (0,0,0),\ e_1 = (p(t),0,0),\ e_2 = (0,q(t),s(t)),\ e_3 = (0,q(t),0),\ e_4 = (0,0,s(t)),$$

where $p(t)$, $q(t)$ and $s(t)$ are the solutions of the following differential equations

$$\frac{d\theta}{dt} = \theta(r_i(t) - b_{ii}(t)\theta),\ \ i = 1,2,3, \tag{1.2}$$

respectively. Noticing the differential equations (1.2) are Bernoulli equation, one can obtain the explicit expressions for $p(t)$, $q(t)$, $s(t)$ as follows

$$p(t) = \frac{p_0 \int_0^t r_1(s)ds}{1 + p_0 \int_0^t b_{11}(s)e^{\int_0^s r_1(\tau)d\tau}ds},\quad p_0 = \frac{\int_0^T r_1(s)ds - 1}{\int_0^T b_{11}(s)e^{\int_0^s r_1(\tau)d\tau}ds},$$

$$q(t)=\frac{q_0\int_0^t r_2(s)ds}{1+q_0\int_0^t b_{22}(s)e^{\int_0^s r_2(\tau)d\tau}ds},\quad q_0=\frac{\int_0^T r_2(s)ds-1}{\int_0^T b_{22}(s)e^{\int_0^s r_2(\tau)d\tau}ds},$$

$$s(t)=\frac{s_0\int_0^t r_3(s)ds}{1+s_0\int_0^t b_{33}(s)e^{\int_0^s r_3(\tau)d\tau}ds},\quad s_0=\frac{\int_0^T r_3(s)ds-1}{\int_0^T b_{33}(s)e^{\int_0^s r_3(\tau)d\tau}ds}.$$

It is direct to check that under the conditions

$$\int_0^T r_2(t)dt<\int_0^T b_{21}(t)p(t)dt,\quad \int_0^T r_3(t)dt<\int_0^T b_{31}(t)p(t)dt, \tag{1.3}$$

the equilibrium e_1 is linearly stable, and the condition

$$\int_0^T r_1(t)dt>\int_0^T (b_{12}(t)q(t)+b_{13}(t)s(t))dt, \tag{1.4}$$

ensures e_2 is unstable. Throughout this paper, we always assume that (1.3) and (1.4) hold true and are interested in such time periodic traveling wave solutions which connect e_1 to e_2. This means $(X,Y,Z)(t,z)$ satisfies the boundary conditions

$$\begin{pmatrix} X(t,-\infty)\\ Y(t,-\infty)\\ Z(t,-\infty)\end{pmatrix}:=\lim_{z\to-\infty}\begin{pmatrix} X(t,z)\\ Y(t,z)\\ Z(t,z)\end{pmatrix}=\begin{pmatrix} p(t)\\ 0\\ 0\end{pmatrix},\quad \begin{pmatrix} X(t,\infty)\\ Y(t,\infty)\\ Z(t,\infty)\end{pmatrix}:=\lim_{z\to\infty}\begin{pmatrix} X(t,z)\\ Y(t,z)\\ Z(t,z)\end{pmatrix}=\begin{pmatrix} 0\\ q(t)\\ s(t)\end{pmatrix}. \tag{1.5}$$

To get a cooperative system so that it is more convenient for analysis, we introduce a pair of new variables $U(t,z),V(t,z),W(t,z)$, which are defined by

$$U(t,z)=\frac{X(t,z)}{p(t)},\quad V(t,z)=1-\frac{Y(t,z)}{q(t)},\quad W(t,z)=1-\frac{Z(t,z)}{s(t)}. \tag{1.6}$$

Substituting (1.6) into (1.1) leads to the following wave profile system

$$\begin{cases} U_t=U_{zz}+cU_z+U[b_{11}(t)p(t)(1-U)-b_{12}(t)q(t)(1-V)-b_{13}(t)s(t)(1-W)],\\ V_t=d_1(t)V_{zz}+cV_z+(1-V)[b_{21}(t)p(t)U-b_{22}(t)q(t)V],\\ W_t=d_2(t)W_{zz}+cW_z+(1-W)[b_{31}(t)p(t)U-b_{33}(t)s(t)W],\\ (U,V,W)(t,z)=(U,V,W)(t+T,z),\\ (U,V,W)(t,-\infty)=(1,1,1),(U,V,W)(t,\infty)=(0,0,0),\ z\in R,\ t\in R^+.\end{cases} \tag{1.7}$$

By use of the abstract results in [29], it follows that there exists a critical number $c_{\min}$ (In most references, it is usually called as the minimal wave speed of (1.1)) such that system (1.7) admits a solution for each c satisfying $c\geq c_{\min}$. Upon this critical number and another speed c_0 (see (4.4)) solved from the linear equation of U-equation in (1.7), one can give the concept of linear or nonlinear selection of the minimal wave speed $c_{\min}$, see for example [3, 14]. More precisely, the minimal wave speed is said to be linear selected provided that $c_{\min}=c_0$, and nonlinear selected provided that $c_{\min}>c_0$.

The present paper is arranged as follows. In section 2, we present the results about nonlinear selection. The results of linear selection are showed in section 3. In section 4, we add an appendix concluding some primary conclusions.

2 The nonlinear selection for the minimal wave speed

In this section, we take an insight to the nonlinear selection for the minimal wave speed. Firstly, we give the following lemma which can be proved with a slight modification by employing the ideas in [36].

Lemma 2.1 *If $U(t,z) = U(t, x-ct)$ is continuous in t and z, non-increasing in z and T-periodic in t for $c \geq c_0 = 2\sqrt{\overline{\Theta(t)}}$, and satisfies $U(t,+\infty) = 0$, $U(t,-\infty) > \max\left\{\frac{b_{22}(t)q(t)}{b_{21}(t)p(t)}, \frac{b_{33}(t)s(t)}{b_{31}(t)p(t)}\right\}$, then the following system*

$$\begin{cases} V_t = d_1(t)V_{zz} + cV_z + (1-V)[b_{21}(t)p(t)U - b_{22}(t)q(t)V], \\ W_t = d_2(t)W_{zz} + cW_z + (1-W)[b_{31}(t)p(t)U - b_{33}(t)s(t)W], \\ (V,W)(t,z) = (V,W)(t+T,z), (V,W)(t,+\infty) = 0, (V,W)(t,-\infty) = 1, \end{cases} \tag{2.1}$$

has a pair of solutions $(V,W)(t,z)$. Furthermore, if we think of V and W as functions of U, then $V(U)$ and $W(U)$ are monotone increasing in U.

To proceed, we need the following theorem saying that the faster decaying rate (at the right far end) of the time periodic wave solution plays a crucial role in determining the nonlinear selection.

Theorem 2.2 *Assume (1.7) has a continuous lower solution $(\underline{U}, \underline{V}, \underline{W})(t,z) \geq 0, z = x - c_1 t$ with $c_1 > c_0$, and is T-periodic in t. Moreover, the first component $\underline{U}(t,z)$ is further supposed to be monotone decreasing in z and satisfies*

$$\lim_{z\to-\infty} \sup \underline{U}(t,z) < 1, \; \underline{U}(t,z) \sim \varphi_{1,\mu_2}(t)e^{-\mu_2 z}, \quad \text{as } z \to \infty, \tag{2.2}$$

where μ_2 is defined in (4.3). Then system (1.7) has no traveling wave solution for the speed c in $[c_0, c_1)$.

