A new study for the global asymptotic stability of a general predator-prey model

Manh Tuan Hoang[1,*]

[a]*Department of Mathematics, FPT University, Hoa Lac Hi-Tech Park, Km29 Thang Long Blvd, Hanoi, Viet Nam*

Abstract

In a previous paper [L. M. Ladino, E. I. Sabogal, Jose C. Valverde, General functional response and recruitment in a predator-prey system with capture on both species, Math. Methods Appl. Sci. 38(2015) 2876-2887], a mathematical model for a predator-prey model with general functional response and recruitment including capture on both species was formulated and analyzed. However, the global asymptotic stability (GAS) of this model was only partially resolved. In the present paper, we provide a rigorously mathematical analysis for the complete GAS of the predator-prey model. By using the Lyapunov stability theory in combination with some nonstandard techniques of mathematical analysis for dynamical systems, the GAS of equilibria of the model is determined fully. The obtained results not only provide an important improvement for the population dynamics of the predator-prey model but also can be extended to study its modified versions in the context of fractional-order derivatives. The theoretical results are supported and illustrated by a set of numerical examples.

Keywords: Predator-prey system; Population dynamics; Global asymptotic stability; Lyapunov stability theorem; Nonlinear dynamics
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1. Introduction

Mathematical models describing the population dynamics of predator-prey systems have played a prominent role in both theory and applications, especially in biology, ecology as well as interdisciplinary sciences [3, 5, 6, 18]. For many years, a large number of mathematical studies for predator-prey systems have been formulated and analyzed by many mathematicians and ecologists [9, 10, 11, 12, 14, 15, 16, 17, 20, 21, 22, 25, 27, 29, 30, 31, 33, 34, 35]. For predator-prey models, the global asymptotic stability (GAS) analysis is one of the important

*Corresponding author
Email address: `tuanhm14@fe.edu.vn; hmtuan01121990@gmail.com` (Manh Tuan Hoang)

problems with many useful applications in real-world situations [3, 6, 16, 18, 20, 29, 31, 34].

In this paper, we revisit a predator-prey model with general functional response and recruitment including capture on both species, which was proposed and analyzed in [22]. The mathematical formulation and dynamical properties of this model will be recalled in Section 2. In [22], the population dynamics of the model was systematically established; however, the GAS of the model was only partially resolved (see Theorem 1). More clearly, only the GAS of the extinction equilibrium point was proved, meanwhile, other equilibria (ecological stability equilibrium point, equilibrium point of extinction of the predator species and equilibrium point of extinction of the prey species) were only confirmed their local asymptotic stability. Nevertheless, numerical studies presented in [8, 22] suggested that the equilibria may be not only locally asymptotically stable but also globally asymptotically stable.

Motivated by the above reasons, in the present paper we will provide a rigorously mathematical analysis for the complete GAS of the predator-prey model (1). By using the Lyapunov stability theory [19, 23, 26] in combination with some nonstandard techniques of mathematical analysis for dynamical systems, the GAS of the equilibria of the predator-prey model is determined fully. The obtained results not only provide an important improvement for the population dynamics of the predator-prey model but also can be extended to study its modified versions in the context of fractional-order derivatives. Moreover, the theoretical results are supported and illustrated by a set of numerical examples.

The paper is organized as follows. In Section 2 we recall from [22] the mathematical model and population dynamics of the predator-prey model. The complete GAS of the model is established in Section 3. Section 4 reports two numerical examples. Some conclusions and open problems are presented in the last section.

2. Mathematical model and its dynamics

In [22], Ladino et al. proposed a mathematical model for a predator-prey system with general functional response and recruitment including capture on both species. The model is represented by

$$\begin{aligned} \dot{x}(t) &= x(t)f(x(t), y(t)) = x(t)\big[r(x(t)) - y(t)\phi(x(t)) - m_1\big], \\ \dot{y}(t) &= y(t)g(x(t), y(t)) = y(t)\big[s(y(t)) + cx(t)\phi(x(t)) - m_2\big], \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ are prey population and predator population at time t, respectively; the functions $r(x), s(y)$ and $\phi(x)$ satisfy

$$\begin{aligned} &\forall x \geq 0, \quad r(x) > 0, \quad r'(x) < 0, \quad [xr(x)]' \geq 0, \quad \text{and} \quad \lim_{x\to\infty} r(x) = 0, \\ &\forall y \geq 0, \quad s(y) > 0, \quad s'(y) < 0, \quad [ys(y)]' \geq 0, \quad \text{and} \quad \lim_{y\to\infty} s(y) = 0, \\ &\forall x \geq 0, \quad \phi(x) > 0, \quad \phi'(x) \leq 0, \quad \text{and} \quad [x\phi(x)]' \geq 0. \end{aligned} \tag{2}$$

and

$$m_1 > 0, \quad m_2 > 0, \quad 0 < c < 1. \tag{3}$$

We refer the readers to [22] for more details of the model (1). It is clear that the region $\Omega = \mathbb{R}^2_+$ is a positively invariant set of the model (1). Qualitative dynamical properties of the model (1) are given in the following results.

Lemma 1. *(The existence of equilibria [22, Proposition 1]) The predator-prey model* (1) *has four distinct kinds of possible equilibrium points in the set* Ω*:*

(i) A trivial equilibrium point $P_0^* = (x_0^*, y_0^*) = (0, 0)$*, for all the values of the parameter.*

(ii) An equilibrium point of the form $P_1^* = (x_1^*, y_1^*) = (K, 0)$*, with* $r(K) = m_1$*, if and only if* $m_1 < r(0)$*.*

(iii) An equilibrium point of the form $P_2^* = (x_2^*, y_2^*) = (0, M)$*, with* $s(M) = m_2$*, if and only if* $m_2 < s(0)$*.*

(iv) *An equilibrium point of the form* $P_3^* = (x_3^*, y_3^*) = (x^*, y^*)$, *where* x^* *satisfies the equation*

$$cx^*\phi(x^*) + s\Big(\frac{r(x^*) - m_1}{\phi(x^*)}\Big) - m_2 = 0,$$

and y^* *is given, as a function of* x^*, *by*

$$y^* = \frac{r(x^*) - m_1}{\phi(x^*)},$$

if and only if (m_1, m_2) *verifies* $m_1 < r(0) - M\phi(0)$ *and* $m_2 < s(0)$ *or* $m_1 < r(0)$ *and* $s(0) < m_2 < s(0) + cK\phi(K)$.

Theorem 1. *(Stability analysis [22])*

(i) *If* $m_1 > r(0)$ *and* $m_2 > s(0)$, *then the extinction equilibrium point* $P_0^* = (0, 0)$ *is locally asymptotically stable, and unstable otherwise.*

(ii) *If* $m_1 \geq r(0)$ *and* $m_2 \geq s(0)$ *then the extinction equilibrium point* P_0^* *is globally asymptotically stable.*

(iii) *If* $m_1 < r(0)$ *and* $m_2 > s(0) + cK\phi(K)$, *then the equilibrium point of the form* $P_1^* = (K, 0)$ *is locally asymptotically stable, and unstable otherwise. The equilibrium point* $(K, 0)$ *is called the equilibrium point of extinction of the predator species.*

(iv) *If* $m_1 > r(0) - M\phi(0)$ *and* $m_2 < s(0)$, *then the equilibrium point of the form* $P_2^* = (0, M)$ *is locally asymptotically stable, and unstable otherwise. The equilibrium point* $(0, M)$ *is called the equilibrium point of extinction of the prey species.*

(v) *If an equilibrium point of the form* $P_3^* = (x^*, y^*)$ *belongs to* Ω, *then it is locally asymptotically stable. This equilibrium point* (x^*, y^*) *is called the ecological stability equilibrium.*

In the next section, we will analyze the GAS of the equilibrium points P_1^*, P_2^* and P_3^*.

3. Global asymptotic stability analysis

In this section, the GAS of the model (1) will be analyzed. For each case of the parameters for the model (1), we denote by $E(P^*)$ the set containing possible equilibrium points but not P^*. For example, when $m_1 < r(0) - M\phi(0)$ and $m_2 < s(0)$ the model (1) has four equilibrium points, which are P_0^*, P_1^*, P_2^* and P_3^*. Hence,

$$\begin{aligned} E(P_0^*) &= \{P_1^*,\ P_2^*,\ P_3^*\}, \\ E(P_1^*) &= \{P_0^*,\ P_2^*,\ P_3^*\}, \\ E(P_2^*) &= \{P_0^*,\ P_1^*,\ P_3^*\}, \\ E(P_3^*) &= \{P_0^*,\ P_1^*,\ P_2^*\}. \end{aligned}$$

3.1. GAS analysis for the equilibrium point of extinction of the predator species

Note that the condition for the existence of the equilibrium point of extinction of the predator species $P_1^* = (K, 0)$ is $m_1 < r(0)$, where K is the unique positive solution of the equation $r(x) = K$.

