Dynamical analysis and numerical simulation of a new fractional-order two-stage species model with recruitment

Manh Tuan Hoang[a,*]

[a]*Department of Mathematics, FPT University, Hoa Lac Hi-Tech Park, Km29 Thang Long Blvd, Hanoi, Viet Nam*

Abstract

The purpose of this work is to propose and analyze a new fractional-order two-stage species model with recruitment to discover memory effects on population dynamics. This fractional-order model is constructed by incorporating a well-known integer-order two-stage species model with the Caputo fractional derivative. Firstly, the positivity and boundedness of solutions of the proposed model are investigated by using some standard comparison results for fractional differential equations. Next, a simple approach is utilized to study stability properties of the fractional-order model. This approach is based on the Lyapunov stability theory and Barbalat's lemma in combination with some nonstandard techniques for fractional dynamical systems. More clearly, we use general quadratic Lyapunov candidate functions and combine them with characteristics of quadratic forms associated with real matrices to establish the stability properties. As an important consequence, global asymptotic stability, uniform and Mittag-Leffler stability and therefore, population dynamics of the proposed fractional-order model are analyzed rigorously. Finally, we extend the Mickens' methodology to construct a dynamically consistent nonstandard finite difference (NSFD) scheme for the purpose of numerical simulation of the fractional-order model. It is proved that the NSFD scheme preserves the positivity and boundedness of the fractional-order model regardless of the values of the step size; moreover, it is also simple and efficient. The theoretical results and advantages of the NSFD scheme are supported by illustrative numerical experiments. The experiments provide strong evidence, which shows that the numerical results are consistent with the theoretical ones.

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*Corresponding author

Email address: `tuanhm14@fe.edu.vn; hmtuan01121990@gmail.com` (Manh Tuan Hoang)

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1. Introduction

It is well-known that the fractional calculus (FC), which generalizes integrals and derivatives of integer-order, has a long history. The FC has many useful applications in not only theory but also practice. In particular, fractional differential equations (FDEs) have been strongly developed in recent decades for the purpose of mathematical modeling and analysis of real-life problems [12, 13, 18, 27, 52, 70]. It was proved in many recognized works that fractional-order models can describe phenomena and processes arising in real-world applications more accurately than integer-order models due to the effective memory function of fractional derivatives [4, 5, 6, 7, 37, 39, 49, 51, 72, 73, 74, 79, 80]. Recently, we have considered some fractional-order systems that mathematically model phenomena and processes arising in biology and ecology, in which the stability problem was mainly focused [41, 42].

Population dynamics of the harvesting and fisheries has an important role in ecology and environment and this topic has been widely studied by many mathematicians as well as biologists and ecologists (see, for example, [14, 16, 17, 67] and references therein). In a previous work [53], Ladino and Valverde applied basic ideas and technical hypotheses, which are motivated by biological and ecological reasons, to propose a mathematical model for examining population dynamics of a two-stage (migratory) fish population. This model is represented by a pair of nonlinear ordinary differential equations of the form:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \widetilde{\delta}y(t) - \frac{\widetilde{\alpha}x(t)}{\widetilde{\beta} + x(t)} - \widetilde{\mu}x(t),\\
\frac{dy(t)}{dt} &= \frac{\widetilde{\alpha}x(t)}{\widetilde{\beta} + x(t)} - (\widetilde{\mu} + \widetilde{F})y(t),
\end{aligned}
\tag{1}
$$

where

- $x(t)$ is the pre-recruit population (eggs, larvae, and juvenile one) at the time t;
- $y(t)$ is the exploitable population (adult fishes) at the time t;

- du/dt denotes the first derivative with respect to the time variable t of a given function $u(t)$;

- all the parameters are positive because of biological reasons.

We refer the readers to the benchmark work [53] for details of the derivation and qualitative study of the model (1). To the best of our knowledge, the system (1) was investigated and developed at some levels (see, for instance, [54]) but their extended versions in the context of fractional derivatives have not been considered.

Motivated and inspired by the important applications of population dynamics of the harvesting and fisheries and advantages of fractional-order derivatives over integer-order ones, we will study the integer-order system (1) in the context of the Caputo fractional-order derivative [18]. More precisely, we will consider the following system of fractional differential equations

$$\begin{aligned} {}_0^C D_t^q x(t) &= \delta y(t) - \frac{\alpha x(t)}{\beta + x(t)} - \mu x(t), \\ {}_0^C D_t^q y(t) &= \frac{\alpha x(t)}{\beta + x(t)} - (\mu + F) y(t), \end{aligned} \tag{2}$$

where ${}_0^C D_t^q z(t)$ with $t > 0$ and $q \in (0, 1)$ is the right-sided Caputo fractional derivative of order q of the function $z(t)$ (see, [18, 27, 52, 70]). Here, the dimension of the parameters in the fractional-order model (2) has been adjusted to ensure that both sides have the same dimension (see [19, 29, 31, 40]). More clearly,

$$\delta = \widetilde{\delta}\tau^{1-q}, \quad \alpha = \widetilde{\alpha}\tau^{1-q}, \quad \widetilde{\beta}\tau^{1-q}, \quad \mu = \widetilde{\mu}\tau^{1-q}, \qquad F = \widetilde{F}\tau^{1-q}, \quad \beta = \widetilde{\beta},$$

where τ is a parameter that has the dimension of time (see [19, 31, 40]).

Formally, the fractional-order system (2) is obtained from the integer-order one (1) by replacing the first classical derivative in (1) by the Caputo fractional one. However, the derivation of the proposed fractional-order model will be explained in terms of memory effects on population dynamics in Section 3; moreover, effects of order q on population dynamics of the model (2) will be examined in numerical examples conducted in Section 5. It will be seen that the fractional-order model is more flexible than the integer-order one. This is completely consistent with analysis in [19, 29]. Besides, it is important to note that the stability analysis of the fractional-order model (2) is now posed. This is an important but not simple problem.

Firstly, we investigate the positivity and boundedness of solutions of the model (2) based on some standard comparison results for fractional differential equations. The result is that we obtain the positivity and boundedness of solutions. This is useful and plays an important role in analyzing stability properties of the model (2).

Secondly, we analyze stability properties of the fractional-order model, including local, global, uniform and Mittag-Leffler stability. Similarly to dynamical systems governed by integer-order differential equations, the stability analysis of fractional-order systems is very important but not a simple task in general. In recent years, Lyapunov stability theory for fractional-order dynamical systems has been widely studied [3, 4, 15, 32, 55, 56, 74]. This can be considered as one of the most powerful and successful approaches to the stability problem of fractional dynamical systems. However, the main challenge when using the Lyapunov stability theory is that we must construct suitable Lyapunov functions but there is no universal method for constructing them. Although some classes of Lyapunov candidate functions such as quadratic and Volterra-type Lyapunov functions can be suitable with a broad range of fractional-order systems [32, 74], finding Lyapunov functions is not easy in general.

In order to examine the stability properties of the model (2), we use a simple approach that is based on the fractional Lyapunov stability theory and Barbalat's lemma in combination with some nonstandard techniques for fractional dynamical systems. More clearly, we use suitable quadratic Lyapunov functions and characteristics of quadratic forms associated with real matrices to establish the stability properties. As an important consequence, local and global, uniform and Mittag-Leffler stability and therefore, population dynamics of the proposed fractional-order model are analyzed rigorously. It is worth noting that the used approach can be extended for general dynamical systems described by fractional-order differential equations.

After performing the dynamical analysis, we extend the Mickens' methodology [61, 62, 63, 64, 65] to construct a dynamically consistent nonstandard finite difference (NSFD) scheme for the model (2) for the purpose of numerical simulation. By rigorous mathematical analysis and numerical experiments, we show that the NSFD scheme can provide reliable numerical approximations, which preserve essential dynamical properties of the fractional-order model (2) for all the finite step sizes. It should be emphasized that NSFD schemes, which were first introduced by Mickens, are a powerful and efficient approach for solving differential equations [61, 62, 63, 64, 65]. It was proved that NSFD schemes have the ability to preserve essential mathematical features of differential equation models; therefore, they can compensate for drawbacks of standard finite differences [61, 62, 63, 64, 65, 68, 69]. On the other hand, NSFD schemes are simple, effective and can be applied to solve a broad class of differential equations. For these reasons, NSFD schemes have been widely used in solving differential equation models arising in real-world applications nowadays [1, 2, 8, 9, 10, 20, 21, 22, 30, 33, 34, 35, 66, 75, 77, 78]. In [23, 24, 25, 26, 43, 44, 45, 46, 47, 48], we have constructed NSFD schemes for some classes of differential equations of both integer and fractional orders, in which the positivity, boundedness and stability of the NSFD schemes were mainly studied.

Lastly, a set of numerical experiments is conducted to support the theoretical results and to show advantages of the NSFD scheme over the standard Grunwald-Letnikov (G-L) method. It is proved that the numerical results are consistent with the theoretical ones. In particular, the NSFD scheme preserves not only the positivity and boundedness of solutions but also the stability properties of the model (2) for all the values of the step size, meanwhile, the Grunwald-Letnikov scheme can generate numerical approximations that are negative and unstable for some specific step sizes.

The plan of this work is as follows:
Some concepts and preliminaries are presented in Section 2. Dynamical analysis is performed in Section 3. The NSFD scheme is constructed and analyzed in Section 4. A set of numerical experiments is reported in Section 5. Some conclusions and open problems are discussed in the last section.

2. Preliminaries

In this section, we provide preliminaries and auxiliary results that will be used in the next sections.

2.1. Caputo fractional derivative and fractional dynamical systems

We recall from [18, 27, 52, 70] the definition of Caputo fractional derivatives and some of its basic properties.