Proof. For the given initial values $(u,v,w)(0,x)$, suppose (1.7) has a traveling wave solution for some constants c satisfying $c_0 \leq c < c_1$. It follows from

$$\lim_{z\to-\infty} \sup \underline{U}(t,z) < 1, \text{ and } U(t,-\infty) = 1,$$

that there exists a sufficiently large negative number x_1 such that $\underline{U}(0,x) < u(0,x)$ for any $x \in (-\infty, x_1)$. In addition, the assumption $\underline{U}(t,z) \sim \varphi_{1,\mu_2}(t)e^{-\mu_2 z}$ as $z \to \infty$ implies that there exists another number x_2 such that $\underline{U}(0,x) < u(0,x)$ for any $x \in (x_2, \infty)$, since $\mu_2(c)$ is monotone increasing in c. Hence, one can get the inequality

$$\underline{U}(0,x) < u(0,x),$$

holds for all $x \in (-\infty,\infty)$, by a shift if it is necessary. From the monotonicities of $V(U)$ and $W(U)$ in U explored in the Lemma 2.1, we can infer that $(\underline{U}, \underline{V}, \underline{W})(0,x) < (U,V,W)(0,x), \forall x \in (-\infty,\infty)$. Recalling $(\underline{U}, \underline{V}, \underline{W})(t, x-c_1 t)$ is a lower solution of the system (1.7) and making use of the comparison principle enable us to get

$$\begin{aligned} \underline{U}(t, x-c_1 t) &\leq U(t, x-ct), \\ \underline{V}(t, x-c_1 t) &\leq V(t, x-ct), \\ \underline{W}(t, x-c_1 t) &\leq W(t, x-ct), \end{aligned} \tag{2.3}$$

where $(t,x) \in (0,\infty) \times (-\infty,\infty)$. Owing to the continuity of $\underline{U}(t,z)$ in z for each t and (2.2), we can take a point σ_1 in the range so that $\underline{U}(t,\sigma_1) > 0$. As a consequence of the first equation of (2.3), we have

$$\underline{U}(t,\sigma_1) \leq U(t, x-ct) = U(t, \sigma_1 + (c_1 - c)t) \to 0, \text{ as } z \to \infty.$$

However, this is contradicted to $\underline{U}(t,\sigma_1) > 0$. Thus, the proof is accomplished. $\square$

Remark 2.3 *Theorem 2.2 implies that the system (1.7) does not admit traveling wave solution when c falls in the interval $[c_0, c_1)$, which in turn means that the wave speed is nonlinear selected.*

According to Theorem 2.2, to establish a principle for nonlinear selection, it is sufficient to find a suitable lower solution with faster decay rate at the positive infinity. With this aim, we set

$$\underline{U}_1(t,z) = \frac{\underline{k}}{1 + \frac{e^{\mu_2 z}}{\varphi(t)}}, \quad \underline{V}_1(t,z) = \underline{W}_1(t,z) = \frac{\underline{U}_1(t,z)}{\underline{k}}, \tag{2.4}$$

where $\mu_2 = \mu_2(c_1)$ is the constant defined in (4.3) with $c_1 = c_0 + \epsilon$ (ϵ is a sufficiently small number), $\varphi(t) := \varphi_{1,\mu_2}(t)$ (see (4.4)) and $\underline{k}$ is a constant satisfying $0 < \underline{k} < 1$.

Theorem 2.4 *If the following inequality*

$$\max\left\{\max_{t\in[0,T]} N_1(t), \max_{t\in[0,T]} N_2(t)\right\} < \min_{t\in[0,T]}\left\{1 - \frac{2\overline{\Theta(t)}}{b_{11}(t)p(t)}\right\},$$

where

$$N_1(t) = \frac{(d_1(t)+1)\overline{\Theta(t)} + b_{22}(t)q(t) + \Theta(t)}{b_{21}(t)p(t)},$$
$$N_2(t) = \frac{(d_2(t)+1)\overline{\Theta(t)} + b_{33}(t)s(t) + \Theta(t)}{b_{31}(t)p(t)},$$

holds true, then the minimal wave speed is nonlinearly selected.

Proof. By the assumption, we can assume that the range of the constant $\underline{k}$ satisfies the following conditions

$$\max\left\{\max_{t\in[0,T]} N_1(t), \max_{t\in[0,T]} N_2(t)\right\} < \underline{k} < \min_{t\in[0,T]}\left\{1 - \frac{2\overline{\Theta(t)}}{b_{11}(t)p(t)}\right\}. \tag{2.5}$$

Plugging (2.4) into the first equation of (1.7), we have

$$\begin{aligned}
&(\underline{U}_1)_{zz} + c_1(\underline{U}_1)_z + \underline{U}_1[b_{11}(t)p(t)(1-\underline{U}_1) - b_{12}(t)q(t)(1-\underline{V}_1) - b_{13}(t)s(t)(1-\underline{W}_1)] - (\underline{U}_1)_t \\
&= \frac{\underline{U}_1^2}{\underline{k}}\left(1 - \frac{\underline{U}_1}{\underline{k}}\right)\left[-2\mu_2^2 + \frac{b_{11}(t)p(t)(1-\underline{k})}{1-\frac{\underline{U}_1}{\underline{k}}}\right] \\
&\geq \frac{\underline{U}_1^2}{\underline{k}}\left(1 - \frac{\underline{U}_1}{\underline{k}}\right)\left[-2\mu_2^2 + b_{11}(t)p(t)(1-\underline{k})\right].
\end{aligned}$$

Let $\epsilon \to 0^+$, then it follows that $\mu_2 \to \sqrt{\overline{\Theta(t)}}$. One can derive from the right part of (2.5) that

$$-2\mu_2^2 + b_{11}(t)p(t)(1-\underline{k}) > 0.$$

Therefore,

$$(\underline{U}_1)_{zz} + c_1(\underline{U}_1)_z + \underline{U}_1[b_{11}(t)p(t)(1-\underline{U}_1) - b_{12}(t)q(t)(1-\underline{V}_1) - b_{13}(t)s(t)(1-\underline{W}_1)] - (\underline{U}_1)_t \geq 0. \tag{2.6}$$

As for the second equation of (1.7), a straightforward computation gives

$$
\begin{aligned}
& d_1(t)(\underline{V}_1)_{zz} + c_1(\underline{V}_1)_z + (1-\underline{V}_1)[b_{21}(t)p(t)\underline{U}_1 - b_{22}(t)q(t)\underline{V}_1] - (\underline{V}_1)_t \\
= & \frac{\underline{U}_1}{\underline{k}}\left(1-\frac{\underline{U}_1}{\underline{k}}\right)\left[d_1(t)\mu_2^2 - 2d_1(t)\mu_2^2\frac{\underline{U}_1}{\underline{k}} - c_1\mu_2 - \frac{\varphi'(t)}{\varphi(t)} + \underline{k}b_{21}(t)p(t) - b_{22}(t)q(t)\right] \\
\geq & \frac{\underline{U}_1}{\underline{k}}\left(1-\frac{\underline{U}_1}{\underline{k}}\right)\left[-d_1(t)\mu_2^2 - c_1\mu_2 - \frac{\varphi'(t)}{\varphi(t)} + \underline{k}b_{21}(t)p(t) - b_{22}(t)q(t)\right].
\end{aligned}
$$