Lemma 2. *Consider the model* (1) *in the case* $m_1 < r(0)$. *Then*

(i) The set

$$\Omega_K = \Big\{(x, y) \in \Omega \Big| x \le K,\ y \ge 0\Big\}$$

is a positively invariant set of the model (1).

(ii) In addition, if $m_2 > s(0) + cK\phi(K)$ *then* $P_1^* = (K, 0)$ *is globally asymptotically stable with respect to the set* $\Omega_K - E(P_1^*)$.

Proof. **Proof of Part (i).** In order to show Ω_K is a positively invariant set of the model (1) we need to show that $\big(x(t), y(t)\big) \in \Omega_k$ for all $t > 0$ whenever $\big(x(0), y(0)\big) \in \Omega_K$, which is equivalent to $x(t) \le K$ and $y(t) \ge 0$ for all $t > 0$. Note that Ω is a positively invariant set of the model (1); hence, $y(t) \ge 0$ for all $t > 0$.

We introduce a new variable $z(t) := K - x(t)$ for $t \geq 0$. From (1) we obtain a new system for z and y

$$\begin{aligned} \dot{z} &= -(K-z)\big[r(K-z) - y\phi(K-z) - m_1\big], \\ \dot{y} &= y\big[s(y(t)) + c(K-z)\phi(K-z) - m_2\big], \end{aligned} \tag{4}$$

which implies that

$$\dot{z}|_{z=0} = -K[r(K) - y\phi(K) - m_1] = Ky\phi(K) + K[m_1 - r(K)] = Ky\phi(K) \geq 0,$$

$$\dot{z}|_{y=0} = 0.$$

Hence, by Proposition B.7 in [32], we have that $z(t) \geq 0$ and $y(t) \geq 0$ whenever $z(0) \geq 0$ and $y(0) \geq 0$. Consequently, $x(t) \leq K$ for all $t > 0$.

Proof of Part (ii). First, the assumption $m_2 > s(0) + cK\phi(K)$ implies that $m_2 > s(0) \geq s(y)$ for all $y \geq 0$. Since $[x\phi(x)]' \geq 0$, for all $(x, y) \in \Omega_K$ we have

$$g(x, y) = s(y) + cx\phi(x) - m_2 \leq s(0) + cK\phi(K) - m_2 < 0. \tag{5}$$

To show the GAS of P_1^* with respect to Ω_K under the condition $m_2 > s(0) + cK\phi(K)$ we consider a Lyapunov function $V : \Omega_K \to \mathbb{R}_+$ given by

$$V(x, y) = \frac{1}{2}y^2. \tag{6}$$

It is clear that $V(x, y)$ is continuously differentiable and positive definite. Moreover, the derivative of $V(x, y)$ along solutions of (1) is

$$\frac{dV}{dt} = y\dot{y} = y^2 g(x, y).$$

By (5), we conclude that $dV/dt \leq 0$ for all $(x, y) \in \Omega_K$ and

$$E := \Big\{(x, y) \in \Omega_K | \dot{V}(x, y) = 0\Big\} \equiv \big\{(x, y) \in \Omega_K | y = 0\big\}.$$

Let us denote by M_E the largest invariant set in E. Then $M_E = E$. In M_E, it is sufficient to consider the sub-system of the system (1)

$$\dot{x} = x\big[r(x) - m_1\big]. \tag{7}$$

Since $r'(x) < 0$ and $m_1 < r(0)$, the equation (7) has a unique positive equilibrium point $x_* = K$. Consider a Lyapunov function defined by

$$L(x) = x - K - K \ln \frac{x}{K}. \tag{8}$$

Then

$$\dot{L}(x) = \frac{x-K}{x}\dot{x} = (x-K)\big[r(x)-m_1\big].$$

Because $r'(x) < 0$, $\dot{L}(x) < 0$ for all $x \geq 0$ except for $x = K$, where $\dot{L}(x) = 0$. Consequently, by the Lyapunov stability theorem we have $x_* = K$ is globally asymptotically stability. Hence, $\lim_{t\to\infty} x(t) = K$.

Now using LaSalle invariant principle [23, 19] and the local asymptotic stability of P_1^*, we conclude that P_1^* is globally asymptotically stable. □

Lemma 3. *Consider the model* (1) *in the case* $m_1 < r(0)$. *If* $m_2 > s(0) + cK\phi(K)$, *then the equilibrium point* P_1^* *is globally asymptotically stable with respect to the set* $\Omega^K - E(P_1^*)$, *where* Ω^K *is defined by*

$$\Omega^K = \Big\{(x,y) \in \Omega \big| x \geq K,\ y \geq 0\Big\}. \tag{9}$$

Proof. From Case (iii) of Theorem 1, we only need to show that $P_1^* = (K, 0)$ is globally attractive with respect to $\Omega^K - E(P_1^*)$, i.e, for any $\big(x(0), y(0)\big) \in \Omega^K$ the solution $\big(x(t), y(t)\big)$ satisfies $\lim_{t\to\infty}\big(x(t), y(t)\big) = (K, 0)$. Indeed, we consider two following cases.

Case 1. There exists a positive number $t_0 > 0$ such that $x(t_0) < K$.

In this case $\big(x(t_0), y(t_0)\big) \in \Omega_K$. Then, by resetting the initial data at $\big(x(t_0), y(t_0)\big)$ and using Lemma 2 we obtain $\lim_{t\to\infty} x(t) = K$ and $\lim_{t\to\infty} y(t) = 0$.

Case 2. $x(t) \geq K$ for all $t > 0$.

In this case, we have $\big(x(t), y(t)\big) \in \Omega^K$ for all $t > 0$. Then, for any $t > 0$

$$\dot{x} = x[r(x) - y\phi(x) - m_1] \leq x[r(K) - m_1] - xy\phi(x) = -x(t)y(t)\phi(x(t)) \leq 0,$$

which implies that $x(t)$ is decreasing and bounded from below by K. On the other hand, if we set $u(t) = x(t) + y(t)$, then

$$\begin{aligned}
\dot{u} = \dot{x} + \dot{y} &= x[r(x) - m_1] + y[s(y) - m_2] + (c-1)xy\phi(x) \\
&\leq x[r(K) - m_1] + y[s(0) - m_2] + (c-1)x(t)y(t)\phi(x) \\
&= y[s(0) - m_2] + (c-1)xy\phi(x) \\
&< y[s(0) + cK\phi(K) - m_2] + (c-1)xy\phi(x) \leq 0.
\end{aligned}$$

Hence, $u(t)$ is also decreasing and bounded from below.

Since $x(t)$ and $u(t) = x(t) + y(t)$ are bounded and decreasing functions, $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ exist. Assume that $\lim_{t\to\infty}(x(t), y(t)) = (e_1, e_2)$. Obviously, (e_1, e_2) must be an equilibrium point of the model (1).

We recall that if $m_1 < s(0)$ and $m_2 > s(0) + cK\phi(K)$ then the equilibrium point $(0, 0)$ is unstable and there exists a unique equilibrium point of the form $(K, 0)$ which is locally asymptotically stable. In this case, the model (1) has no equilibrium points of the form $(0, M)$ or (x^*, y^*). However, $x(t)$ is bounded from below by K. So, $(e_1, e_2) \neq (0, 0)$. This implies that $(e_1, e_2) = (K, 0)$.

From Theorem 1, $P_1^* = (K, 0)$ is locally asymptotically stable if $m_1 < s(0)$ and $m_2 > s(0) + cK\phi(K)$. Combining this with the global attraction of P_1^*, we obtain its GAS. $\square$

Because $\Omega = \Omega^K \bigcup \Omega_k$, from Lemmas 2 and 3 we obtain the complete GAS of P_1^*.

Theorem 2. *The equilibrium point of extinction of the predator species P_1^* is locally asymptotically stable with respect to the set $\Omega - E(P_1^*)$ if $m_1 < s(0)$ and $m_2 > s(0) + cK\phi(K)$.*

3.2. GAS analysis for the equilibrium point of extinction of the predator species

We recall that $P_2^* = (0, M)$ exists only if $m_2 < s(0)$, where M is the unique positive solution of the equation $s(M) = K$.