Let $AC[a, b]$ $(a < b)$ be the space of absolutely continuous functions on the interval $[a, b]$. Caputo fractional derivatives of order $\alpha \in \mathbb{C}$ of a function $f(t)$ belonging to $AC[a, b]$ are defined via the Riemann-Liouville fractional derivatives. In particular, the right-sided Caputo fractional derivative of order $\alpha \in (0,1)$ can be given by (see [52, Theorem 2.1 in Section 2.4], or also [18, 27, 70])

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)d\tau}{(t-\tau)^\alpha}, \quad t > a $$

Theorem 1. *([28, Theorem 2.2]) Assume that $f \in C^1[a,b]$ is such that ${}_a^C D_t^\alpha f(t) \geq 0$ for all $t \in [a,b]$ and all $\alpha \in (\alpha_0, 1)$ with some $\alpha_0 \in (0,1)$. Then, f is monotone increasing. Similarly, if ${}_a^C D_t^\alpha f(t) \leq 0$ for all t and α mentioned above, then f is monotone decreasing.*

Consider a general dynamical system governed by the Caputo fractional differential equations of the form

$$ {}_{t_0}^C D_t^\alpha y(t) = f(t,y), \quad y(t_0) = y_0, \quad \alpha \in (0,1). \tag{3} $$

Definition 1. *([56]). A point y^* is called an equilibrium point of the Caputo fractional dynamical system (3) if and only if $f(t, y^*) = 0$.*

Definition 2. *(Class-$\mathcal{K}$ functions [50]) A continuous function $\alpha : [0,t) \to [0,\infty)$ is said to belong to class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.*

Theorem 2. *(Lyapunov stability and uniform stability of fractional order systems [32]) Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system (3). Let us assume that there exists a continuous Lyapunov function $V(y(t),t)$ and a scalar class-$\mathcal{K}$ function $\gamma_1(.)$ such that, $\forall y \neq 0$*

$$ \gamma_1(\|y(t)\|) \leq V(y(t),t) $$

and

$$ {}_{t_0}^C D_t^\beta y(t) \leq 0, \quad \textit{with} \quad \beta \in (0,1] $$

then the origin of the system (3) is Lyapunov stable (stable).

If, furthermore, there is a scalar class-$\mathcal{K}$ function $\gamma_2(.)$ satisfying

$$ V(y(t),t) \leq \gamma_2(\|y\|) $$

then the origin of the system (3) is Lyapunov uniformly stable (uniformly stable).

Theorem 3. *(Fractional order Barbalat's lemma [76, Theorem 3]) If a scalar function $V(t, y(t))$ is positive semi-definite and the Caputo fractional derivative of $V(t, y(t))$ along the solution $y(t)$ of the system* (3) *satisfies* ${}_{t_0}^{C}D_t^{\alpha}V(t, y(t)) \leq -\varphi(\|y(t)\|)$, *where $\varphi(.)$ belongs to class-$\mathcal{K}$, then $y(t) \to 0$ as $t \to +\infty$ if $y_i(t)$ $i = 1, 2, \ldots, n$ are uniformly continuous.*

Corollary 1. *([76, Corollary 3]) If a scalar function $V(t, y(t))$ is positive semi-definite and the Caputo fractional derivative of $V(t, y(t))$ along the solution $y(t)$ of the system* (3) *satisfies* ${}_{t_0}^{C}D_t^{\alpha}V(t, y(t))$ *is negative semi-define, then $y(t) \to 0$ as $t \to +\infty$ if $f_i(t, y(t))$ $i = 1, 2, \ldots, n$ for the system* (3) *are bounded.*

Definition 3 (Mittag–Leffler Stability [56])**.** *The solution of* (3) *is said to be Mittag-Leffler stable if*

$$\|y(t)\| \leq \{m[y(t_0)]E_{\alpha}(-\lambda(t - t_0)^{\alpha})\}^b$$

where t_0 is the initial time, $\alpha \in (0, 1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \in \mathbb{R}^n$ with Lipschitz constant m_0.

Theorem 4 (Theorem 5.1 in [56])**.** *Let $y = 0$ be an equilibrium point for the system* (3) *and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, y(t)) : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to y such that*

$$\begin{aligned} &\alpha_1\|y\|^a \leq V(t, y(t)) \leq \alpha_2\|y\|^{ab}, \\ &{}_{0}^{C}D_t^{\beta}V(t, y(t)) \leq -\alpha_3\|y\|^{ab}, \end{aligned}$$

where $t \geq 0$, $x \in \mathbb{D}$, $\beta \in (0, 1)$, $\alpha_1, \alpha_2, \alpha_3, a$ and b are arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $y = 0$ is globally Mittag-Leffler stable.

Lemma 1. *(A fractional comparison principle [56, Lemma 6.1]) Let $x(0) = y(0)$ and* ${}_{t_0}^{C}D_t^{\beta}x(t) \geq {}_{t_0}^{C}D_t^{\beta}y(t)$, *where $\beta \in (0, 1)$. Then $x(t) \geq y(t)$.*

Lemma 2. *([4, Lemma 1]). Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}x^2(t) \leq x(t)\,{}_{t_0}^{C}D_t^{\alpha}x(t), \quad \forall\alpha \in (0, 1).$$

2.2. The Grunwald-Letnikov definition and Grunwald-Letnikov numerical method

Assume that the function $D_t^q z(\tau)$ satisfies necessary conditions of smoothness in every finite interval $(0,t)$. We partition the interval $[0,t]$ by

$$0 = \tau_0 < \tau_1 < \ldots < \tau_{n+1} = t = (n+1)\Delta t, \qquad \tau_{n+1} - \tau_n = \Delta t.$$

Using the classical notation of finite differences

$$\begin{aligned} &\frac{1}{\Delta t^q}\Delta_{\Delta t}^q z(t) = \frac{1}{\Delta t^q}\left(z(\tau_{n+1}) - \sum_{\nu=1}^{n+1} c_\nu^q z(\tau_{n+1-\nu}) \right), \\ &c_\nu^q = (-1)^{q-1}\binom{q}{\nu}, \qquad \binom{q}{\nu} := \frac{q(q-1)(q-2)\ldots(q-\nu+1)}{\nu!}. \end{aligned} \tag{4}$$

Then, the Grunwald-Letnikov definition reads [70]

$${}_0^C D_t^q z(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^q}\Delta_{\Delta t}^q z(t).$$

Based on the Grunwald-Letnikov definition, we obtain the explicit Grunwald-Letnikov (GL) method for the equation (3) ([71])

$$y_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu} - r_{n+1}^q y_0 = \Delta t^q f(t_n, y_n),$$

where

$$r_{n+1}^q = \gamma_{0,-1}^1 (n+1)^q, \quad \gamma_{\mu,k}^q = \frac{\Gamma(\mu q+1)}{\Gamma(kq+1)}$$

is a correction term, which tends to 0 as $n \to \infty$. Note that the binomial coefficients c_ν^q can be recursively defined by

$$\begin{aligned} c_\nu^1 &= q, \\ c_\nu^q &= \left(1 - \frac{q+1}{\nu}\right) c_{\nu-1}^q, \qquad \nu > 1. \end{aligned}$$

Properties of the explicit G-L can be found in [71].

2.3. Real quadratic forms

A quadratic form is a homogeneous polynomial of the second degree in n variables $x_1, x_2, \dots, x_n$ and it always can be represented in the form (see [36, Chapter X])

$$\sum_{i,k=1}^{n} a_{ik} x_i x_k, \quad a_{ik} = a_{ki}; \qquad i, k = 1, 2, \dots, n,$$

where $A = (a_{ik})_{n\times n}$ is a symmetric matrix.

If we denote by x the column matrix $(x_1, x_2, \dots, x_n)$ and denote the quadratic form by

$$A(x, x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k,$$

then, we have

$$A(x, x) = x^T A x.$$

Definition 4 (Definitions 3 and 4, Chapter X in [36])**.** *(i) A real quadratic form $A(x, x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k$ is called positive (negative) semidefinite if for arbitrary of the variables:*

$$A(x, x) \geq 0, \qquad (\leq 0).$$

(ii) A quadratic form $A(x, x)$ is called positive (negative) definite if for arbitrary of the variables, not all zero, ($x \neq 0$)

$$A(x, x) > 0 \qquad (< 0).$$

For the sake of convenience, we now only focus on necessary and sufficient conditions for 2×2 matrices to be negative semidefinite/definite.

Let $A(x, x)$ be a quadratic form associated with a 2×2 real matrix $A = (a_{ij})$ $(1 \leq i, j \leq 2)$. From Theorems 5 and 6 in Chapter X in [36] we obtain:

Theorem 5. *(i) A quadratic form $A(x, x)$ is negative definite if and only if the following inequalities hold:*

$$a_{11} < 0, \qquad \det(A) > 0.$$

(ii) A quadratic form $A(x, x)$ is negative semidefinite if and only if

$$a_{11} \leq 0, \qquad a_{22} \leq 0, \qquad \det(A) \geq 0.$$

2.4. Nonstandard finite difference schemes

Let

$$D_{\Delta t}(y_k) = F_{\Delta t}(f; y_k), \tag{5}$$

be a general finite difference scheme using a step size Δt, which numerically solves the initial value problem

$$\frac{dy}{dt} = f(y), \quad 0 \le t \le T, \quad y(0) = y_0 \in \mathbb{R}^n. \tag{6}$$

Here, $D_{\Delta t}(y_k) \approx dy/dt$, $F_{\Delta t}(f; y_k) \approx f(y)$ and $t_k = k\Delta t$. We recall from [60, 61, 62, 63, 64, 65] that an NSFD scheme for the equation (6) is a discrete model constructed based on a set of six rules, which is originally proposed by Mickens. In particular, NSFD schemes for (6) can be defined as follows [8, 9].

Definition 5. *The finite difference scheme* (5) *is called an NSFD scheme if at least one of the following conditions is satisfied:*

- $D_{\Delta t}(y_k) = \dfrac{y_{k+1} - y_k}{\phi(\Delta t)}$, *where* $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ *is a non-negative function and is called a nonstandard denominator function;*
- $F_{\Delta t}(f; y_k) = g(y_k, y_{k+1}, \Delta t)$, *where* $g(y_k, y_{k+1}, \Delta t)$ *is a non-local approximation of the right-hand side of the system* (6)*.*

The main advantage of NSFD schemes is that they can correctly preserve important mathematical features of differential equations for all the values of the step size. This advantage is shown in the following definitions.