Note that $c_1 \to 2\sqrt{\overline{\Theta(t)}}$ and $\mu_2 \to \sqrt{\overline{\Theta(t)}}$ as $\epsilon \to 0^+$. The left part of (2.5) and (4.2) ensure that

$$
d_1(t)(\underline{V}_1)_{zz} + c_1(\underline{V}_1)_z + (1-\underline{V}_1)[b_{21}(t)p(t)\underline{V}_1 - b_{22}(t)q(t)\underline{V}_1] - (\underline{V}_1)_t \geq 0, \tag{2.7}
$$

and

$$
d_2(t)(\underline{W}_1)_{zz} + c_1(\underline{W}_1)_z + (1-\underline{W}_1)[b_{31}(t)p(t)\underline{U}_1 - b_{33}(t)s(t)\underline{W}_1] - (\underline{W}_1)_t \geq 0. \tag{2.8}
$$

Therefore, it can be seen from (2.6), (2.7) and (2.8) that the set of functions defined in (2.4) is a lower solution that satisfies all of the assumptions in Theorem 2.2. So the proof is done. □

3 The linear selection for the minimal wave speed

As far as the linear selection is concerned, one can immediately get a result by adapting the ideas of [30]. To present our results, for conciseness, we use the following notations hereafter

$$
\begin{aligned}
\Theta(t) :&= b_{11}(t)p(t) - b_{12}(t)q(t) - b_{13}(t)s(t),\ \overline{\Theta(t)} := \frac{1}{T}\int_0^T \Theta(t)dt, \\
\Theta_1(t) :&= b_{21}(t)p(t) - b_{22}(t)q(t),\ \Theta_2(t) := b_{31}(t)p(t) - b_{33}(t)s(t),
\end{aligned} \tag{3.1}
$$

where T is the smallest positive period, and make the following assumption:
(H) $b_{ij} \in C^\theta(R)$ for some θ with $0 < \theta < 1$.

Theorem 3.1 *Suppose that the assumptions (1.3)-(1.4) and (H) hold. Then the minimal wave speed of (1.1) is linear selected provided that* $0 < d_i(t) \leq 1, i = 1, 2$ *and* $\Theta(t) \geq \Theta_1(t) \geq \Theta_2(t) \geq 0$ *for any* $t \in [0, T]$.

Proof. As mentioned above, the proof follows similarly from [30, Theorems 2.5 and 2.6]. We state it here for completeness. We firstly focus on the case $c > c_0$, and set

$$
\varphi_1(t) := \exp\left(\int_0^t \Theta(\tau)d\tau - t\overline{\Theta(t)}\right),\ \nu(t,z) := m\varphi_1(t)e^{-\mu_1 z}, \tag{3.2}
$$

where $\mu_1 = \frac{c-\sqrt{c^2-4\overline{\Theta(t)}}}{2}$. For each $c > c_0$, it is easy to see that $\mu_1 > 0$. By a simple calculation, one can obtain

$$
\varphi_1'(t) = \Theta(t)\varphi_1(t) - \overline{\Theta(t)}\varphi_1(t).
$$

Next, we apply the upper/lower method to show the existence of the traveling wave solutions of (1.7). Take $(\overline{U}, \overline{V}, \overline{W}) := \min\{(\nu,\nu,\nu)(t,z), (1,1,1)\}$. Firstly, we want to show that $(\overline{U}, \overline{V}, \overline{W})$ is an upper solution to the system (1.7). It is sufficient to prove that $(\overline{U},\overline{V},\overline{W})$ satisfies

$$\begin{cases} \overline{U}_t \geq \overline{U}_{zz} + c\overline{U}_z + \overline{U}[b_{11}(t)p(t)(1-\overline{U}) - b_{12}(t)q(t)(1-\overline{V}) - b_{13}(t)s(t)(1-\overline{W})], \\ \overline{V}_t \geq d_1(t)\overline{V}_{zz} + c\overline{V}_z + (1-\overline{V})[b_{21}(t)p(t)\overline{U} - b_{22}(t)q(t)\overline{V}], \\ \overline{W}_t \geq d_2(t)\overline{W}_{zz} + c\overline{W}_z + (1-\overline{W})[b_{31}(t)p(t)\overline{U} - b_{33}(t)s(t)\overline{W}]. \end{cases} \tag{3.3}$$

In fact, when $\nu(t,z) \geq 1$, we have $(\overline{U},\overline{V},\overline{W}) = (1,1,1)$ and clearly (3.3) holds in this case. When $\nu(t,z) \leq 1$, we have $(\overline{U},\overline{V},\overline{W}) = (\nu,\nu,\nu)(t,z)$ and

$$\begin{aligned} &\nu_{zz} + c\nu_z - \nu_t + \nu(1-\nu)\Theta(t) \leq \nu(\mu_1^2 - c\mu_1 + \overline{\Theta(t)}) = 0, \\ &d_1\nu_{zz} + c\nu_z - \nu_t + \nu(1-\nu)\Theta_1(t) \leq \nu(\mu_1^2 - c\mu_1 + \overline{\Theta(t)}) = 0, \\ &d_2\nu_{zz} + c\nu_z - \nu_t + \nu(1-\nu)\Theta_2(t) \leq \nu(\mu_1^2 - c\mu_1 + \overline{\Theta(t)}) = 0, \end{aligned}$$

which indicates that $(\overline{U},\overline{V},\overline{W})$ is an upper solution.

For the construction of a lower solution of (3.3), we let $\psi_{d_1}(t)$ and $\psi_{d_2}(t)$ be defined as the periodic solutions of the following two equations respectively,

$$b_{21}(t)p(t)\varphi_1(t) - (b_{22}(t)q(t) + \overline{\Theta(t)} + (1-d_1(t))\mu_1^2)\alpha - \frac{d\alpha}{dt} = 0, \tag{3.4}$$

$$b_{31}(t)p(t)\varphi_1(t) - (b_{33}(t)s(t) + \overline{\Theta(t)} + (1-d_2(t))\mu_1^2)\beta - \frac{d\beta}{dt} = 0. \tag{3.5}$$

In view of $\overline{b_{22}(t)q(t) + \overline{\Theta(t)} + (1-d_1(t))\mu_1^2} > 0$ and $\overline{b_{33}(t)q(t) + \overline{\Theta(t)} + (1-d_2(t))\mu_1^2} > 0$, the solutions $\psi_{d_1}(t)$ and $\psi_{d_2}(t)$ are unique. In particular, we denote the solutions of (3.4) and (3.5) as $d_1(t) = 1$ and $d_2(t) = 1$ respectively by $\psi_1(t)$ and $\psi_2(t)$. Let $\vartheta = -[(\mu_1+\epsilon)^2 - c(\mu_1+\epsilon) + \overline{\Theta(t)}]$, where ϵ is a sufficient small positive number. It can be seen that $\vartheta > 0$. Fix n_1, n_2, n_3 such that $n_1 \geq 1$ and

$$n_2 = \max\left\{n_1, n_1 \max_{t\in\mathbb{R}^+} \frac{\psi_{d_1}}{\psi_1}\right\},\ n_3 = \max\left\{n_1, n_1 \max_{t\in\mathbb{R}^+} \frac{\psi_{d_2}}{\psi_2}\right\}.$$