Lemma 4. *Consider the model* (1) *in the case $m_2 < s(0)$. Then*

(i) The set

$$\Omega^M = \Big\{(x, y) \in \Omega \big| x \geq 0,\ y \geq M\Big\}$$

is a positively invariant set of the model (1).

(ii) Additionally, if $m_1 > s(0) - M\phi(0)$ then $P_2^* = (K, 0)$ is globally asymptotically stable with respect to the set $\Omega^M - E(P_2^*)$.

Proof. **Proof of Part (i)**. In order to show Ω^M is a positively invariant set of the model (1), we need to show that $\big(x(t), y(t)\big) \in \Omega^M$ for all $t > 0$ whenever $\big(x(0), y(0)\big) \in \Omega^M$. Since Ω is a positively invariant set of the model (1), $x(t) \geq 0$ for all $t > 0$.

Set $v(t) := y(t) - M$ for $t \geq 0$. The system (1) implies that

$$\begin{aligned} \dot{x} &= x\big[r(x) - (v+M)\phi(x) - m_1\big], \\ \dot{v} &= (v+M)\big[s(v+M) + cx\phi(x) - m_2\big]. \end{aligned} \tag{10}$$

Consequently,

$$\begin{aligned} \dot{x}|_{x=0} &= 0, \\ \dot{v}|_{v=0} &= M\big[s(M) + cx\phi(x) - m_2\big] = cx\phi(x) \geq 0. \end{aligned}$$

Proposition B.7 in [32] implies that $v(t) \geq 0$ for all $t > 0$ if $v(0) \geq 0$. Therefore, $y(t) \geq M$ for all $t \geq 0$.

Proof of Part (ii). Since $m_1 > r(0) - M\phi(0)$, $f_x(x,y) < 0$ and $f_y(x,y) < 0$, for $(x,y) \in \Omega^M$ we have

$$\begin{aligned} f(x,y) &= r(x) - y\phi(x) - m_2 \leq f(0,y) \\ &= r(0) - y\phi(0) - m_1 \leq r(0) - M\phi(0) - m_1 < 0. \end{aligned} \tag{11}$$

To prove the GAS of P_2^* with respect to Ω^M under the assumption $m_1 > r(0) - M\phi(M)$ we consider a Lyapunov function $V : \Omega_K \to \mathbb{R}$ defined by

$$V(x,y) = \frac{1}{2}x^2. \tag{12}$$

The derivative of $V(x,y)$ along solutions of (1) is

$$\frac{dV}{dt} = x\dot{x} = x^2 f(x,y).$$

The estimate (11) imply that $\dot{V}(x,y) \leq 0$ for all $(x,y) \in \Omega^M$ and $\dot{V} = 0$ if and only if $x = 0$.

Repeating the proof of Part (ii) of Lemma 2, it suffices to consider the following sub-system of (1)

$$\dot{y} = y\big[s(y) - m_2\big]. \tag{13}$$

Since $m_2 < s(0)$ and $s'(y) < 0$, the equation (13) has a unique positive equilibrium $y_* = M$ and it is globally asymptotically stable. Hence, $\lim_{t\to\infty} y(t) = M$. Then, by using LaSalle invariant principle [23, 19] and the local asymptotic stability of P_2^*, we conclude that P_2^* is globally asymptotically stable. $\square$

Lemma 5. *Consider model* (1) *in the case* $m_2 < s(0)$. *If* $m_1 > r(0) - M\phi(0)$, *then the equilibrium point* P_2^* *is globally asymptotically stable with respect to the set* $\Omega_M - E(P_2^*)$, *where* Ω^M *is defined by*

$$\Omega_M = \Big\{(x, y) \in \Omega \big| x \geq 0,\ y \leq M\Big\}.$$

Proof. Thanks to Part (iv) of Theorem 1, it is sufficient to prove that P_2^* is globally attractive with respect to $\Omega_M - E(P_2^*)$. We need to show that for any $\big(x(0), y(0)\big) \in \Omega_M$, the solution $\big(x(t), y(t)\big)$ satisfies $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = M$. Indeed, consider two following cases.

Case 1. There exists $t_0 > 0$ for which $y(t_0) > M$.

In this case $\big(x(t_0), y(t_0)\big) \in \Omega^M$. Then by resetting the initial data at $\big(x(t_0), y(t_0)\big)$ and using Lemma 4 we obtain $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = M$.

Case 2. $y(t) \leq M$ for all $t > 0$.

In this case $(x(t), y(t)) \in \Omega_M$ for all $t > 0$. Consequently,

$$\dot{y} = y[s(y) + cx\phi(x) - m_2] \geq y[s(M) - m_2] + cxy\phi(x) = cx(t)y\phi(x) \geq 0,$$

which implies that $y(t)$ is increasing and bounded from above by M. Hence, $\lim_{t\to\infty} y(t)$ exists. We will prove that $\lim_{t\to\infty} x(t)$ also exists. Consider the following sub-cases.

(i) If $m_1 \geq r(0)$, then

$$\dot{x} = x[r(x) - y\phi(x) - m_1] \leq x[r(0) - m_1 - y\phi(x)] \leq -xy\phi(x) \leq 0,$$

which means that $x(t)$ is decreasing and bounded from below. Hence, $\lim_{t\to\infty} x(t)$ exists.

(ii) If $m_1 < r(0)$, then the equilibrium point $P_1^* = (K, 0)$ exists. By Lemma 2, Ω_K is a positively invariant set of (1); consequently, if $x(0) \leq K$ then

$x(t) \le K$ for all $t > 0$ and hence,

$$\begin{aligned}\dot{x} + \frac{1}{c}\dot{y} &= x\big[r(x) - y\phi(x) - m_1\big] + \frac{1}{c}y\big[s(y) + cx\phi(x) - m_2\big]\\ &= x\big[r(x) - m_1\big] + \frac{1}{c}\big[s(y) - m_2\big]\\ &> x[r(K) - m_1] + \frac{1}{c}[s(M) - m_2] = 0.\end{aligned}$$

Hence, $x(t) + \frac{1}{c}y(t)$ is increasing and bounded from below by $K + \frac{M}{c}$. This implies that the function $x(t) + \frac{1}{c}y(t)$ has a finite limit as $t \to \infty$. Therefore, $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ exist.

Otherwise, if $x(0) > K$ then it is sufficient to consider $x(t) \ge K$ for all $t > 0$ (*). Then,

$$\dot{x} = x[r(x) - y\phi(x) - m_1] \le x[r(K) - m_1 - y\phi(x)] = -xy\phi(y) \le 0.$$

Hence, $x(t)$ is decreasing and bounded from below by K. Consequently, $\lim_{t\to\infty} x(t)$ exists.

So, we have proved that $x(t)$ and $y(t)$ have finite limits as $t \to \infty$. Suppose $\lim_{t\to\infty}(x(t), y(t)) = (e_1, e_2)$. Obviously, (e_1, e_2) must be an equilibrium point of (1).

It is important to note that if $m_2 < s(0)$ and $r(0) - M\phi(0) < m_1 < r(0)$ then the equilibrium point $(0, 0)$ is unstable and there exist a unique equilibrium point of the form $(K, 0)$ and a unique equilibrium point of the form $(0, M)$ both in Ω being $(K, 0)$ unstable and $(0, M)$ locally asymptotically stable. In this case, the system (1) has no equilibrium point of the form $(x^*, y^*) \in \Omega$. Since $y(t)$ is increasing and bounded from above by M, $(e_1, e_2) \ne (K, 0)$ and $(e_1, e_2) \ne (0, 0)$. On the other hand, if $m_1 > r(0)$ and $m_2 < s(0)$, then the extinction equilibrium point $(0, 0)$ is unstable and also there exists a unique equilibrium point of the form $(0, M)$, which is locally asymptotically stable. In this case, the system (1) has no equilibrium point of the form $(K, 0)$ or (x^*, y^*) in Ω. Thus, $(e_1, e_2) \ne (0, 0)$.

Hence, we conclude that $(e_1, e_2) = (0, M)$ is globally attractive. This is the desired conclusion. $\square$

Remark 1. *In* $(*)$ *we supposed that* $x(t) \geq K$ *for all* $t > 0$ *to show that* $\lim_{t\to\infty} x(t)$ *exists. However, we also concluded that if* $\lim_{t\to\infty} x(t)$ *exists then* $\lim_{t\to\infty} x(t) = 0$*. Hence,* $(*)$ *cannot occur, i.e, there is no solution* $x(t)$ *satisfying* $x(t) \geq K$ *for all* $t > 0$*.*

Since $\Omega = \Omega^M \bigcup \Omega_M$, combining Lemmas 4 and 5 we obtain the complete GAS of P_2^*.