Definition 6 ([9, 10])**.** *Assume that the solutions of the equation* (6) *satisfy some property* $\mathcal{P}$*. The numerical scheme* (5) *is called (qualitatively) stable with respect to property* $\mathcal{P}$ *(or* $\mathcal{P}$*-stable), if for every value of* $\Delta t > 0$ *the set of solutions of* (5) *satisfies property* $\mathcal{P}$*.*

Definition 7 ([8, 9, 60])**.** *Consider the differential equation* $dy/dt = f(y)$*. Let a finite difference scheme for the equation be* $y_{k+1} = F(y_k; \Delta t)$*. Let the differential equation and/or its solutions have property* $\mathcal{P}$*. The discrete model equation is dynamically consistent with the differential equation if it and/or its solutions also have property* $\mathcal{P}$*.*

Similarly NSFD schemes for first-order integer-order systems, NSFD schemes for fractional-order systems are also constructed based on the methodology in Definition 5, namely using nonstandard denominator functions or non-local approximations for right-hand side functions (see, for example, [11]).

3. Qualitative dynamical study

3.1. Positivity and boundedness of solutions and equilibrium points and its local asymptotic stability

In this subsection, we analyze positivity, boundedness and possible equilibrium points of the model (2). Note that the existence and uniqueness of solutions can be obtained by directly applying [58, Theorem 3.1] and [58, Remark 3.2].

First, we explain the derivation of the model (2) in terms of memory effects on population dynamics and point out the difference between the fractional-order model and the original integer-order one. A simple approach for doing this is the use of finite difference formulas to discretize the model (2).

Consider the models (1) and (2) on a finite time interval $[0, T]$. We partition this interval by a uniform mesh given by

$$0 = t_0 < t_1 < \ldots < t_N = T,$$

where $\Delta t = T/N$ is the step size and $t_{n+1} = t_n + \Delta t$ for $n \geq 0$. It follows from the system (2) that

$$\begin{aligned} {}_0^C D_t^q x(t)\Big|_{t=t_n} &= \delta y(t_n) - \frac{\alpha x(t_n)}{\beta + x(t_n)} - \mu x(t_n), \\ {}_0^C D_t^q y(t)\Big|_{t=t_n} &= \frac{\alpha x(t_n)}{\beta + x(t_n)} - (\mu + F) y(t_n), \end{aligned} \tag{7}$$

By using the Grunwald-Letnikov definition for the Caputo fractional derivative, we obtain

$$\begin{aligned} {}_0^C D_t^q x(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t^q} \Delta_{\Delta t}^q x(t), \\ {}_0^C D_t^q y(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t^q} \Delta_{\Delta t}^q y(t), \end{aligned} \tag{8}$$

Combining (8) with (7) we deduce that

$$\begin{aligned} \frac{1}{\Delta t^q}\left(x(\tau_{n+1}) - \sum_{\nu=1}^{n+1} c_\nu^q x(\tau_{n+1-\nu}) \right) &\approx \delta y(t_n) - \frac{\alpha x(t_n)}{\beta + x(t_n)} - \mu x(t_n), \\ \frac{1}{\Delta t^q}\left(y(\tau_{n+1}) - \sum_{\nu=1}^{n+1} c_\nu^q y(\tau_{n+1-\nu}) \right) &\approx \frac{\alpha x(t_n)}{\beta + x(t_n)} - (\mu + F) y(t_n), \end{aligned} \tag{9}$$

for Δt small enough. Consequently,

$$\begin{aligned} x(\tau_{n+1}) &\approx \sum_{\nu=1}^{n+1} c_\nu^q x(\tau_{n+1-\nu}) + \Delta t^q \left(\delta y(t_n) - \frac{\alpha x(t_n)}{\beta + x(t_n)} - \mu x(t_n) \right), \\ y(\tau_{n+1}) &\approx \sum_{\nu=1}^{n+1} c_\nu^q y(\tau_{n+1-\nu}) + \Delta t^q \left(\frac{\alpha x(t_n)}{\beta + x(t_n)} - (\mu + F) y(t_n) \right), \end{aligned} \tag{10}$$

which implies that the values of $x(\tau_{n+1})$ and $y(\tau_{n+1})$ are determined by all the past values of $x(t_i)$ and $y(t_i)$ for $0 \le i \le n$. Meanwhile, for the integer-order (1) we have

$$\begin{aligned} x(\tau_{n+1}) &\approx x(\tau_n) + \Delta t \left(\delta y(t_n) - \frac{\alpha x(t_n)}{\beta + x(t_n)} - \mu x(t_n) \right), \\ y(\tau_{n+1}) &\approx y(\tau_n) + \Delta t \left(\frac{\alpha x(t_n)}{\beta + x(t_n)} - (\mu + F) y(t_n) \right), \end{aligned} \tag{11}$$

which indicates that the values $x(\tau_{n+1})$ and $y(\tau_{n+1})$ only depend on $x(\tau_n)$ and $y(\tau_n)$. So, the difference between the models (2) and (1) is pointed out by finite difference formulas.

Let us denote by

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 | x, y \ge 0\}.$$

Lemma 3 (Positivity and boundedness of solutions). *The fractional-order model* (2) *admits the set* $\mathbb{R}_+^2$ *as a positively invariant set, that is,* $\big(x(t), y(t)\big) \in \mathbb{R}_+^2$ *for all* $t > 0$ *whenever* $\big(x(0), y(0)\big) \in \mathbb{R}_+^2$. *Furthermore, the following estimate holds*

$$\limsup_{t\to\infty} y(t) \le \frac{\alpha}{\mu + F}, \qquad \limsup_{t\to\infty} x(t) \le \frac{\delta\alpha}{\mu(\mu + F)}. \tag{12}$$

Proof. First, it follows from the system (2) that

$${}_0^C D_t^q x|_{x=0} = \delta y, \quad {}_0^C D_t^q y|_{y=0} = \frac{\alpha x}{\beta + x}.$$

It follows from Theorem 1 that if $x(0) \ge 0$ and $y(0) \ge 0$ then $x(t)$ and $y(t)$ cannot escape from the hyperplanes of $x = 0$ and $y = 0$, and on each hyperplane the vector field is tangent to that hyperplane or points toward the interior of $\mathbb{R}^2_+$. So, $x(t), y(t) \geq 0$ for $t > 0$.

Next, from the second quation of (2) we have

$$ {}_0^C D_t^q y = \frac{\alpha x}{\beta + x} - (\mu + F)y \leq \alpha - (\mu + F)y. $$

Consider the auxiliary linear differential equation

$$ {}_0^C D_t^q z = \alpha - (\mu + F)z, \qquad z(0) = y(0). \tag{13} $$

Then, it follows from Lemma 1 that $y(t) \leq z(t)$ for $t \geq 0$. On the other hand, the equation (13) has a unique equilibrium point $z^* = \alpha/(\mu + F)$. By using representation formula of linear Caputo differential equations [70], we obtain that x^* is globally asymptotically stable. Hence, $\lim_{t\to\infty} z(t) = z^*$. Consequently, due to the fact that $y(t) \leq z(t)$ we have

$$ \limsup_{t\to\infty} y(t) \leq \frac{\alpha}{(\mu + F)}. $$

Lastly, we deduce from the first equation of (13) that

$$ {}_0^C D_t^q x = \delta y(t) - \frac{\alpha x(t)}{\beta + x(t)} - \mu x(t) \leq \frac{\delta\alpha}{\mu + F} - \mu x $$

for t large enough. The second estimate of (12) is established by repeating the above arguments for its first estimate. $\square$

From Definition 1 of equilibrium points of fractional-order dynamical systems, we see that the sets of equilibrium points of the models (2) and (1) are identical. The analysis in [53] leads to the following result.

Lemma 4. *The fractional-order* (2) *always has a trivial equilibrium point* $E_0 = (x_0, y_0) = (0, 0)$ *for all the values of the parameters. In the case when* $\dfrac{\delta}{\mu + F} - \dfrac{\mu\beta}{\alpha} > 1$ *the model has a non-trivial (positive) equilibrium point* $E_* = (x_*, y_*)$, *where* x_* *and* y_* *are given by*

$$ x_* = \frac{\alpha}{\mu}\left(\frac{\delta}{\mu + F} - \frac{\mu\beta}{\alpha} - 1\right), \quad y_* = \frac{\alpha}{\delta - (\mu + F)}\left(\frac{\delta}{\mu + F} - \frac{\mu\beta}{\alpha} - 1\right). \tag{14} $$

Similar to [53], let us denote

$$\mathcal{R}_0 = \frac{\delta}{\mu + F} - \frac{\mu\beta}{\alpha}. \tag{15}$$

As will be seen later, $\mathcal{R}_0$ plays not only as a threshold of the existence of the equilibrium points but also a stability threshold of the model (2).

We now examine the local asymptotic stability of the possible equilibrium points of the model (2). From the linearization theorem for fractional dynamical systems [57], it is sufficient to consider linearized equations around equilibria of the system (2). Then, by the stability results of linear systems [59], we will obtain the local asymptotic stability of the equlibria. As a consequence, the following result is obtained thanks to the local stability analysis for the integer-order (1) presented in [53].