Set

$$\begin{aligned} \Lambda(t) &:= [b_{11}(t) + b_{21}(t) + b_{31}(t)]p(t) + b_{12}(t)q(t) + b_{13}(t)s(t), \\ \triangle_\vartheta &:= \frac{\vartheta \min\{n_1 \min_t \varphi, n_2 \min_t \psi_1, n_3 \min_t \psi_2\}}{(1 + n_2 \max_t\{\frac{\psi_1}{\psi_{d_1}}\} + n_3 \max_t\{\frac{\psi_2}{\psi_{d_2}}\}) \max_t\{\varphi^2 + \psi_{d_1}^2 + \psi_{d_2}^2\} \max_t\{\Lambda(t)\}}, \end{aligned}$$

and define

$$\begin{aligned} \underline{U}(t,z) &= \min\{\sigma e^{-\mu_1 z}\varphi_1(t)(1 - n_1 e^{-\epsilon z}), 0\}, \\ \underline{V}(t,z) &= \min\left\{\sigma e^{-\mu_1 z}\psi_{d_1}(t)\left(1 - n_2 \frac{\psi_1(t)}{\psi_{d_1}(t)} e^{-\epsilon z}\right), 0\right\}, \\ \underline{W}(t,z) &= \min\left\{\sigma e^{-\mu_1 z}\psi_{d_2}(t)\left(1 - n_3 \frac{\psi_2(t)}{\psi_{d_2}(t)} e^{-\epsilon z}\right), 0\right\}, \end{aligned} \tag{3.6}$$

where $\sigma \in (0, \triangle_\vartheta]$. According to (3.6), it is easy to see that $(\underline{U},\underline{V},\underline{W}) \leq (0,0,0)$ for all (t,z) in the region $\{(t,z)|t \in \mathbb{R}^+, z < z^0 = \frac{\ln n_1}{\epsilon}\}$. When $(t,z) \in \mathbb{R}^+ \times [z^0,\infty)$, upon the substitution of $(U,V,W)(t,z) = (\underline{U},\underline{V},\underline{W})(t,z)$ into (3.6), one has

$$
\begin{aligned}
&\underline{U}[b_{11}(t)p(t)(1-\underline{U})-b_{12}(t)q(t)(1-\underline{V})-b_{13}(t)s(t)(1-\underline{W})]+\underline{U}_{zz}+c\underline{U}_z-\underline{U}_t\\
&=\sigma e^{-\mu_1 z}\Big\{\Theta(t)\varphi_1(t)(1-n_1e^{-\epsilon z})-b_{11}(t)p(t)e^{-\mu_1 z}[\varphi_1(t)(1-n_1e^{-\epsilon z})]^2-\varphi_1'(t)(1-n_1e^{-\epsilon z})\\
&\quad+(d_1(t)\mu_1^2-c\mu_1)\varphi_1(t)-n_1\varphi_1(t)e^{-\epsilon z}[(\mu_1+\epsilon)^2-c(\mu_1+\epsilon)]\Big\}+b_{12}(t)q(t)\underline{U}\cdot\underline{V}+b_{13}(t)s(t)\underline{U}\cdot\underline{W}\\
&\geq\sigma e^{-\mu_1 z}\Big\{[\Theta(t)-\overline{\Theta(t)}]\varphi_1(t)-\varphi_1'(t)-n_1e^{-\epsilon z}[\Theta(t)\varphi_1(t)+d_1(t)(\mu_1+\epsilon)^2\varphi_1(t)-c(\mu_1+\epsilon)\varphi_1(t)\\
&\quad-\varphi_1'(t)]-\sigma\varphi_1(t)e^{-\mu_1 z}(1-n_1e^{-\epsilon z})\Big[b_{11}(t)p(t)\varphi_1(t)(1-n_1e^{-\epsilon z})\\
&\quad+b_{12}(t)q(t)\psi_{d_1}(t)\left(1-n_2\frac{\psi_1(t)}{\psi_{d_1}(t)}e^{-\epsilon z}\right)+b_{13}(t)s(t)\psi_{d_2}(t)\left(1-n_3\frac{\psi_2(t)}{\psi_{d_2}(t)}e^{-\epsilon z}\right)\Big]\Big\}\\
&=\sigma\varphi_1(t)e^{-\mu_1 z}\Big\{n_1\vartheta e^{-\epsilon z}-\sigma e^{-\mu_1 z}(1-n_1e^{-\epsilon z})\Big[b_{11}(t)p(t)\varphi_1(t)(1-n_1e^{-\epsilon z})\\
&\quad+b_{12}(t)q(t)\psi_{d_1}(t)\left(1-n_2\frac{\psi_1(t)}{\psi_{d_1}(t)}e^{-\epsilon z}\right)+b_{13}(t)s(t)\psi_{d_2}(t)\left(1-n_3\frac{\psi_2(t)}{\psi_{d_2}(t)}e^{-\epsilon z}\right)\Big]\Big\}\geq 0,
\end{aligned}
$$

for the U-equation. For the V-equation, we have

$$
\begin{aligned}
&(1-\underline{V})[b_{21}(t)p(t)U-b_{22}(t)q(t)\underline{V}]+d_1(t)\underline{V}_{zz}+c\underline{V}_z-\underline{V}_t\\
&=\sigma e^{-\mu_1 z}\Big\{b_{21}(t)p(t)\varphi_1(t)(1-n_1e^{-\epsilon z})-b_{22}(t)q(t)(\psi_{d_1}(t)-n_2\psi_1(t)e^{-\epsilon z})\\
&\quad-\sigma b_{21}(t)p(t)e^{-\mu_1 z}\left[\varphi_1(t)(1-n_1e^{-\epsilon z})\psi_{d_1}(t)\left(1-n_2\frac{\psi_1(t)}{\psi_{d_1}(t)}e^{-\epsilon z}\right)\right]+(d_1(t)\mu_1^2-c\mu_1)\psi_{d_1}(t)\\
&\qquad-n_2\psi_1(t)e^{-\epsilon z}[d_1(t)(\mu_1+\epsilon)^2-c(\mu_1+\epsilon)]-(\psi_{d_1}'(t)-n_2\psi_1'(t)e^{-\epsilon z})\Big\}+b_{22}(t)q(t)(\underline{V})^2\\
&\geq\sigma e^{-\mu_1 z}\Big\{b_{21}(t)p(t)\varphi_1(t)-(b_{22}(t)q(t)+\overline{\Theta(t)}+(1-d_1(t))\mu_1^2)\psi_{d_1}(t)-\psi_{d_1}'(t)\\
&\qquad\qquad-\sigma b_{21}(t)p(t)e^{-\mu_1 z}\left[\varphi_1(t)(1-n_1e^{-\epsilon z})\psi_{d_1}(t)\left(1-n_2\frac{\psi_1(t)}{\psi_{d_1}(t)}e^{-\epsilon z}\right)\right]\Big\}\\
&\quad-n_2\sigma e^{-(\mu_1+\epsilon)z}\Big\{b_{21}(t)p(t)\varphi_1(t)-[b_{22}(t)q(t)+(\mu_1+\epsilon)^2-c(\mu_1+\epsilon)]\psi_1(t)\\
&\qquad\qquad-\psi_1'(t)+(d_1(t)-1)(\mu_1+\epsilon)^2\psi_1(t)\Big\}\\
&\geq\sigma e^{-\mu_1 z}\Big\{n_2\vartheta\psi_1(t)e^{-\epsilon z}-\sigma b_{21}(t)p(t)e^{-\mu_1 z}\left[\varphi_1(t)(1-n_1e^{-\epsilon z})\psi_{d_1}(t)\left(1-n_2\frac{\psi_1(t)}{\psi_{d_1}(t)}e^{-\epsilon z}\right)\right]\Big\}\\
&\geq 0.
\end{aligned}
$$

The calculation of lower solution on the W-equation is similar to the V-equation, we omit it for convenience. Therefore, we get

$$
\begin{aligned}
&(1-\underline{W})[b_{31}(t)p(t)\underline{U}-b_{33}(t)s(t)\underline{W}]+d_2(t)\underline{W}_{zz}+c\underline{W}_z-\underline{W}_t\\
&\geq\sigma e^{-\mu_1 z}\Big\{n_3\vartheta\psi_2(t)e^{-\epsilon z}-\sigma b_{31}(t)p(t)e^{-\mu_1 z}\left[\varphi_1(t)(1-n_1e^{-\epsilon z})\psi_{d_2}(t)\left(1-n_3\frac{\psi_2(t)}{\psi_{d_2}(t)}e^{-\epsilon z}\right)\right]\Big\}\geq 0.
\end{aligned}
$$

Thus, $(\underline{U},\underline{V},\underline{W})$ satisfies (3.3).