Theorem 3. *The equilibrium point of extinction of the predator species* $P_2^* = (0, M)$ *is globally asymptotically stable with respect to the set* $\Omega - E(P_2^*)$ *if* $m_2 < s(0)$ *and* $m_2 > s(0) + cK\phi(K)$ *.*

3.3. Global asymptotic stability of the ecological stability equilibrium

In this subsection, the GAS of the equilibrium point P_3^* is analyzed. Generally, it is very difficult to investigate the GAS of P_3^* with general functional response and recruitment; hence, we will consider an essential case for the functional response and recruitment. More precisely, we assume that a recruitment of the form Beverton-Holt for both species and a predator functional response Holling type II. In this sense, the functions representing the per capita recruitment rates of prey and predators and the predator functional response are given by [22]

$$\begin{aligned} xr(x) &= \frac{\alpha_1 x}{\beta_1 + x}, \quad \alpha_1,\ \beta_1 > 0, \\ x\phi(x) &= \frac{\alpha_2 x}{\beta_2 + x}, \quad \alpha_2,\ \beta_2 > 0, \\ ys(y) &= \frac{\alpha_3 y}{\beta_3 + y}, \quad \alpha_3,\ \beta_3 > 0, \end{aligned} \tag{14}$$

respectively. In [22], a specific case of (14) was considered for numerical simulations. Importantly, the GAS of P_3^* for this case was also confirmed by numerical studies in [8, 22].

Theorem 4. *Consider the predator-prey model* (1) *with a recruitment of the form Beverton-Holt for both species and a predator functional response Holling* *type II:*

$$\begin{aligned}\dot{x} &= x\left(\frac{\alpha_1}{\beta_1+x}-\frac{\alpha_2 y}{\beta_2+x}-m_1\right),\\ \dot{y} &= y\left(\frac{\alpha_3}{\beta_3+y}+c\frac{\alpha_2 x}{\beta_2+x}-m_2\right).\end{aligned}\tag{15}$$

Then the ecological stability equilibrium point $P_3^*=(x^*,y^*)$ is globally asymptotically stable with respect to the set $\Omega - E(P_3^*)$ whenever it exists.

Proof. Since $P_3^*=(x^*,y^*)$ is the positive equilibrium point, we have

$$\begin{aligned}m_1 &= \frac{\alpha_1}{\beta_1+x^*}-\frac{\alpha_2 y^*}{\beta_2+x^*},\\ m_2 &= \frac{\alpha_3}{\beta_3+y^*}+c\frac{\alpha_2 x^*}{\beta_2+x^*}.\end{aligned}\tag{16}$$

Consequently, the model (15) can be written in the form

$$\begin{aligned}\dot{x} &= x\left(\frac{\alpha_1}{\beta_1+x}-\frac{\alpha_2 y}{\beta_2+x}-\frac{\alpha_1}{\beta_1+x^*}+\frac{\alpha_2 y^*}{\beta_2+x^*}\right),\\ \dot{y} &= y\left(\frac{\alpha_3}{\beta_3+y}+c\frac{\alpha_2 x}{\beta_2+x}-\frac{\alpha_3}{\beta_3+y^*}-c\frac{\alpha_2 x^*}{\beta_2+x^*}\right),\end{aligned}\tag{17}$$

or equivalently to

$$\begin{aligned}\dot{x} &= x\left[\left(\frac{\alpha_1}{(\beta_1+x)(\beta_1+x^*)}-\frac{\alpha_2 y^*}{(\beta_2+x)(\beta_2+x^*)}\right)(x^*-x)-\frac{\alpha_2\beta_2+\alpha_2 x^*}{(\beta_2+x)(\beta_2+x^*)}(y-y^*)\right],\\ \dot{y} &= y\left[\frac{c\alpha_2\beta_2}{(\beta_2+x)(\beta_2+x^*)}(x-x^*)-\frac{\alpha_3}{(\beta_2+x)(\beta_2+x^*)}(y-y^*)\right].\end{aligned}\tag{18}$$

Consider a Lyapunov function of the form

$$V(x,y)=\tau_1\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\tau_2\left(y-y^*-y^*\ln\frac{y}{y^*}\right),\tag{19}$$

where τ_1 and τ_2 are undetermined positive real numbers. From (18) and (19), the derivative of V along solutions of (14) satisfies

$$\begin{aligned}
\dot{V} &= \frac{x-x^*}{x}\dot{x} + \frac{y-y^*}{y}\dot{y} \\
&= \left[\left(\frac{\alpha_1}{(\beta_1+x)(\beta_1+x^*)} - \frac{\alpha_2 y^*}{(\beta_2+x)(\beta_2+x^*)}\right)(x^*-x) - \frac{\alpha_2\beta_2+\alpha_2 x^*}{(\beta_2+x)(\beta_2+x^*)}(y-y^*)\right](x-x^*) \\
&+ \left[\frac{c\alpha_2\beta_2}{(\beta_2+x)(\beta_2+x^*)}(x-x^*) - \frac{\alpha_3}{(\beta_2+x)(\beta_2+x^*)}(y-y^*)\right](y-y^*) \\
&= -\tau_1\left[\frac{\alpha_1}{(\beta_1+x)(\beta_1+x^*)} - \frac{\alpha_2 y^*}{(\beta_2+x)(\beta_2+x^*)}\right](x^*-x)^2 - \tau_2\frac{\alpha_3}{(\beta_2+x)(\beta_2+x^*)}(y-y^*)^2 \\
&+ \left[-\tau_1\frac{\alpha_2\beta_2+\alpha_2 x^*}{(\beta_2+x)(\beta_2+x^*)} + \tau_2\frac{c\alpha_2\beta_2}{(\beta_2+x)(\beta_2+x^*)}\right](x-x^*)(y-y^*).
\end{aligned} \tag{20}$$

If τ_1 and τ_2 satisfy

$$\tau_1(\alpha_2\beta_2+\alpha_2 x^*) = \tau_2 c\alpha_2\beta_2,$$

then

$$\begin{aligned}
\dot{V} = -\tau_1&\left[\frac{\alpha_1}{(\beta_1+x)(\beta_1+x^*)} - \frac{\alpha_2 y^*}{(\beta_2+x)(\beta_2+x^*)}\right](x-x^*)^2 \\
-\tau_2&\frac{\alpha_3}{(\beta_2+x)(\beta_2+x^*)}(y-y^*)^2.
\end{aligned} \tag{21}$$

From the hypothesis $f_x(x,y) = r'(x) - y\phi'(x) < 0$ for all $x, y \geq 0$ we obtain

$$\frac{\alpha_1}{(\beta_1+x)(\beta_1+x^*)} - \frac{\alpha_2 y^*}{(\beta_2+x)(\beta_2+x^*)} > 0.$$

This implies that the function V satisfies the Lyapunov stability theorem. Consequently, the GAS of P_3^* is confirmed. □

Remark 2. *The GAS of the model* (1) *with general functional response and recruitment can be established similarly to Theorem 4. Numerical examples in the next section will show the GAS of the model* (1) *with some general functional response and recruitment.*

4. Numerical examples

In this section, we report two numerical examples to illustrate the theoretical results.

Example 1 (The predator-prey model with a recruitment of the form Beverton-Holt for both species and a predator functional response Ivlev type).

Consider the predator-prey model (1) with a recruitment of the form Beverton-Holt for both species and a predator functional response Ivlev type

$$xr(x) = \frac{15x}{x+10}, \quad ys(y) = \frac{5y}{y+10}, \quad x\phi(x) = \frac{1-e^{-x}}{30},$$

with $c = 0.003$ and 6 cases of the parameters of m_1 and m_2 which correspond to 6 cases listed in Corollary 1 and Figure 2 in [22]. We use the classical fourth-order Runge-Kutta method [4] with step size $h = 10^{-5}$ to solve the model (1) on the interval $t \in [0, 500]$.

Table 1: The parameters m_1 and m_2 in Example 1.