Lemma 5 (Local stability analysis)**.** *(i) The trivial equilibrium point is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*
(ii) The positive equilibrium point E_* of the fractional-order model (2) *is locally asymptotically stable provided that it exists.*

3.2. Global stability analysis

Our main investigation in this subsection is focused on the global stability of the model (2). As a consequence of Lemma 3, it is sufficient to study dynamics of the model (2) on a feasible set defined by

$$\Omega = \left\{(x, y) : 0 \le x \le \frac{\delta\alpha}{\mu(\mu + F)}, \quad 0 \le y \le \frac{\alpha}{\mu + F}\right\}. \tag{16}$$

Theorem 6. *If $\mathcal{R}_0 < 1$, then the trivial equilibrium point E_0 is not only locally asymptotically stable but also globally asymptotically stable.*

Proof. Consider a Lyapunov function candidate defined by

$$V_0(x, y) = \frac{1}{2}L_1(x + y)^2 + \frac{1}{2}L_2y^2, \tag{17}$$

where L_1 and L_2 are undetermined positive real numbers that will be chosen later. Then by Lemma 2, the derivative of V_0 along solutions of (2) satisfies

$$\begin{aligned}
{}_0^C D_t^q V_0 &\leq L_1(x+y)\left[{}_0^C D_t^q (x+y)\right] + L_2 y\left({}_0^C D_t^q y\right) \\
&= L_1(x+y)\left[(\delta - F)y - \mu(x+y)\right] + L_2 y\left[\frac{\alpha x}{\beta + x} - (\mu + F)y\right] \\
&= L_1(\delta - F)(x+y)y - L_1\mu(x+y)^2 + L_2\frac{\alpha x y}{\beta + x} - L_2(\mu+F)y^2 \\
&\leq L_1(\delta - F)(x+y)y - L_1\mu(x+y)^2 + L_2\frac{\alpha x y}{\beta} - L_2(\mu+F)y^2 \\
&= L_1(\delta - F)(x+y)y - L_1\mu(x+y)^2 + L_2\frac{\alpha(x+y)y}{\beta} - L_2\frac{\alpha y^2}{\beta} - L_2(\mu+F)y^2 \\
&= v^T A v,
\end{aligned}$$

where v and A are given by

$$v = \begin{pmatrix} x+y \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} -L_1\mu & \dfrac{L_1(\delta - F) + L_2\dfrac{\alpha}{\beta}}{2} \\ \dfrac{L_1(\delta - F) + L_2\dfrac{\alpha}{\beta}}{2} & -L_2(\mu + F) - L_2\dfrac{\alpha}{\beta} \end{pmatrix}. \tag{18}$$

We will show that there always exist L_1 and L_2 for which the quadratic form $v^T A v$ associated with the matrix A is negative definite. Indeed, it is clear that $D_1 = -L_1\mu < 0$ for all $L_1 > 0$. On the other hand,

$$D_2 = \det(A) = -\frac{(\delta - F)^2}{4}L_1^2 - \frac{(\alpha/\beta)^2}{4}L_2^2 - \left[\mu(\mu + F) + \frac{\mu\alpha}{\beta} - \frac{(\delta - F)\alpha/\beta}{2}\right]L_1 L_2.$$

Consider D_2 as a function of L_1. Then, its discrimination is given by

$$\Delta_{D_2} = \left[\mu(\mu + F) + \frac{\mu\alpha}{\beta}\right]\left[\mu(\mu + F) + \frac{\mu\alpha}{\beta} - (\delta - F)\frac{\alpha}{\beta}\right]L_2^2.$$

It is important to note that $\Delta_{D_2} > 0$ for all $L_2 > 0$ if and only if $\mathcal{R}_0 < 1$. Consequently, if $\mathcal{R}_0 < 1$ there always exist L_1 and L_2 for which $v^T A v$ is negative define. Using Theorem 2 and Barbalat's lemma (Corollary 1) we obtain the global asymptotic stability of E_0. The proof is complete. $\square$

Assume that $\mathcal{R}_0 > 1$. Then, $x_* > 0$. For $x \geq 0$, we define

$$\begin{aligned}
A &= -\frac{\delta^2}{4}, \\
B(x) &= \mu(\mu+F) + \frac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)} - \frac{1}{2}\frac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)}, \\
C(x) &= -\frac{1}{4}\frac{(\alpha\beta)^2}{(\beta+x)^2(\beta+x_*)^2}, \\
\Delta(x) &= B^2(x) - 4AC(x).
\end{aligned} \tag{19}$$

It it clear that $A < 0$ and $C(x) < 0$ for $x \geq 0$.

Lemma 6. *Let A, $B(x)$ and $C(x)$ be defined in* (19). *If $\mathcal{R}_0 > 1$, then the following estimates hold:*

$$\begin{aligned}
&\Delta(0) = 0, \\
&\Delta(x) > 0 \quad \textit{for} \quad x > 0, \\
&B(x) > 0 \quad \textit{for} \quad x \geq 0.
\end{aligned}$$

Proof. First, it is easy to verify that

$$\Delta(x) = \left[\mu(\mu+F)+\frac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)}\right]\left[\mu(\mu+F)+\frac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)}-\frac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)}\right].$$

On the other hand, by simple algebraic manipulations

$$\mu(\mu+F) + \frac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)} - \frac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)} = \frac{\mu(\mu+F)x}{\beta+x}. \tag{20}$$

Consequently, we obtain the first two estimates.

Note that we deduce from (20) that $B(x) > 0$ for $x \geq 0$. The proof is complete. $\square$

For each $x \geq 0$, we consider a quadratic polynomial of z given by

$$P_2(z) = Az^2 + B(x)z + C(x), \tag{21}$$

where A, $B(x)$ and $C(x)$ are defined in (19). As a consequence of Lemma 6, $P_2(z)$ has two distinct positive roots $r_1(x) < r_2(x)$, where

$$\begin{aligned}
r_2(x) &= \frac{-B(x) - \sqrt{B^2 - 4AC}}{2A} \\
&= -\frac{\mu(\mu+F) + \dfrac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)} - \dfrac{1}{2}\dfrac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)}}{\dfrac{\delta^2}{2}} \\
&\quad - \frac{\sqrt{\left[\mu(\mu+F) + \dfrac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)}\right]\left[\mu(\mu+F) + \dfrac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)} - \dfrac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)}\right]}}{\dfrac{\delta^2}{2}}, \\
r_1(x) &= \frac{-B(x) + \sqrt{B^2 - 4AC}}{2A} \\
&= -\frac{\mu(\mu+F) + \dfrac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)} - \dfrac{1}{2}\dfrac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)}}{\dfrac{\delta^2}{2}} \\
&\quad + \frac{\sqrt{\left[\mu(\mu+F) + \dfrac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)}\right]\left[\mu(\mu+F) + \dfrac{\alpha\beta(\mu+F)}{(\beta+x)(\beta+x_*)} - \dfrac{\delta\alpha\beta}{(\beta+x)(\beta+x_*)}\right]}}{\dfrac{\delta^2}{2}},
\end{aligned} \tag{22}$$

In order to investigate the global asymptotic stability of the positive equilibrium point E_*, we introduce the following technical hypothesis

$$r := \max_{x\in\Omega_1} r_1(x) \le R := \min_{x\in\Omega_1} r_2(x), \quad \Omega_1 = \left\{x : 0 \le x \le \frac{\alpha\delta}{\mu(\mu+F)}\right\}. \tag{23}$$

Theorem 7. *If $\mathcal{R}_0 > 1$ and the condition (23) is satisfied, then the positive equilibrium point E_* is not only locally asymptotically stable but also globally asymptotically stable.*

Proof. Consider a Lyapunov function candidate $V(x,y) : \Omega \to \mathbb{R}_+$ defined by

$$V(x,y) = \frac{\tau}{2}(x-x_*)^2 + \frac{1}{2}(y-y_*)^2, \tag{24}$$

where τ is a positive real number, which will be determined later. Since (x_*, y_*) is a equilibrium point of the system (2), we have

$$\delta y_* - \frac{\alpha x_*}{\beta + x_*} - \mu x_* = 0, \quad \frac{\alpha x_*}{\beta + x_*} - (\mu + F) y_* = 0.$$

Hence, the system (2) can be rewritten in the form

$$\begin{aligned} {}_0^C D_t^q x &= \delta(y - y_*) - \frac{\alpha\beta}{(\beta + x)(\beta + x_*)}(x - x_*) - \mu(x - x_*), \\ {}_0^C D_t^q y &= \frac{\alpha\beta}{(\beta + x)(\beta + x^*)}(x - x_*) - (\mu + F)(y - y_*), \end{aligned} \tag{25}$$

Then, by Lemma 2 and (25) the derivative of the function V along solutions of (2) satisfies

$$\begin{aligned} {}_0^C D_t^q V(x, y) &\leq \tau(x - x_*)({}_0^C D_t^q x) + (y - y_*)({}_0^C D_t^q y) \\ &= \tau(x - x_*)\left[\delta(y - y_*) - \frac{\alpha\beta}{(\beta + x)(\beta + x_*)}(x - x_*) - \mu(x - x_*)\right] \\ &+ (y - y_*)\left[\frac{\alpha\beta}{(\beta + x)(\beta + x^*)}(x - x_*) - (\mu + F)(y - y_*)\right] \\ &= -\left[\frac{\alpha\beta}{(\beta + x)(\beta + x_*)}\tau + \mu\tau\right](x - x_*)^2 - (\mu + F)(y - y_*)^2 \\ &+ \left[\tau\delta + \frac{\alpha\beta}{(\beta + x)(\beta + x_*)}\right](x - x_*)(y - y_*) \\ &:= z^T P z, \end{aligned} \tag{26}$$

where z and P are given by

$$z = \begin{pmatrix} y - y_* \\ x - x_* \end{pmatrix}, \qquad P = \begin{pmatrix} -(\mu + F) & \dfrac{\tau\delta + \dfrac{\alpha\beta}{(\beta + x)(\beta + x_*)}}{2} \\ \dfrac{\tau\delta + \dfrac{\alpha\beta}{(\beta + x)(\beta + x_*)}}{2} & -\left[\dfrac{\alpha\beta}{(\beta + x)(\beta + x_*)}\tau + \mu\tau\right] \end{pmatrix}$$

We now show that there always exists a constant $\tau > 0$ for which $P(z, z) = z^T P z$ is negative semidefinite. Indeed, $D_1(P) = -(\mu + F) < 0$ and

$$\det(P) = A\tau^2 + B(x)\tau + C(x),$$

where A, $B(x)$ and $C(x)$ are given in (19). Hence, for each $x \geq 0$ there is a positive real number $\tau(x)$ depending on x for which $\det(P) \leq 0$. Here, $r_1(x) \leq \tau(x) \leq r_2(x)$, where $r_1(x)$ and $r_2(x)$ are defined in (22). The condition (23) implies that there always exists a constant $\tau^* \in [r, R]$ for which $z^T P z$ is negative semidefinite. Consequently, it follows from the Barbalat's lemma that

$$\lim_{t\to\infty} \big(x(t), y(t)\big) = (x_*, y_*).$$

Combining this with the stability of E_* established in Lemma 5 we conclude that E_* is globally asymptotically stable. The proof is complete. □

We now reduce the condition (23) to a simple one that is easily to be verified.