Noticing that $(\underline{U},\underline{V},\underline{W})$ and $(\overline{U},\overline{V},\overline{W})$ are periodic in t, one can increase m such that $(\overline{U},\overline{V},\overline{W}) \geq (\underline{U},\underline{V},\underline{W})$ for all $(t,z) \in \mathbb{R}^+ \times \mathbb{R}$. By [30, Lemma 2.4], there exists a time periodic solution (U^c, V^c, W^c) for (1.7) for each $c > c_0$. For the verification that the derived solution (U^c, V^c, W^c) satisfies the boundary conditions involved in (1.7), we refer the reader to [30, Theorems 2.5]. Moreover, for the existence of the solution of (1.7) as $c = c_0$, we can follow the same lines presented in [30, Theorems 2.6]. Since there is no essential difference, we omit the details. The proof is thus complete. □

By taking a look at the proof of Theorem 3.1, we find that the pair of functions $(\underline{U},\underline{V},\underline{W})$ to be a lower solution needs the requirements $d_i(t) \in (0,1], i = 1,2$ for all $t \in [0,T]$. Hence, for the sake of simplicity, we always assume (1.3)-(1.4) and (H) hold as well as $d_i(t) \in (0,1], i = 1,2$ for all $t \in [0,T]$.

Theorem 3.2 *Assume that*

$$\frac{b_{21}(t)p(t)}{Q_1(t)} < M_1 < \frac{\overline{\Theta(t)}}{b(t)}, \quad \frac{b_{31}(t)p(t)}{Q_2(t)} < M_2 < \frac{\overline{\Theta(t)}}{b(t)}, \tag{3.7}$$

where

$$b(t) = \max\{b_{12}(t)q(t), b_{13}(t)s(t)\},$$
$$Q_1(t) = (1 - d_1(t))\overline{\Theta(t)} + \Theta(t) + b_{22}(t)q(t),$$
$$Q_2(t) = (1 - d_2(t))\overline{\Theta(t)} + \Theta(t) + b_{33}(t)s(t).$$

The functions $\Theta(t)$ and $\overline{\Theta(t)}$ are the ones defined in (3.1). Then the minimal wave speed of system (1.7) is linearly selected.

Proof. Define

$$\overline{U}(t,z) = \frac{1}{1 + \frac{e^{\mu_1 z}}{\varphi_1(t)}}, \tag{3.8}$$

where $z = x - c_0 t$, $\mu_1 := \mu_1(c_0)$ with $c_0 = 2\sqrt{\overline{\Theta(t)}}$ (see (4.4)) and $\varphi_1(t) = \varphi_{1,\mu_1}(t)$ is a positive characteristic function defined in (4.3). Let

$$\overline{V}(t,z) = \min\{1, M_1\overline{U}(t,z)\} = \begin{cases} 1, & z \leq z_2(t), \\ M_1\overline{U}(t,z), & z > z_2(t), \end{cases}$$

$$\overline{W}(t,z) = \min\{1, M_2\overline{U}(t,z)\} = \begin{cases} 1, & z \leq z_3(t), \\ M_2\overline{U}(t,z), & z > z_3(t). \end{cases}$$

Without loss of generality, we suppose that $z_3(t) > z_2(t)$ which indicates that $\overline{U}(t,z_2) > \overline{U}(t,z_3)$ and $M_2 > M_1 > 1$. While $z_3(t) \leq z_2(t)$, we can obtain the same results and so we omit it for conciseness. We shall prove that $(\overline{U},\overline{V},\overline{W})$ is a generalized upper solution for three cases.

(i). When $z \leq z_2(t)$, we have $\overline{V}(t,z) = \overline{W}(t,z) = 1$. It is easy to check that the V-equation and W-equation hold true.

(ii). When $z_2(t) < z < z_3(t)$, it follows that $\overline{V}(t,z) = M_1\overline{U}(t,z)$, where $\overline{U} \in [0, \frac{1}{M_1}]$. Substituting it into V-equation, we have

$$d_1(t)\overline{V}_{zz} + c_0\overline{V}_z + (1-\overline{V})[b_{21}(t)p(t)\overline{U} - b_{22}(t)q(t)\overline{V}] - \overline{V}_t =: M_1\overline{U} \cdot D_1(\overline{U}), \tag{3.9}$$

where

$$D_1(\overline{U}) = (1-\overline{U})\left[d_1(t)\mu_1^2 - c_0\mu_1 - \frac{\varphi_1'(t)}{\varphi_1(t)} - 2d_1(t)\mu_1^2\overline{U}\right] + (1-M_1\overline{U})\left(\frac{b_{21}(t)p(t)}{M_1} - b_{22}(t)q(t)\right).$$

It is direct to calculate that $D_1''(\overline{U}) = 4d_1(t)\mu_1^2 > 0$. Therefore, for each $t \in \mathbb{R}^+$, $D_1(\overline{U})$ is a concave function in $\overline{U} \in [0, \frac{1}{M_1}]$ and $D_1(\overline{U})$ exists a minimum. Thus we only need to verify that the values of $\overline{U}$ at the boundary are negative. Upon substitution of $\overline{U} = 0$ and $\overline{U} = \frac{1}{M_1}$ in $D_1(\overline{U})$, we have

$$\begin{aligned} D_1(0) &= d_1(t)\mu_1^2 - c_0\mu_1 - \frac{\varphi_1'(t)}{\varphi_1(t)} + \frac{b_{21}(t)p(t)}{M_1} - b_{22}(t)q(t), \\ D_1\left(\frac{1}{M_1}\right) &= \left(1-\frac{1}{M_1}\right)\left[d_1(t)\mu_1^2 - c_0\mu_1 - \frac{\varphi_1'(t)}{\varphi_1(t)} - 2d_1(t)\mu_1^2\frac{1}{M_1}\right]. \end{aligned} \tag{3.10}$$

Keeping $\mu_1 = \sqrt{\overline{\Theta(t)}}$ and $c_0 = 2\sqrt{\overline{\Theta(t)}}$ as well as $\frac{\varphi_1'(t)}{\varphi_1(t)} = \mu_1^2 - c_0\mu_1 + \Theta(t)$ (see (4.2)) in mind, the wanted inequality $D_1(0) < 0$ results in the left side of the first condition of (3.7). So the left of (3.9) is less than zero. While $D_1(\frac{1}{M_1}) < 0$ is always valid since $d_1(t) < 1$. Thus, $D_1(\overline{U}) < 0$, that is

$$d_1(t)\overline{V}_{zz} + c_0\overline{V}_z + (1-\overline{V})[b_{21}(t)p(t)\overline{U} - b_{22}(t)q(t)\overline{V}] - \overline{V}_t < 0. \tag{3.11}$$

On the other side, thanks to $\overline{W}(t,z) = 1$ in this case, the W-equation holds true.