Case	m_1	m_2	Source	Verified conditions	GAS equilibrium point
1	1.53	0.622	[22]	$m_1 > r(0)$ and $m_2 > s(0)$	$P_1^* = (0, 0)$
2	1.53	0.4789	[22]	$m_1 > r(0)$ and $m_2 < s(0)$	$P_2^* = (0, 0.4406)$
3	1.4925	0.4789	[22]	$m_2 < s(0)$ and $r(0) - M\phi(0) < m_1 < r(0)$	$P_2^* = (0, 0.4406)$
4	1.38	0.4789	[22]	$m_2 < s(0)$ and $m_1 < r(0) - M\phi(0)$	$P_3^* = (0.80, 0.44)$
5	0.3	0.501	[22]	$m_1 < r(0)$ and $s(0) < m_2 < s(0) + cK\phi(K)$	$P_3^* = (39.6, 0.06)$
6	1.38	0.622	[22]	$m_1 < r(0)$ and $m_2 > s(0) + cK\phi(K)$	$P_1^* = (0.86, 0)$

Phase planes for the predator-prey model that correspond to 6 cases of (m_1, m_2) are depicted in Figures 1-6, respectively. In all the figures, each blue curve represents a phase plane corresponding to a specific initial data, the green circle represents the globally asymptotically stable equilibrium point (GAS equilibrium point) and the red arrows represent the evolution of two species (predator x and prey y).

It is clear that all the solutions are stable and converge to the GAS equilibrium points. In other words, the GAS of the predator-prey (1) is confirmed in this example.

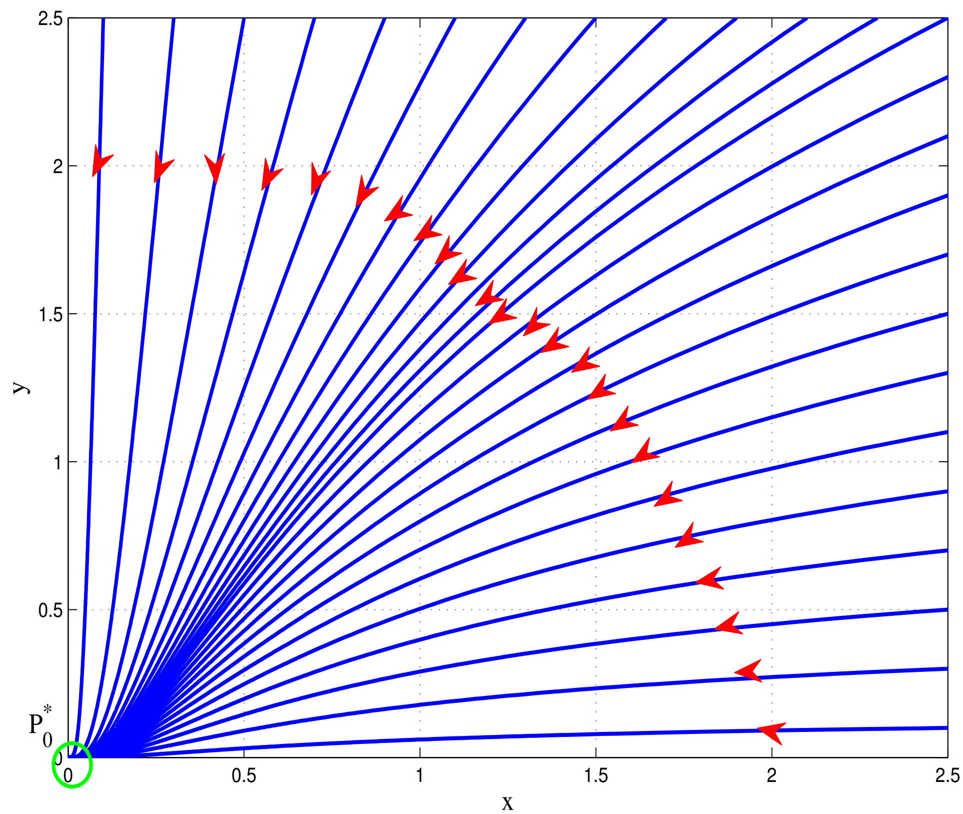

Figure 1: The phase planes of the predator-prey for Case 1 of Example 1.

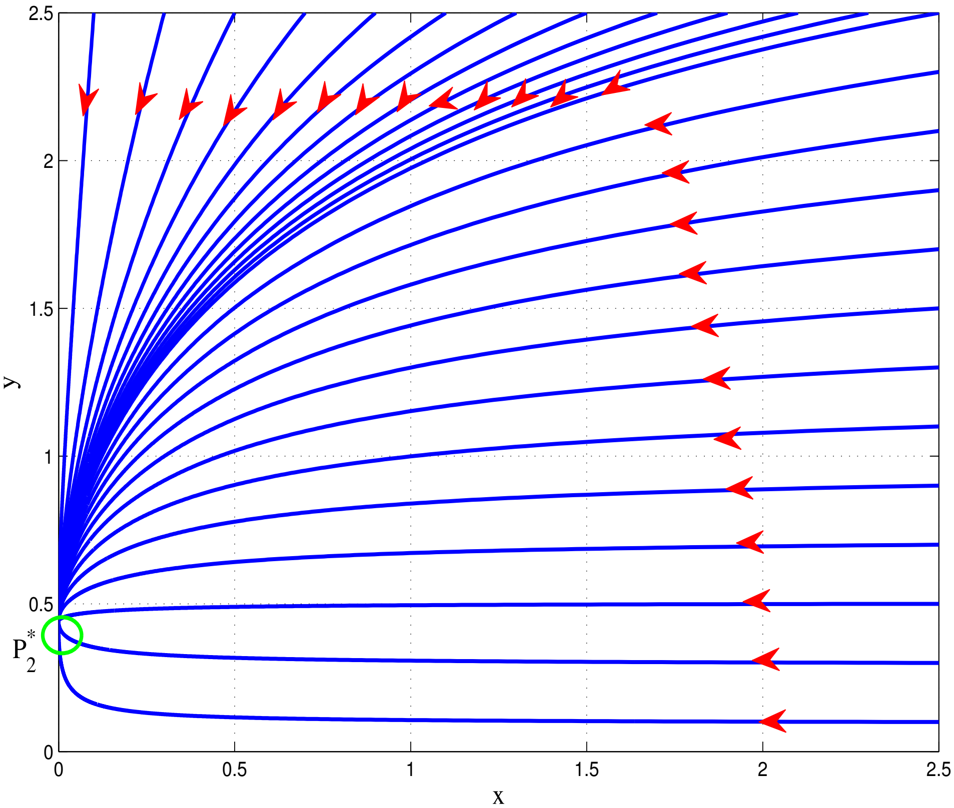

Figure 2: The phase planes of the predator-prey model for Case 2 of Example 1.

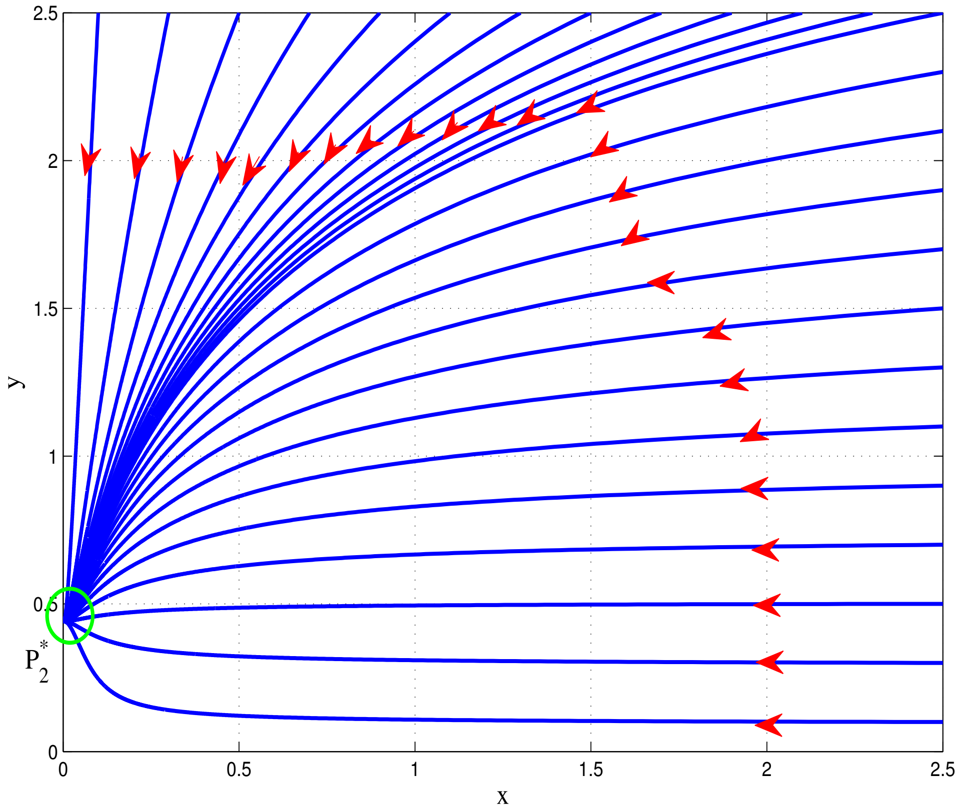

Figure 3: The phase planes of the predator-prey model for Case 3 of Example 1.