Lemma 7. *Assume that* $\mathcal{R}_0 > 1$. *Then, the condition* (23) *is satisfied if*

$$\frac{\delta}{\mu + F} > 2. \tag{27}$$

Proof. First, we have

$$B'(x) = \left[\frac{\alpha\beta(\mu + F)}{\beta + x_*} - \frac{1}{2}\frac{\delta\alpha\beta}{\beta + x_*}\right]\frac{-1}{(\beta + x)^2}.$$

So, the condition (27) implies that $B'(x) > 0$ for $x \geq 0$.

Next, the derivative of $r_2(x)$ is given by

$$\begin{aligned}
r_2'(x) &= \frac{1}{2A}\left(- B'(x) - \frac{B(x)B'(x) - 2AC'(x)}{\sqrt{B^2(x) - 4AC'(x)}}\right) \\
&= \frac{B'(x)(-B(x) - \sqrt{B^2(x) - 4AC(x)})}{2A\sqrt{B^2(x) - 4AC(x)}} + \frac{C'(x)}{\sqrt{B^2(x) - 4AC}} \\
&= B'(x)r_2(x)\frac{1}{\sqrt{B^2(x) - 4AC(x)}} + \frac{C'(x)}{\sqrt{B^2(x) - 4AC(x)}},
\end{aligned}$$

which implies that $r_2'(x) > 0$ for $x \geq 0$. Similarly, we have that

$$r_1'(x) = -B'(x)r_1(x)\frac{1}{\sqrt{B^2 - 4AC(x)}} - \frac{C'(x)}{\sqrt{B^2 - 4AC(x)}}.$$

Consequently, $r_1'(x) < 0$ for $x \geq 0$.

We deduce from $r_1'(x) < 0$ and $r_2'(x) > 0$ for $x \geq 0$ that

$$\max_{x \geq 0} r_1(x) = r_1(0) \leq r_2(0) = \min_{x \geq 0} r_2(x).$$

This is the desired conclusion. The proof is complete. □

Lemma 7 and Theorem 7 lead to the following result.

Theorem 8. *If $\mathcal{R}_0 > 1$ and the condition (27) is satisfied, then the positive equilibrium point E_* is not only locally asymptotically stable but also globally asymptotically stable.*

Remark 1. *The condition $\mathcal{R}_0 > 1$ can be written in the form*

$$\frac{\delta}{\mu + F} > 1 + \frac{\mu\beta}{\alpha}. \tag{28}$$

Therefore, the assumption of Theorem 8 is equivalent to

$$\frac{\delta}{\mu + F} > \max\left\{2, 1 + \frac{\mu\beta}{\alpha}\right\}. \tag{29}$$

This condition is easy to be verified.
In numerical examples performed in Section 5, we will show that the condition (23) is not in conflict with the condition $\mathcal{R}_0 > 1$.

3.3. Uniform stability and Mittag-Leffler stability

By combining the quadratic Lyapunov functions (17) and (24) proposed in the proofs of Theorems 6 and 7 with Theorems 2 and 3, we obtain the uniform stability and Mittag-Leffler stability of the fractional-order model (2) as follows.

Theorem 9 (Uniform stability and Mittag-Leffler stability)**.** *(i) If $\mathcal{R}_0 < 1$, then the trivial equilibrium point E_0 of the model (2) is uniform stability and Mittag-Leffler stability.*
(ii) If $\mathcal{R}_0 > 1$ and the condition (23) is satisfied, then the positive equilibrium point E_* is uniformly stable and Mittag-Leffler stable.

Theorem 10 (A simple condition for uniform stability and Mittag-Leffler stability of E_*)**.** *If the condition (29) holds, then the positive equilibrium point E_* is uniform stability and Mittag-Leffler stability.*

4. Construction of positivity-preserving NSFD scheme

In this section, we utilize the Mickens' methodology to construct a NSFD scheme preserving the positivity of the model (2) and compare it with the G-L method.

First, in order to obtain a positivity-preserving NSFD scheme, we modify the system (9) by replacing its right-hand side function by a nonlocal approximation given by

$$\begin{aligned} &\frac{\alpha x(t_n)}{\beta + x(t_n)} - (\mu + F)y(t_n) \approx \frac{\alpha x_n}{\beta + x_n} - (\mu + F)y_{n+1}, \\ &\delta y(t_n) - \frac{\alpha x(t_n)}{\beta + x(t_n)} - \mu x(t_n) \approx \delta y_{n+1} - \frac{\alpha x_{n+1}}{\beta + x_n} - \mu x_{n+1}. \end{aligned} \tag{30}$$

Note that the order of variables x and y was changed in (30). Then, we obtain an NSFD scheme

$$\begin{aligned} &\frac{1}{\Delta t^q}\left(y_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu}\right) = \frac{\alpha x_n}{\beta + x_n} - (\mu + F)y_{n+1}, \\ &\frac{1}{\Delta t^q}\left(x_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu}\right) = \delta y_{n+1} - \frac{\alpha x_{n+1}}{\beta + x_n} - \mu x_{n+1}, \end{aligned} \tag{31}$$

The NSFD scheme (31) can be transformed to the explicit form

$$\begin{aligned} y_{n+1} &= \frac{\displaystyle\sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu} + \Delta t^q \frac{\alpha x_n}{\beta + x_n}}{1 + \Delta t^q(\mu + F)}, \\ x_{n+1} &= \frac{\displaystyle\sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu} + \Delta t^q \delta y_{n+1}}{1 + \Delta t^q \dfrac{\alpha}{\beta + x_n} + \Delta t^q \mu}. \end{aligned} \tag{32}$$

In the NSFD scheme (32), y_{n+1} is calculated first by the first equation and then, it is immediately used in the second equation to evaluate x_{n+1}.

By using mathematical induction, we obtain from (32) that $x_{n+1} \geq 0$ and $y_{n+1} \geq 0$ whenever $x_0 \geq 0$ and $y_0 \geq 0$. In other words, the NSFD scheme (32) preserves the positivity of the solutions of the model (2) for all the values of the step size. On the other hand, it follows from the first equation of the system (31) that

$$\frac{1}{\Delta t^q}\left(y_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu}\right) = \frac{\alpha x_n}{\beta + x_n} - (\mu + F)y_{n+1} \leq \alpha - (\mu + F)y_{n+1},$$

or equivalently

$$y_{n+1} \leq \frac{\displaystyle\sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu} + \Delta t^q \alpha}{1 + \Delta t^q(\mu + F)},$$

which implies that

$$\limsup_{n\to\infty} y_n \leq \frac{\alpha}{\mu + F}.$$

Furthermore, from the second equation of (31) we obtain

$$\frac{1}{\Delta t^q}\left(x_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu}\right) = \delta y_{n+1} - \frac{\alpha x_{n+1}}{\beta + x_n} - \mu x_{n+1} \leq \frac{\delta\alpha}{\mu + F} - \mu x_{n+1}$$

for n large enough. This implies that

$$\limsup_{n\to\infty} x_n \leq \frac{\alpha\delta}{\mu(\mu + F)}.$$

Then, we obtain the following theorem.

Theorem 11. *The NSFD scheme* (31) *preserves the positivity and boundedness of the solutions of the model* (2) *for all the values of the step size. In other words, it is dynamically consistent with respect to the positivity and boundedness of the model* (2)*.*

Remark 2. *(i) Note that the convergence of the NSFD scheme* (31) *can be established similarly to the analysis in [21].*
(ii) Following the Mickens' methodology on non-local approximations, we can obtain the following NSFD schemes for the model (2)

$$\begin{aligned} \frac{1}{\Delta t^q}\left(x_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu}\right) &= \delta y_n - \frac{\alpha x_{n+1}}{\beta + x_n} - \mu x_{n+1}, \\ \frac{1}{\Delta t^q}\left(y_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu}\right) &= \frac{\alpha x_{n+1}}{\beta + x_n} - (\mu + F)y_{n+1}, \end{aligned} \tag{33}$$

and

$$\begin{aligned}
\frac{1}{\Delta t^q}\left(x_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu}\right) &= \delta y_n - \frac{\alpha x_{n+1}}{\beta + x_n} - \mu x_{n+1}, \\
\frac{1}{\Delta t^q}\left(y_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu}\right) &= \frac{\alpha x_n}{\beta + x_n} - (\mu + F) y_{n+1},
\end{aligned} \tag{34}$$

It is easy to verify that the NSFD schemes (33) *and* (34) *are all dynamically consistent with the positivity of solutions. Similarly to the NSFD scheme* (31)*, the value of* x_{k+1} *determined by the first equation of* (33) *is immediately used in the second equation* (33) *to calculate the value of* y_{k+1}*. However, it is not easy to establish the boundedness of this NSFD scheme comparing with* (31)*. The NSFD scheme* (34) *also preserves the boundedness of solutions as the NSFD scheme* (34)*, but* x_{k+1} *or* y_{k+1} *is not immediately utilized to compute the remaining component.*