(iii). When $z \geq z_3(t)$, we have $\overline{V}(t,z) = M_1\overline{U}(t,z)$, $\overline{W}(t,z) = M_2\overline{U}(t,z)$. Due to $\overline{U} \in [0, \frac{1}{M_2}] \subset [0, \frac{1}{M_1}]$, we know that (3.11) is still true. By performing a similar discussion, it follows from the second condition of (3.7) that

$$d_2(t)\overline{W}_{zz} + c_0\overline{W}_z + (1-\overline{W})[b_{31}(t)p(t)\overline{U} - b_{33}(t)s(t)\overline{W}] - \overline{W}_t < 0.$$

Finally, for the U-equation, we can easily verify that

$$\begin{aligned} &\overline{U}_{zz} + c_0\overline{U}_z + \overline{U}[b_{11}(t)p(t)(1-\overline{U}) - b_{12}(t)q(t)(1-\overline{V}) - b_{13}(t)s(t)(1-\overline{W})] - (\overline{U})_t \\ &= \overline{U}^2(1-\overline{U})\left[-2\mu_1^2 + \frac{b_{12}(t)q(t)(\overline{V}-\overline{U}) + b_{13}(t)s(t)(\overline{W}-\overline{U})}{\overline{U}(1-\overline{U})}\right] \\ &< \overline{U}^2(1-\overline{U})[-2\overline{\Theta(t)} + b(t)G(t,z)], \end{aligned}$$

where

$$b(t) = \max\{b_{12}(t)q(t), b_{13}(t)s(t)\},\ G(t,z) = \frac{\overline{V}-\overline{U}+\overline{W}-\overline{U}}{\overline{U}(1-\overline{U})}.$$

$G(t,z)$ can be estimated as

$$G(t,z) = \begin{cases} \dfrac{2}{\overline{U}} \leq 2M_1, & z \leq z_2(t), \\ \dfrac{M_1-1}{1-\overline{U}} + \dfrac{1}{\overline{U}} \leq 2M_2, & z_2(t) < z < z_3(t), \\ \dfrac{M_1+M_2-2}{1-\overline{U}} \leq 2M_2, & z \geq z_3(t). \end{cases}$$

As a result, we can get

$$\overline{U}_{zz} + c_0\overline{U}_z + \overline{U}[b_{11}(t)p(t)(1-\overline{U}) - b_{12}(t)q(t)(1-\overline{V}) - b_{13}(t)s(t)(1-\overline{W})] - \overline{U}_t < 0.$$

By a standard argument of the upper-lower solution (see for example [2]), one can see that the time periodic traveling wave exists under the assumptions in this theorem when $c = c_0$. This means the minimal wave speed is linearly selected. $\square$

Theorem 3.3 *If*

$$b_{11}(t)p(t) \leq \frac{3}{2}\overline{\Theta(t)}, \tag{3.12}$$

and

$$\begin{cases} \left(\dfrac{d_1(t)}{4} - \dfrac{1}{2}\right)\overline{\Theta(t)} - b_{22}(t)q(t) - \dfrac{1}{2}\Theta(t) < 0, \\ \left(\dfrac{d_2(t)}{4} - \dfrac{1}{2}\right)\overline{\Theta(t)} - b_{33}(t)s(t) - \dfrac{1}{2}\Theta(t) < 0, \\ b_{21}(t)p(t) - \dfrac{d_1(t)}{2}\overline{\Theta(t)} < 0, \\ b_{31}(t)p(t) - \dfrac{d_2(t)}{2}\overline{\Theta(t)} < 0, \end{cases} \tag{3.13}$$

or

$$\begin{cases} b_{21}(t)p(t) - \dfrac{d_1(t)}{2}\overline{\Theta(t)} > 0, \\ b_{31}(t)p(t) - \dfrac{d_2(t)}{2}\overline{\Theta(t)} > 0, \\ \left(-\dfrac{d_1(t)}{4} - \dfrac{1}{2}\right)\overline{\Theta(t)} - b_{22}(t)q(t) - \dfrac{1}{2}\Theta(t) + b_{21}(t)p(t) \leq 0, \\ \left(-\dfrac{d_2(t)}{4} - \dfrac{1}{2}\right)\overline{\Theta(t)} - b_{33}(t)s(t) - \dfrac{1}{2}\Theta(t) + b_{31}(t)p(t) \leq 0, \end{cases} \tag{3.14}$$

then the minimal wave speed of system (1.7) is linearly selected, where $\Theta(t)$ and $\overline{\Theta(t)}$ are defined in (3.1).

Proof. Let $\overline{V}(t,z) = \overline{W}(t,z) = \overline{U}^{\frac{1}{2}}(t,z)$, where $\overline{U}(t,z)$ is defined by (3.8). By a simple calculation, we have

$$\begin{aligned} &\overline{U}_{zz} + c_0\overline{U}_z + \overline{U}[b_{11}(t)p(t)(1-\overline{U}) - b_{12}(t)q(t)(1-\overline{V}) - b_{13}(t)s(t)(1-\overline{W})] - \overline{U}_t \\ &\quad = \overline{U}^{\frac{3}{2}}(1-\overline{U}^{\frac{1}{2}})\left[-\frac{3}{2}\overline{\Theta(t)} + b_{11}(t)p(t)\right], \\ &d_1(t)\overline{V}_{zz} + c_0\overline{V}_z + (1-\overline{V})[b_{21}(t)p(t)\overline{U} - b_{22}(t)q(t)\overline{V}] - \overline{V}_t \\ &\quad = \overline{V}(1-\overline{V})\left\{\left[\left(\frac{d_1(t)}{4} - \frac{1}{2}\right)\overline{\Theta(t)} - b_{22}(t)q(t) - \frac{1}{2}\Theta(t)\right] + \left(b_{21}(t)p(t) - \frac{d_1(t)}{2}\overline{\Theta(t)}\right)\overline{V}\right\}, \\ &d_2(t)\overline{W}_{zz} + c_0\overline{W}_z + (1-\overline{W})[b_{31}(t)p(t)\overline{U} - b_{33}(t)s(t)\overline{W}] - \overline{W}_t \\ &\quad = \overline{W}(1-\overline{W})\left\{\left[\left(\frac{d_2(t)}{4} - \frac{1}{2}\right)\overline{\Theta(t)} - b_{33}(t)s(t) - \frac{1}{2}\Theta(t)\right] + \left(b_{31}(t)p(t) - \frac{d_2(t)}{2}\overline{\Theta(t)}\right)\overline{W}\right\}. \end{aligned} \tag{3.15}$$

If the conditions (3.12)-(3.14) hold, the right sides of three equations of (3.15) are negative. Therefore, $(\overline{U}(t,z), \overline{V}(t,z), \overline{W}(t,z))$ is an upper solution which indicates that the minimal wave speed is linear selected. The proof is complete. $\square$

From Theorems 3.2 and 3.3, one can see that once the lower solution is found, then the linear selection is totally dependent on the expression of upper solution. Roughly speaking, an upper solution leads to a sufficient condition such that the minimal wave speed is linearly selected. We will end this section by giving another upper solution.