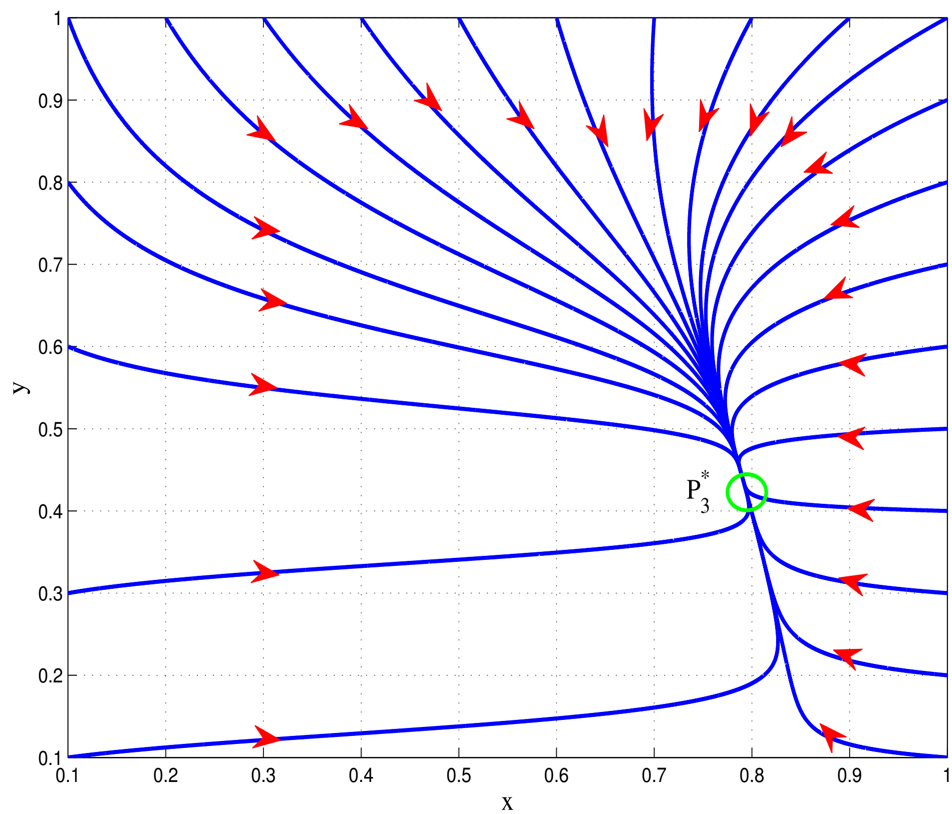

Figure 4: The phase planes of the predator-prey model for Case 4 of Example 1.

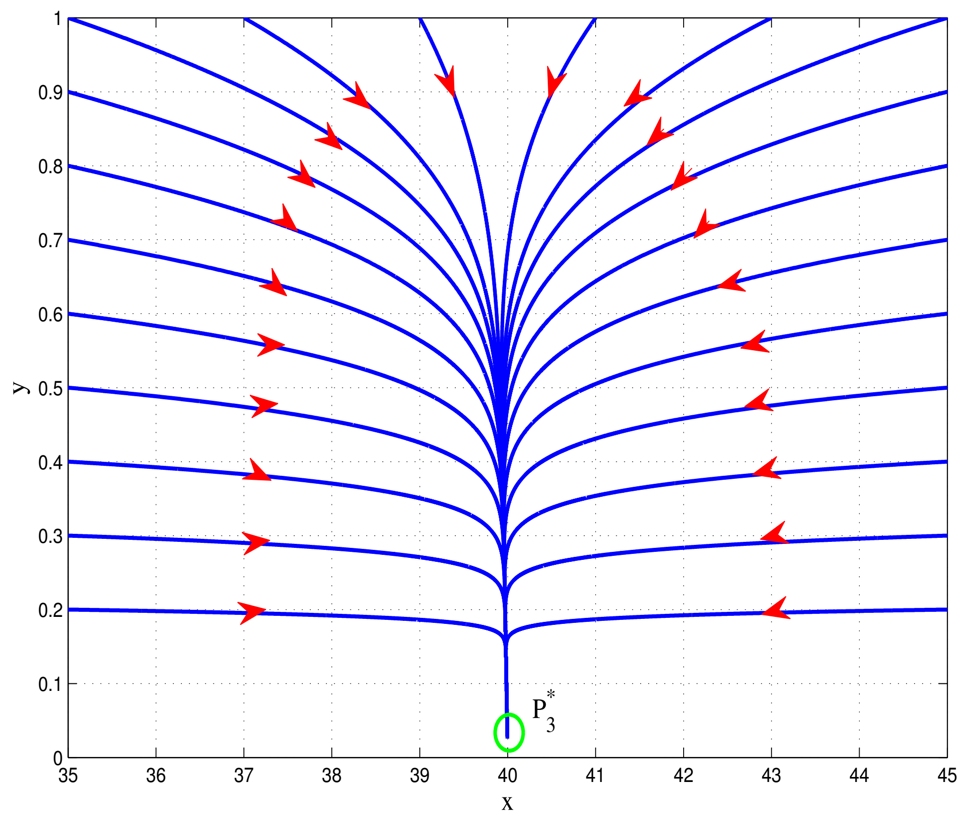

Figure 5: The phase planes of the predator-prey model for Case 5 of Example 1.

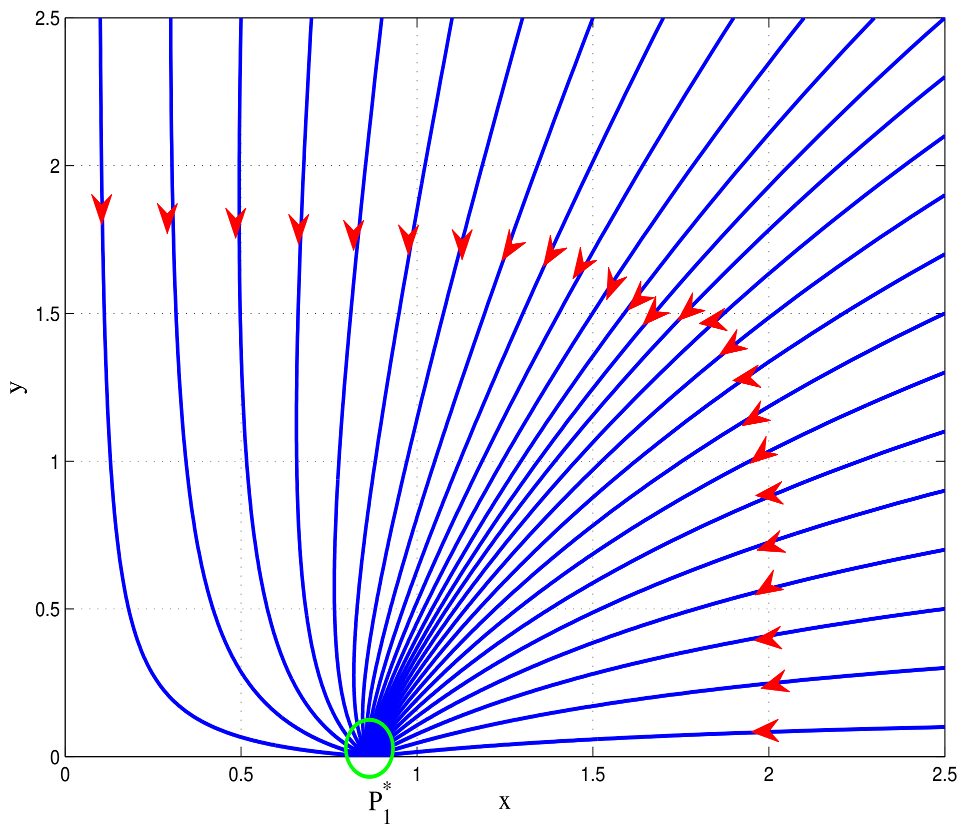

Figure 6: The phase planes of the predator-prey model for Case 6 of Example 1.