We now apply the G-L method for the fractional-order model (2). Then, we obtain

$$\begin{aligned}
y_{n+1} &= \sum_{\nu=1}^{n+1} c_\nu^q y_{n+1-\nu} + r_{n+1}^q y_0 + \Delta t^q \left[\frac{\alpha x_n}{\beta + x_n} - (\mu + F) y_n\right], \\
x_{n+1} &= \sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu} + r_{n+1}^q x_0 + \Delta t^q \left[\delta y_n - \frac{\alpha x_n}{\beta + x_n} - \mu x_n\right].
\end{aligned} \tag{35}$$

Lemma 8. *Let* (x_0, y_0) *with* $x_0, y_0 \geq 0$ *be any initial data for the initial value problem* (2) *and* $\{(x_n, y_n)\}_{n\geq 1}$ *be the approximation generated by the G-L scheme* (35)*. Then,* $x_n, y_n \geq 0$ *for* $n \geq 1$ *provided that*

$$\Delta t < \sqrt[q]{\min\left\{\frac{c_1^q}{\alpha + \mu}, \quad \frac{c_1^q}{\mu + F}\right\}}. \tag{36}$$

Proof. The lemma is provided thanks to the following estimates

$$\frac{\alpha x_n}{\beta + x_n} \leq \alpha, \quad x_n \geq 0$$

and due to the fact that if (36) holds then

$$c_1^q - \Delta t^q(\alpha + \mu) > 0, \qquad c_1^q - \Delta t^q(\mu + F) > 0.$$

□

Remark 3. *The condition* (36) *means that the G-L scheme only preserves the positivity of the solutions of* (2) *if the step size is small enough. This leads to large volumes of computations when investigate the model* (2) *over long time periods. Meanwhile, the NSFD scheme has the ability to preserves the positivity and boundedness of solutions for all the values of the step size.*

5. Numerical experiments

In this section, we report some illustrative numerical examples to support the theoretical results and advantages of the NSFD scheme.

5.1. Numerical dynamics of the NSFD and G-L schemes

In this section, we observe approximations generated by the NSFD and G-L schemes to show advantages of the NSFD scheme. For this purpose, consider the model (2) with the following set of the parameters:

$$\alpha = 25, \quad \beta = 70, \quad \mu = 0.8, \quad F = 0.75, \quad \delta = 1.6, \quad q = 0.5 \tag{37}$$

associated with the initial data

$$x(0) = 25, \qquad y(0) = 10. \tag{38}$$

In this case, $\mathcal{R}_0 = -1.2077 < 1$. Therefore, the trivial equilibrium point $E_0 = (0, 0)$ is globally asymptotically stable and also uniformly and Mittag-Leffler stable.

We first use the G-L scheme using the step size $\Delta t = 1.0$ to solve the model (2). The numerical approximations are depicted in Figures 1-3. It is clear that these numerical approximations are negative and unstable, and oscillate around the equilibrium position with increasing amplitude. Hence, the essential mathematical features of the fractional-order model are not preserved. However, we see from Figures 4 and 5 that the numerical approximations generated by the NSFD scheme preserve the dynamical properties of the fractional-order model even when using large values of the step size; furthermore, the dynamics of the NSFD scheme does not depend on the step size. So, the advantages of the NSFD scheme are confirmed.

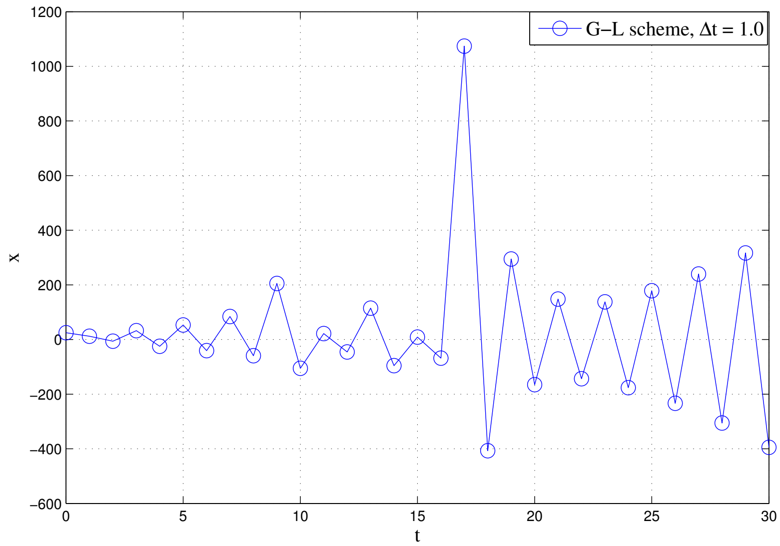

Figure 1: The x-component generated by the G-L method scheme for $\Delta t = 1.0$ after 30 iterations.

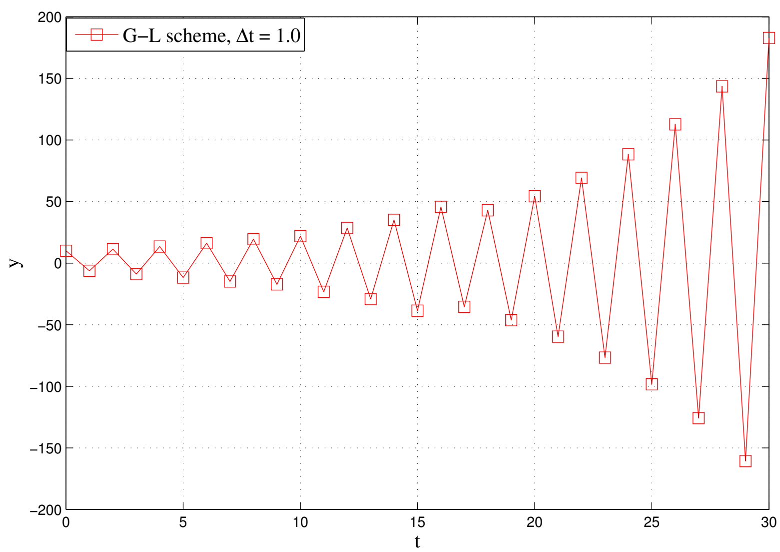

Figure 2: The y-component generated by the G-L method for $\Delta t = 1.0$ after 30 iterations.

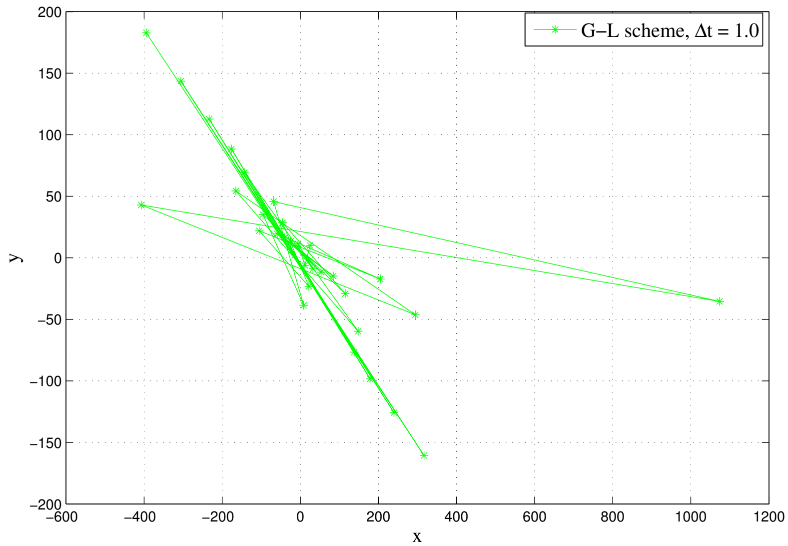

Figure 3: The phase plane generated by the G-L method for $\Delta t = 1.0$ after 30 iterations.

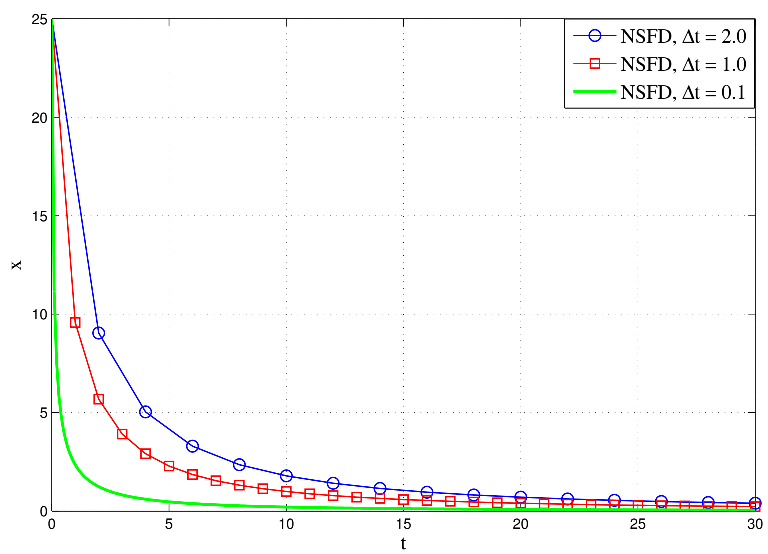

Figure 4: The x-component generated by the NSFD scheme.

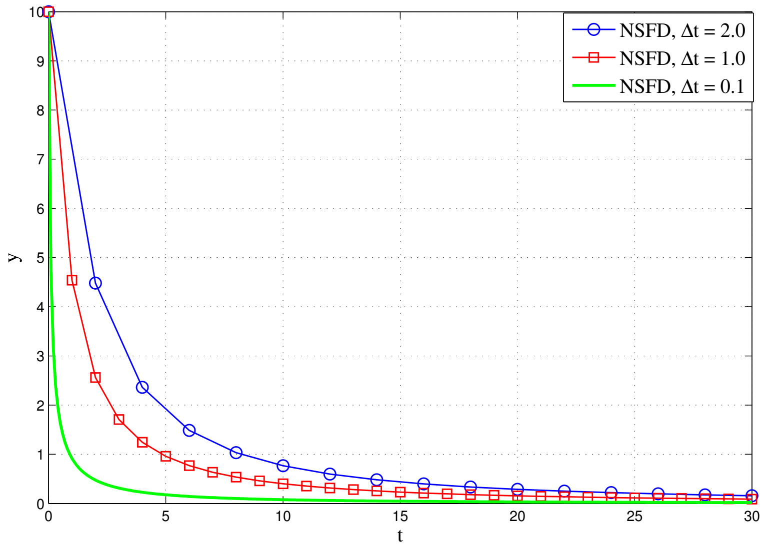

Figure 5: The y-component generated by the NSFD scheme.