Theorem 3.4 *If there is $m = \frac{1}{n}(n \in Z, n \geq 2)$ such that*

$$\left(1-\frac{1}{m}\right)\Theta(t) - 2\left(1-\frac{1}{m}\right)\Theta(t) - \frac{\Theta(t)}{m} + \frac{b_{11}(t)p(t)}{m} < 0, \tag{3.16}$$

and

$$\begin{cases} \left(2d_1(t)m^2-2\right)\overline{\Theta(t)} - 4(m-1)\overline{\Theta(t)}^{\frac{3}{2}} - 2\Theta(t) - \Theta_1(t) < 0, \\ \left(2d_2(t)m^2-2\right)\overline{\Theta(t)} - 4(m-1)\overline{\Theta(t)}^{\frac{3}{2}} - 2\Theta(t) - \Theta_2(t) < 0, \end{cases} \tag{3.17}$$

then the minimal wave speed of system (1.7) is linearly selected, where $\Theta(t)$ and $\overline{\Theta(t)}$ are defined in (3.1).

Proof. We first define the functions

$$\overline{U} = \left(1 + \frac{e^{m\mu_1 z}}{\varphi_1(t)}\right)^{-\frac{1}{m}}, \overline{V} = \overline{W} = 1 - \left(1 - \overline{U}^m\right)^2, \tag{3.18}$$

where $0 < m \leq \frac{1}{2}$. Next, we shall verify that $(\overline{U}, \overline{V}, \overline{W})$ is an upper solution of (1.7). Substituting (3.18) into the first equation of (1.7), we have

$$\begin{aligned} &\overline{U}_{zz} + c_0\overline{U}_z + \overline{U}[b_{11}(t)p(t)(1-\overline{U}) - b_{12}(t)q(t)(1-\overline{V}) - b_{13}(t)s(t)(1-\overline{W})] - \overline{U}_t \\ &= \overline{U}(1-\overline{U}^m)\left[\mu_1^2 - c_0\mu_1 + \Theta(t) - \frac{\varphi_1'(t)}{m\varphi_1(t)} - (1+m)\mu_1^2\overline{U}^m \right. \\ &\qquad \left. + (b_{12}(t)q(t) + b_{13}s(t))\overline{U}^m + b_{11}(t)p(t)\frac{\overline{U}^m - \overline{U}}{1-\overline{U}^m}\right] \\ &\leq \overline{U}(1-\overline{U}^m)\left[\left(1-\frac{1}{m}\right)\mu_1^2 - \left(1-\frac{1}{m}\right)\mu_1 - \frac{\Theta(t)}{m} + \frac{b_{11}(t)p(t)}{m}\right]. \end{aligned} \tag{3.19}$$

Thanks to $\mu_1 = \sqrt{\Theta(t)}$, we get the condition (3.16) to make the last term in (3.19) is non-positive.

Substituting (3.18) into V-equation and W-equation and combining with $\overline{U}^{1-m} \leq \overline{U}^m$, we have

$$\begin{aligned} &d_1(t)\overline{V}_{zz} + c_0\overline{V}_z + (1-\overline{V})[b_{21}(t)p(t)\overline{U} - b_{22}(t)q(t)\overline{V}] - \overline{V}_t \\ &= \overline{U}^m(1-\overline{U}^m)^2\left[2d_1(t)m^2\mu_1^2 - 6d_1(t)m^2\mu_1^2\overline{U}^m - 2mc_0\mu_1 - \frac{2\varphi_1'(t)}{\varphi_1(t)} \right. \\ &\qquad \left. + b_{21}(t)p(t)\overline{U}^{1-m} - b_{22}(t)q(t)(2-\overline{U}^m)\right] \\ &\leq \overline{U}^m(1-\overline{U}^m)^2\left[2m^2d_1(t)\overline{\Theta(t)} - 4m\overline{\Theta(t)}^{\frac{3}{2}} - \frac{2\varphi_1'(t)}{\varphi_1(t)} - b_{22}(t)q(t) + b_{21}(t)p(t)\right], \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &d_2(t)\overline{W}_{zz} + c_0\overline{W}_z + (1-\overline{W})[b_{31}(t)p(t)\overline{U} - b_{33}(t)s(t)\overline{W}] - \overline{W}_t \\ &= \overline{U}^m(1-\overline{U}^m)^2\left[2m^2d_2(t)\mu_1^2 - 6m^2d_2(t)\mu_1^2\overline{U}^m - 2c_0m\mu_1 - \frac{2\varphi_1'(t)}{\varphi_1(t)} \right. \\ &\qquad \left. + b_{31}(t)p(t)\overline{U}^{1-m} - b_{33}(t)s(t)(2-\overline{U}^m)\right] \\ &\leq \overline{U}^m(1-\overline{U}^m)^2\left[2m^2d_2(t)\overline{\Theta(t)} - 4m\overline{\Theta(t)}^{\frac{3}{2}} - \frac{2\varphi_1'(t)}{\varphi_1(t)} - b_{33}(t)s(t) + b_{31}(t)p(t)\right]. \end{aligned} \tag{3.21}$$

With the help of the conditions (3.17), and we can verify that

$$\begin{cases} d_1(t)\overline{V}_{zz} + c_0\overline{V}_z + (1-\overline{V})[b_{21}(t)p(t)\overline{U} - b_{22}(t)q(t)\overline{V}] - \overline{V}_t < 0, \\ d_2(t)\overline{W}_{zz} + c_0\overline{W}_z + (1-\overline{W})[b_{31}(t)p(t)\overline{U} - b_{33}(t)s(t)\overline{W}] - \overline{W}_t < 0. \end{cases}$$

The above discussions show that $(\overline{U}, \overline{V}, \overline{W})$ is an upper solution of (1.7). Therefore, the proof is completed. □

4 Appendix: the local behavior of the traveling wavefront around e_1

The minimal wave speed selection relies heavily on the asymptotic behaviors of the traveling wave solutions near the unstable equilibrium $e_1 = (0, 0, 0)$. In fact, the corresponding linearized system is represented by

$$\begin{cases} U_{zz} + cU_z + \Theta(t)U - U_t = 0, \\ d_1(t)V_{zz} + cV_z + [b_{21}(t)p(t)U - b_{22}(t)q(t)V] - V_t = 0, \\ d_2(t)W_{zz} + cW_z + [b_{31}(t)p(t)U - b_{33}(t)s(t)W] - W_t = 0. \end{cases} \tag{4.1}$$

Making an ansatz $U(t, z) = \varphi(t)e^{-\mu z}$, where $\varphi(t)$ is a T-periodic function. We insert it into the first equation of (4.1) yield

$$[\mu^2 - c\mu + \Theta(t)]\varphi(t) - \varphi'(t) = 0. \tag{4.2}$$

Dividing by $\varphi(t)$ on both sides and integrating the result from 0 to T, we have

$$\mu^2 - c\mu + \overline{\Theta(t)} = 0.$$

Hence, the characteristic roots are

$$\mu_1 := \mu_1(c) = \frac{c - \sqrt{c^2 - 4\overline{\Theta(t)}}}{2}, \; \mu_2 := \mu_2(c) = \frac{c + \sqrt{c^2 - 4\overline{\Theta(t)}}}{2}. \tag{4.3}$$

Therefore, one can obtain the linear speed as below

$$c_0 = 2\sqrt{\overline{\Theta(t)}}. \tag{4.4}$$

When $c = c_0$ and $0 < \mu_1(c) < \mu_2(c)$, it is clear that $\mu_1 = \mu_2 = \sqrt{\overline{\Theta(t)}}$. When $c > c_0$, for the constructions of lower/upper solutions, we need to solve the eigenfunctions $\varphi_{1,\mu_i}(t)$ relating to the eigenvalues μ_i, $i = 1, 2$. Indeed, it can be derived directly from (4.2) that

$$\varphi_{1,\mu_i}(t) = \varphi_{1,\mu_i}(0) \exp\left\{ \int_0^t (\Theta(s) - \overline{\Theta(t)})ds \right\}, \; i = 1, 2. \tag{4.5}$$

By the way, the asymptotic behaviors of $U(t, z)$ can be described as

$$U(t, z) \sim C_1\varphi_{1,\mu_1}(t)e^{-\mu_1 z} + C_2\varphi_{1,\mu_2}(t)e^{-\mu_2 z}, \; z \to \infty,$$

where $C_1 > 0$ or $C_1 = 0, C_2 > 0$.