Example 2 (The predator-prey model with a recruitment of the form Beverton-Holt for both species and a predator functional response Holling type III)**.**

Consider the model (1) with a recruitment of the form Beverton-Holt for both species and a predator functional response Holling type III

$$xr(x) = \frac{15x}{x+10}, \quad ys(y) = \frac{5y}{y+10}, \quad x\phi(x) = \frac{x^2}{x^2+30},$$

with $c = 0.003$ and 5 cases of the parameters m_1 and m_2 given in Table 2.

Table 2: The parameters (m_1, m_2) in Example 2.

Case	m_1	m_2	Source	Verified conditions	GAS equilibrium point
1	1.53	0.622	[22]	$m_1 > r(0)$ and $m_2 > s(0)$	$P_1^* = (0, 0)$
2	1.53	0.4789	[22]	$m_1 > r(0)$ and $m_2 < s(0)$	$P_2^* = (0, 0.44)$
3	1.38	0.4789	[22]	$m_2 < s(0)$ and $m_1 < r(0) - M\phi(0)$	$P_3^* = (0.78, 0.44)$
4	0.3	0.501	[22]	$m_1 < r(0)$ and $s(0) < m_2 < s(0) + cK\phi(K)$	$P_3^* = (39.8, 0.04)$
5	1.38	0.622	[22]	$m_1 < r(0)$ and $m_2 > s(0) + cK\phi(K)$	$P_1^* = (0.8696, 0)$

Phase planes for the predator-prey model that correspond to 5 cases of (m_1, m_2) are depicted in Figures 7-11, respectively. Similarly to Example 1, the GAS of the predator-prey model is shown in this example. Hence, Examples 1 and 2 provide good illustrations for the theoretical results, especially for the GAS of the predator-prey model.

It is important to note that Examples 1 and 2 suggest that the ecological stability equilibrium point may be globally asymptotically stable for general cases of the recruitment and predator functional response. Therefore, it is reasonable to give the following conjecture.

Conjecture 1. *The ecological stability equilibrium point of the predator-prey model* (1) *is globally asymptotically stable whenever it exists.*

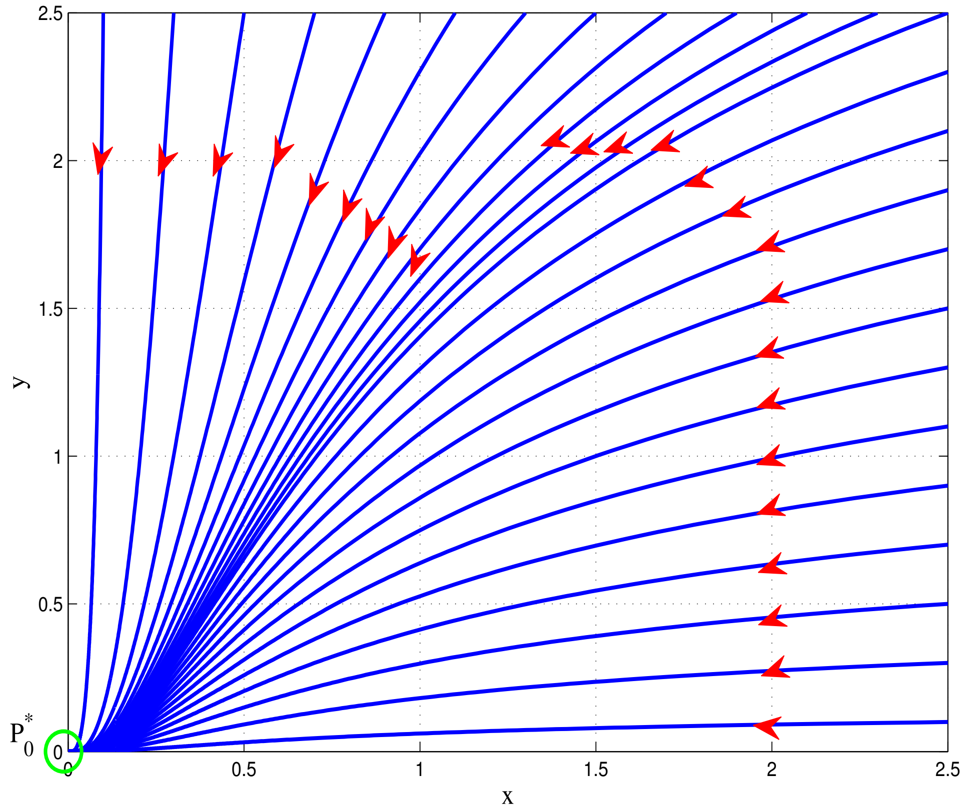

Figure 7: The phase planes of the predator-prey model for Case 1 of Example 2.

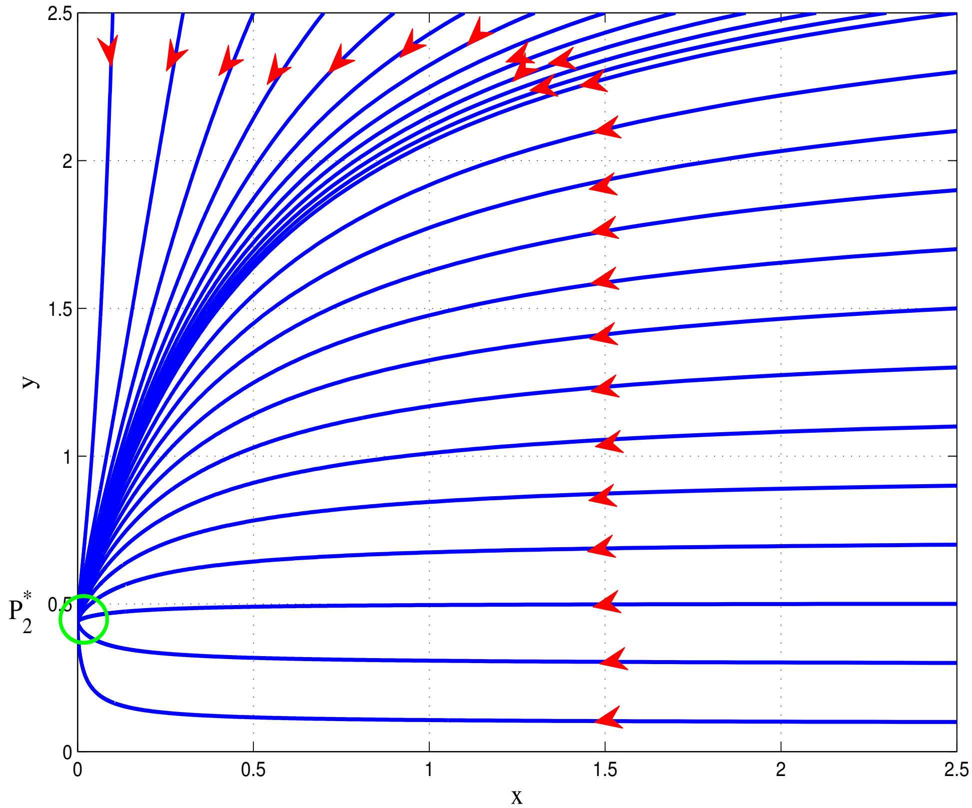

Figure 8: The phase planes of the predator-prey model for Case 2 of Example 2.

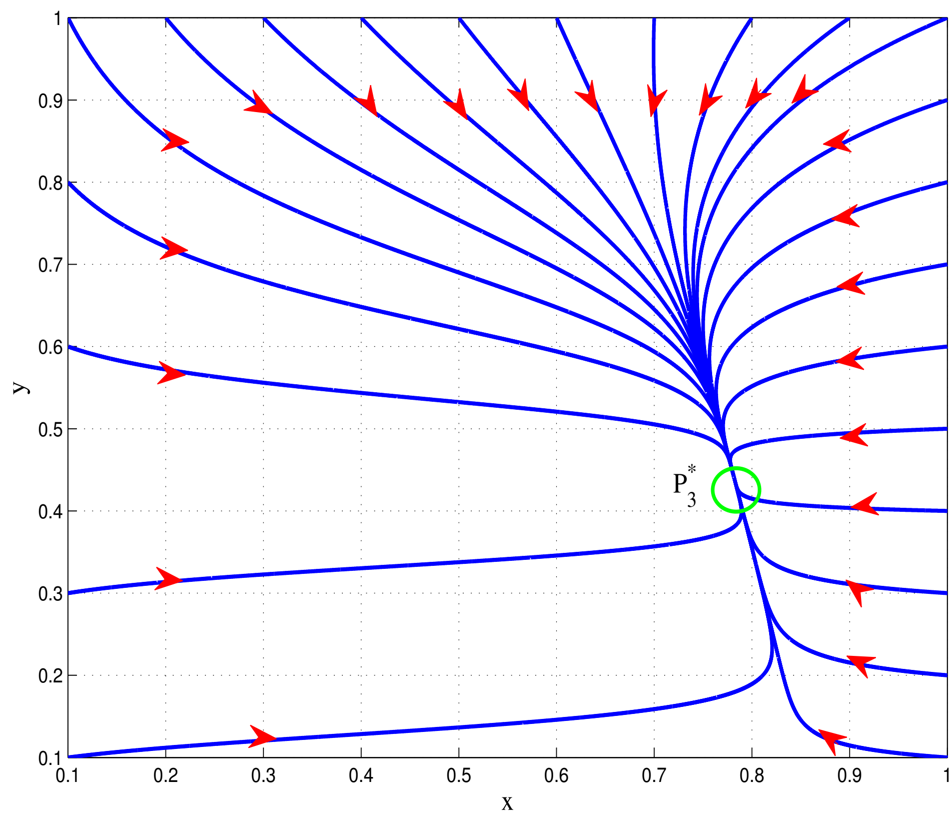

Figure 9: The phase planes of the predator-prey model for Case 3 of Example 2.