5.2. Dynamics of the fractional-order model when $\mathcal{R}_0 < 1$

In this subsection, we observe the dynamics of the model (2) in the case $\mathcal{R}_0 < 1$. For this reason, we consider the model (2) with the parameters given in (37). We use the NSFD scheme with a small step size, namely $\Delta t = 10^{-3}$ for numerical simulation. The obtained phase planes corresponding to several initial values are sketched in Figures 6-10. In these figures, each blue curve represents a phase plane corresponding to an initial data while the red arrows show the evolution of the solutions. Clearly, the numerical solutions provide strong evidence for the dynamical analysis of the model (2) performed in Section 3. It is important to note that the dynamics of the model (2) depend on the values of the order p.

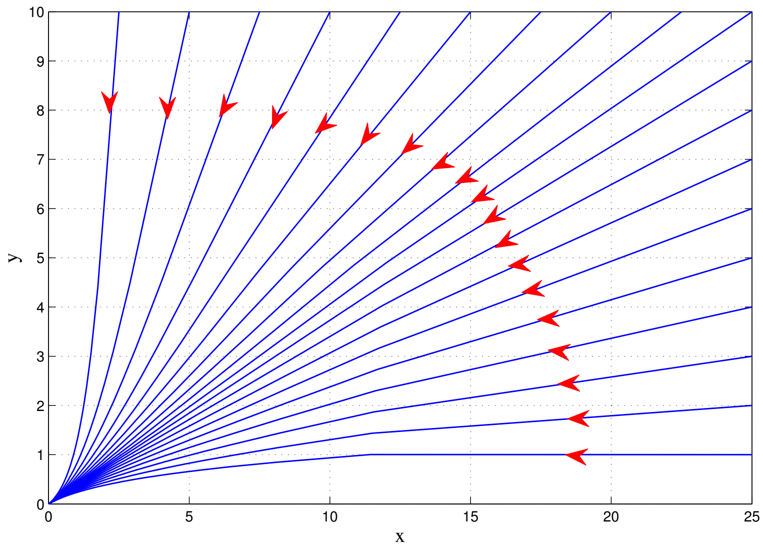

Figure 6: Dynamics of the fractional-order model (2) with $\alpha = 0.5$.

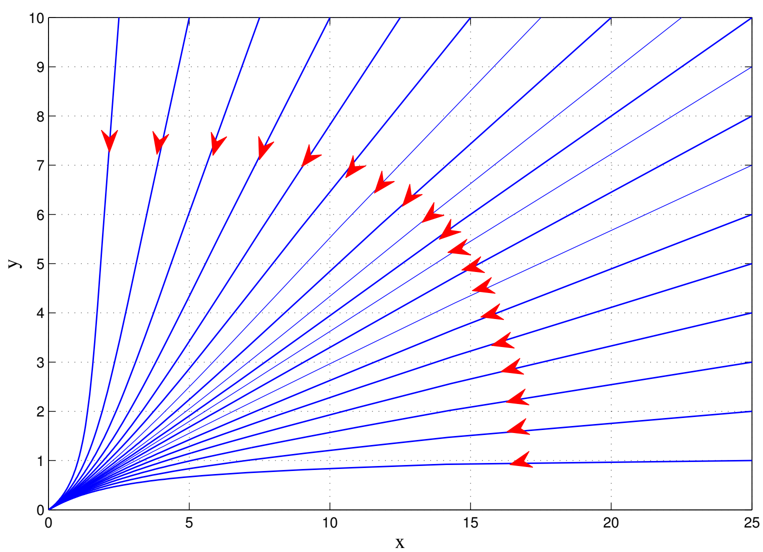

Figure 7: Dynamics of the fractional-order model (2) with $\alpha = 0.6$.

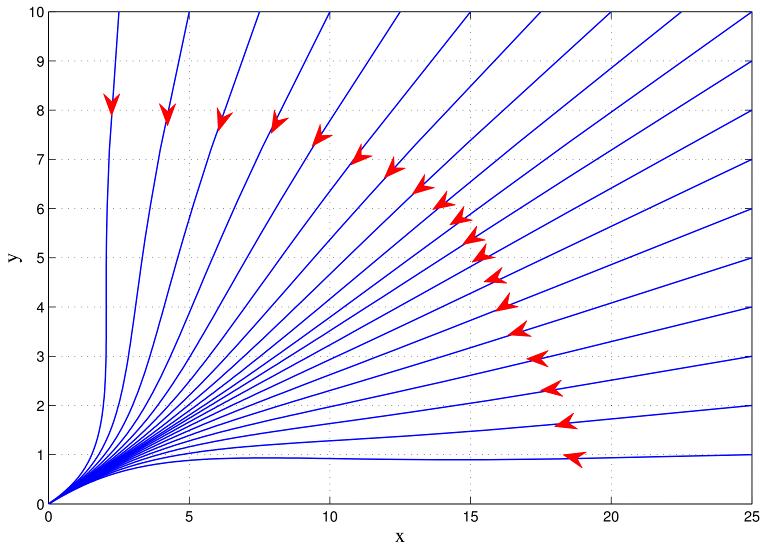

Figure 8: Dynamics of the fractional-order model (2) with $\alpha = 0.75$.

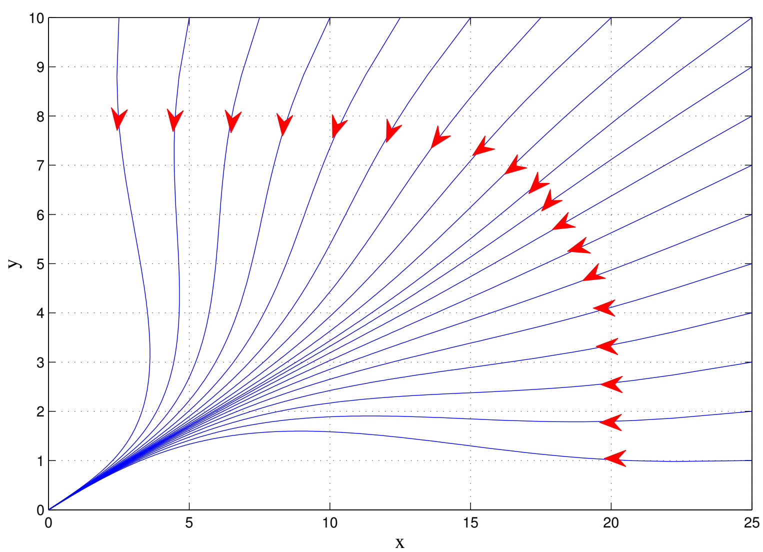

Figure 9: Dynamics of the fractional-order model (2) with $\alpha = 0.90$.

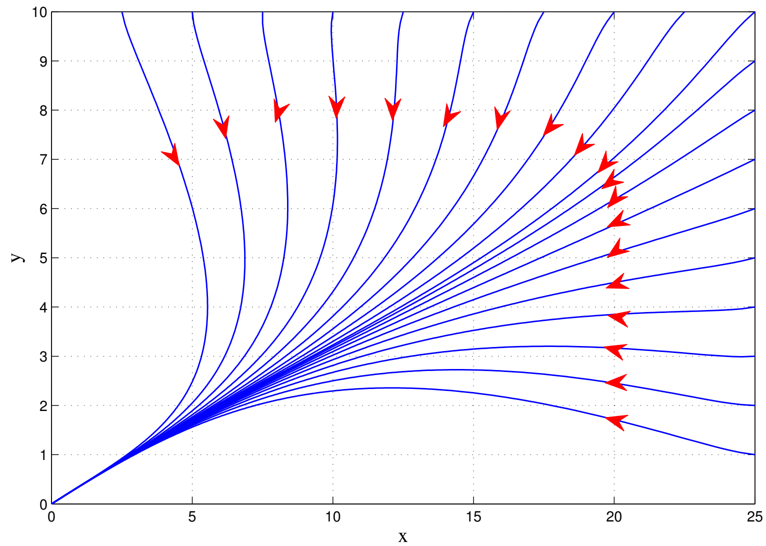

Figure 10: Dynamics of the fractional-order model (2) with $\alpha = 0.99$.

5.3. *Dynamics of the fractional-order model when $\mathcal{R}_0 > 1$*

This subsection provides some numerical examples to investigate the dynamics of the model (2) when $\mathcal{R}_0 > 1$. First, consider the model (2) with the following values of the parameters

$$\alpha = 20, \quad \beta = 80, \quad \mu = 0.8, \quad F = 0.6, \quad \delta = 9.$$

In this case, we have $\mathcal{R}_0 = 3.2286 > 1$ and the model has a unique positive equilibrium point $E_* = (55.7143, 5.8647)$. It is clear that the condition (29) is satisfied. Hence, E_* is globally asymptotically stable and also is uniformly and Mittag-Leffler stable. It is easy to determine the functions $r_1(x)$ and $r_2(x)$ by simple calculations. Their graphs are depicted in Figures 11. Moreover, we have

$$\max_{x \geq 0} r_1(x) = r_1(0) = 0.0164 = \min_{x \geq 0} r_2(x) = r_2(0) = 0.0164.$$

So, we can choose $\tau = r_1(0) = r_2(0)$ for the Lyapunov function V given in (24).