5 Discussion

In this paper, by using the upper and lower solution method, the linear and nonlinear selections of the minimal wave speed of three-species time-periodic Lotka-Volterra competition system are studied. Within the context of three-species constant-coefficient Lotka-Volterra competition system, the issue about the linear selection has been studied and some sufficient conditions were established in [10] and [37]. Here, we only want to compare our results with the ones obtained in [37], since the works of [37] extend the previous ones of [10].

The results obtained in [37] can be summarized as follows:

For the system

$$\begin{cases} u_t = u_{xx} + u(1 - u - b_{12} + b_{12}v - b_{13} + b_{13}w), \\ v_t = d_1 v_{xx} + \alpha(1 - v)(b_{21}u - v), \\ w_t = d_2 w_{xx} + \beta(1 - w)(b_{31}u - w), \end{cases} \tag{5.1}$$

linear selection of the minimal wave speed of (5.1) is realized provided that

$$0 \le d_1, d_2 \le 2,\ 1 - b_{12} - b_{13} > \frac{b_{12}b_{21} + b_{13}b_{31}}{2}, \tag{5.2}$$

or

$$\begin{aligned} &0 \le d_1 < 2, b_{12} < \frac{1 - b_{13}}{2}, \alpha < \frac{(2 - d_1)(1 - b_{12} - b_{13})^2}{b_{12}b_{21} - (1 - b_{12} - b_{13})}, \\ &0 \le d_2 < 2, b_{13} < \frac{1 - b_{12}}{2}, \beta < \frac{(2 - d_2)(1 - b_{12} - b_{13})^2}{b_{13}b_{31} - (1 - b_{12} - b_{13})} \end{aligned} \tag{5.3}$$

is true. While nonlinear selection of the minimal wave speed of (5.1) is realized if

$$\max\left\{\frac{(d_1 + 2)(1 - b_{12} - b_{13}) + \alpha}{\alpha b_{21}}, \frac{(d_2 + 2)(1 - b_{12} - b_{13}) + \beta}{\beta b_{31}}\right\} < 1 - 2(1 - b_{12} - b_{13}) \tag{5.4}$$

is true.

To proceed, we transform the periodic parameters of system (1.1) into the corresponding constants of system (5.1) as follows

$$\begin{aligned} &d_1(t) = d_1,\ d_2(t) = d_2,\ b_{11}(t) = 1,\ b_{12}(t) = b_{12},\ b_{13}(t) = b_{13}, r_1(t) = 1, \\ &r_2(t) = \alpha,\ r_3(t) = \beta,\ b_{21}(t) = b_{21}\alpha,\ b_{31}(t) = b_{31}\beta,\ b_{22}(t) = \alpha,\ b_{33}(t) = \beta. \end{aligned} \tag{5.5}$$

We perform the comparison of the linear selection by three cases.

Case 1. In Theorem 3.2, by taking $M_1 = b_{21}$, $M_2 = b_{31}$, we have

$$0 < d_1, d_2 < 1, 1 - b_{12} - b_{13} > \max\left\{b_{21}b, b_{31}b, \frac{1 - \alpha}{2 - d_1}, \frac{1 - \beta}{2 - d_2}\right\}, \tag{5.6}$$

with $b = \max\{b_{12}, b_{13}\}$. As observed from (5.3) and (5.6), we only need to compare the values of $\frac{b_{12}b_{21}+b_{13}b_{31}}{2}$ and $\max\left\{b_{21}b, b_{31}b, \frac{1-\alpha}{2-d_1}, \frac{1-\beta}{2-d_2}\right\}$. It is easy to calculate that $\max\left\{b_{21}b, b_{31}b, \frac{1-\alpha}{2-d_1}, \frac{1-\beta}{2-d_2}\right\} \ge \frac{b_{12}b_{21}+b_{13}b_{31}}{2}$, so the conditions (5.6) is contained in (5.3). Particularly, when $\max\left\{b_{21}b, b_{31}b, \frac{1-\alpha}{2-d_1}, \frac{1-\beta}{2-d_2}\right\} = \frac{b_{12}b_{21}+b_{13}b_{31}}{2}$, (5.6) is the same as (5.3) if $d_i, i = 1, 2$ are restricted to $0 < d_i < 1, i = 1, 2$, which gives a justification of Theorem 3.2.

Case 2. In Theorem 3.2, by choosing $M_1 = \frac{1-b_{12}-b_{13}}{b_{12}}$, $M_2 = \frac{1-b_{12}-b_{13}}{b_{13}}$, from the left parts of two inequalities of (3.7), we can obtain

$$\alpha < \frac{(2 - d_1)(1 - b_{12} - b_{13})^2}{b_{12}b_{21} - (1 - b_{12} - b_{13})},\ \beta < \frac{(2 - d_2)(1 - b_{12} - b_{13})^2}{b_{13}b_{31} - (1 - b_{12} - b_{13})}, \tag{5.7}$$

where $0 < d_i < 1, i = 1, 2$. It is not hard to see that the upper bounds of α and β in our results possess the same expressions as the ones in (5.3), which also verifies the correctness of Theorem 3.2.

Case 3. According to Theorem 3.4, it follows that

$$0 < d_1, d_2 < 1, 1 - b_{12} - b_{13} \geq \frac{1}{m}, \alpha > \frac{(1 - b_{12} - b_{13})(4m - 2d_1 m^2)}{1 - b_{21}}, \beta > \frac{(1 - b_{12} - b_{13})(4m - 2d_2 m^2)}{1 - b_{31}}. \tag{5.8}$$

Next, we shall compare the ranges of α and β in (5.8) with the ones in (5.3). In fact, it is obvious that $\alpha > \frac{4-2d_1 m}{1-b_{21}}$, $\beta > \frac{4-2d_2 m}{1-b_{31}}$ in (5.8). If we take $d_1 = 0.25$, $d_2 = 0.3$, $b_{12} = b_{13} = \frac{5}{12}$, $b_{21} = b_{31} = \frac{1}{2}$, $m = 6$, then we get $\alpha < \frac{7}{6}$, $\beta < \frac{17}{15}$ from (5.3), and $\alpha > 2$, $\beta > 0.8$ from (5.8). Therefore, we are able to conclude that our results are novel.

For the nonlinear selection, through Theorem 2.4, we get the same conclusion as (5.4), which confirms the correctness of Theorem 2.4.

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