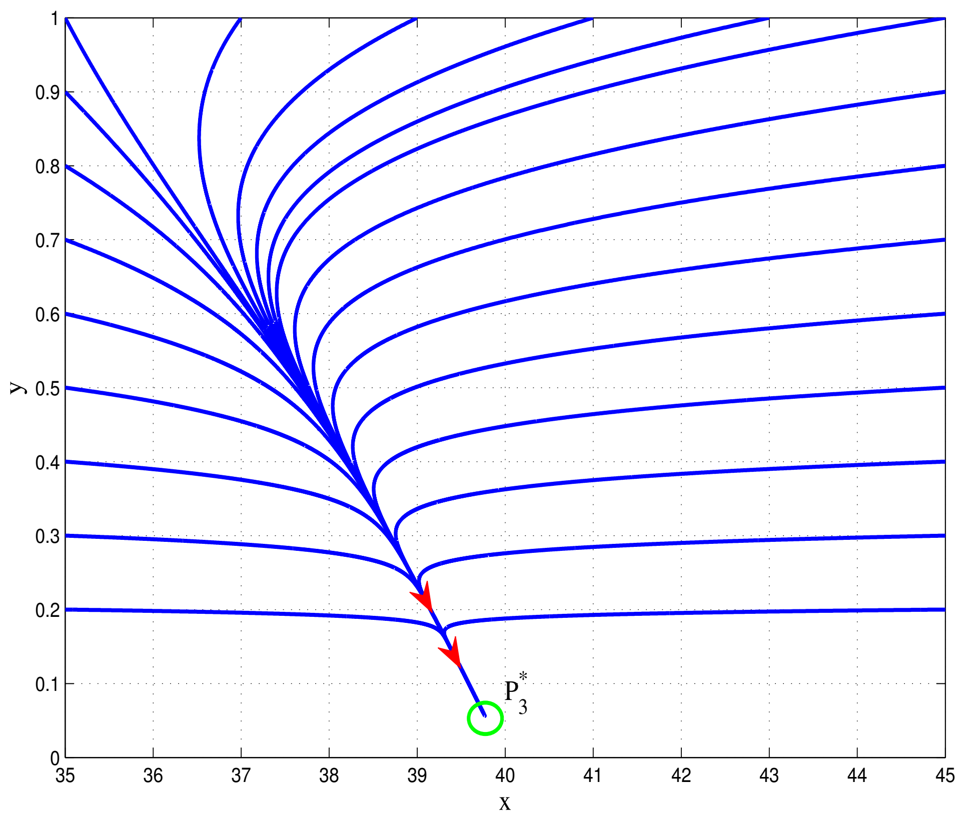

Figure 10: The phase planes of the predator-prey model for Case 4 of Example 2.

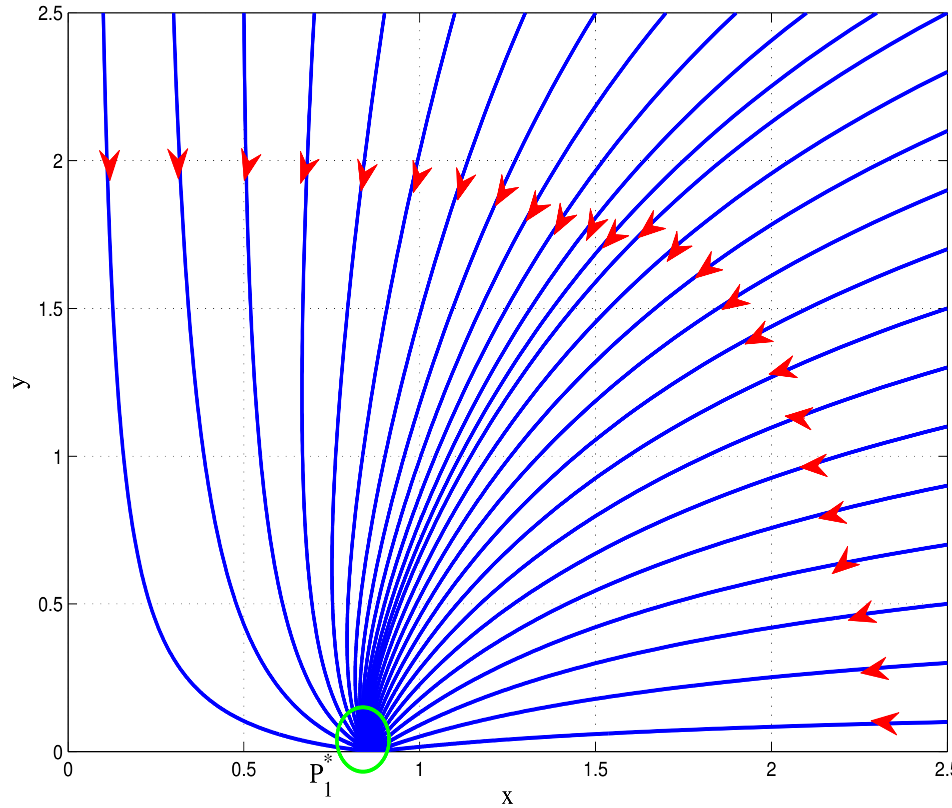

Figure 11: The phase planes of the predator-prey model for Case 5 of Example 2.

5. Some conclusions and open problems

In this work, a rigorously mathematical analysis for the complete GAS of the predator-prey model (1) has been performed. By using the Lyapunov stability theory in combination with some nonstandard techniques of mathematical analysis for dynamical system, we have fully determined the GAS of equilibria of the predator-prey. The obtained results not only improve the results constructed in the benchmark work but also provide an important study for the population dynamics of the predator-prey model. Furthermore, the theoretical results are supported by a set of numerical examples.

Although Theorem 4 only demonstrated the GAS of the ecological stability equilibrium point of the predator-prey model with the recruitment of the form Beverton-Holt for both species and the predator functional response Holling type II (the model (15)), the numerical examples showed the GAS of the model in some general cases. This suggests that the model may be globally asymptotically stable in general cases of the recruitment and predator functional response (see Conjecture 1).

The approach used in this work can be extended to study extensions of the predator-prey model (1) under fractional-order derivatives. For example, we can consider the model (1) in the context of the Caputo fractional derivative:

$$\begin{aligned} \frac{d^\alpha x(t)}{dt} &= x(t)f(x(t),y(t)) = x(t)\big[r(x(t)) - y(t)\phi(x(t)) - m_1\big], \\ \frac{d^\alpha y(t)}{dt} &= y(t)g(x(t),y(t)) = y(t)\big[s(y(t)) + cx(t)\phi(x(t)) - m_2\big], \end{aligned} \tag{22}$$

where $\alpha \in (0,1)$ and $d^\alpha f(t)/dt$ stands for the fractional Caputo derivative of the function $f(t)$ [7]. After that, the GAS of the model (22) can be analyzed by using comparison results [24, 28] and the Lyapunov stability theorem for fractional-order dynamical systems [1, 2, 13]. The GAS problem for the predator-prey model in the context of other fractional-order derivatives can be studied by the same approach.

In the near future, we will analyze the GAS of the model (1) in general cases of recruitment and predator functional response (Conjecture 1). Population dynamics of extended versions of the model predator-prey model (1) in the context of fractional-order derivatives and their applications will be also studied.

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Code availability: Not applicable

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