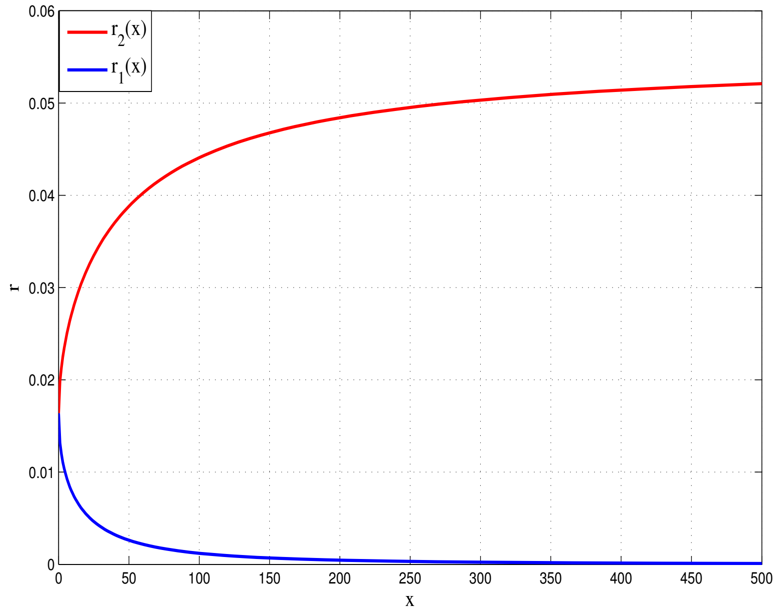

Figure 11: The graphs of the functions $r_1(x)$ and $r_2(x)$.

Some phase planes with $t \in [0, 100]$ provided by the NSFD schemes are given in Figures 12-15. It is clear that the unique positive equilibrium point E_* is stable. On the other hand, the long time behaviour of the fractional-order model depends on the values of α.

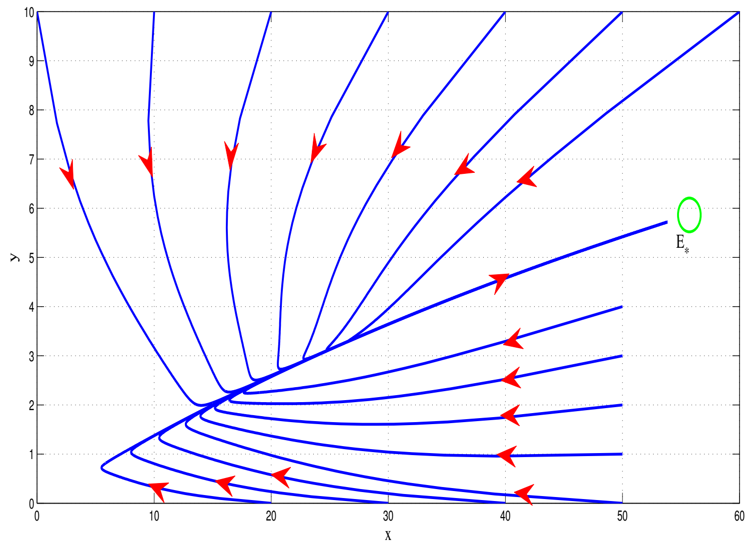

Figure 12: Long time behaviour of the fractional-order model (2) with $\alpha = 0.80$.

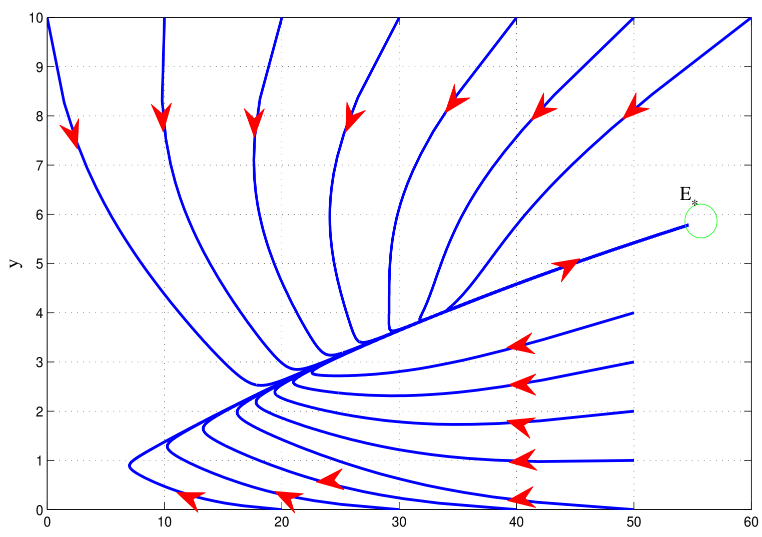

Figure 13: Long time behaviour of the fractional-order model (2) with $\alpha = 0.85$.

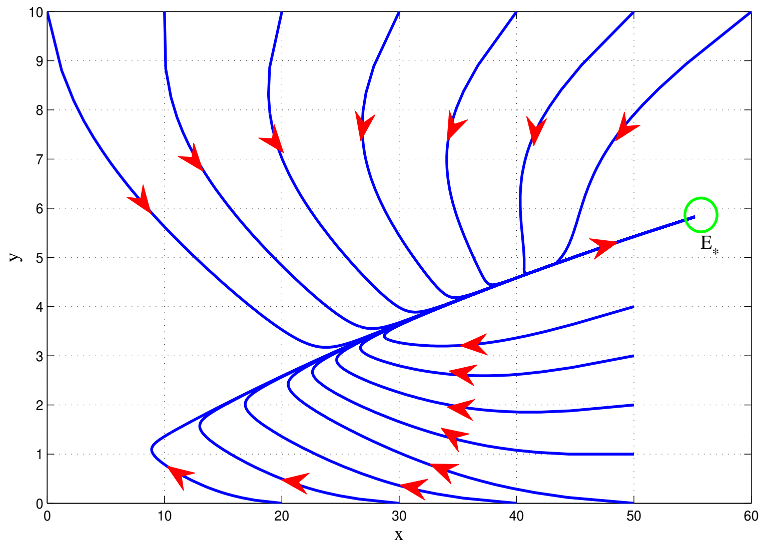

Figure 14: Long time behaviour of the fractional-order model (2) with $\alpha = 0.90$.

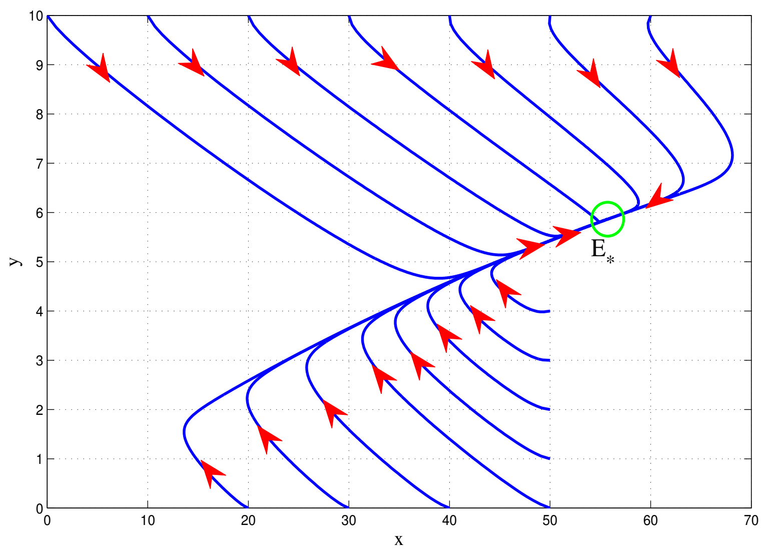

Figure 15: Long time behaviour of the fractional-order model (2) with $\alpha = 0.99$.

6. Conclusions and discussions

As the main conclusion of this work, we have proposed a new fractional-order two-stage species model with recruitment, which is derived from the well-known integer-order model (1) and the Caputo fractional derivative, and have studied dynamical analysis and numerical simulation of this model. The theoretical results and advantages of the NSFD scheme are supported by illustrative numerical experiments.

In the first part of this work, we have performed a rigorous mathematical analysis of qualitative dynamical properties of the proposed fractional-order model. The main result is that we obtained the positivity, boundedness and stability properties of solutions of the model. It is worth noting that the stability properties, including the global, uniform and Mittaf-Leffler stability were established by using a simple approach, which is based on the Lyapunov stability theory and Barbalat's lemma in combination with some nonstandard techniques for fractional dynamical systems. More clearly, we use general quadratic Lyapunov functions and combine them with characteristics of quadratic forms associated with real matrices to establish the stability properties. The present approach is simple and can be applied for a board range of fractional-order dynamical systems.

In the second part of this work, we have extended the Mickens' methodology to construct a dynamically consistent nonstandard finite difference (NSFD) scheme for the model (2) for the purpose of numerical simulation. By rigorous mathematical analysis and numerical experiments, we show that the NSFD scheme can provide reliable numerical approximations preserving all the essential dynamical properties of the fractional-order model (2) for all the finite step sizes. Meanwhile, the Grunwald-Letnikov scheme can generate numerical approximations that are negative and unstable for some specific step sizes.

In the third part of this work, a set of illustrative numerical examples is reported to support the theoretical results and advantages of the NSFD scheme. The experiments provide strong evidence, which shows that the numerical results are consistent with the theoretical ones. It is important to note that the numerical examples suggested that the behaviour of the fractional-order model depends on the values of the order p. This is a main difference between the fractional-order model and integer-order one and also means that the fractional-order model is more flexible. This is very useful in the parameter estimation problem.

In the near future, we will extend the present approach to study dynamical analysis and numerical simulation of fractional-order systems that mathematically model important real-life problems. The combination of the Lyapunov stability theory with real quadratic forms for studying stability problems of fractional-order dynamical systems will be mainly focused. High-order NSFD schemes for the model (2) in particular and for fractional-order systems in general will be also considered.

Ethical Approval: Not applicable.

Availability of supporting data: The data supporting the findings of this study are available within the article [and/or] its supplementary materials.

Conflicts of Interest: We have no conflicts of interest to disclose.

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Authors' contributions: Manh Tuan Hoang: Conceptualization, Methodology, Software, Formal analysis, Writing- Original draft preparation, Methodology, Writing - Review & Editing, Supervision.

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