Received XXXXXX; | Revised XXXXXXX; | Accepted XXXXXX

DOI: xxx/xxxx

ARTICLE TYPE

The relationship between the conservation laws and multi-Hamiltonian structures of the Kundu equation†

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Abstract

By the Lagrangian multiplier and constraint variational derivative, a relationship between conserved quantities and multi-Hamiltonian structures is built. Making using the relation, a method is founded to prove the infinite-dimensional Liouville integrability of evolution equations with continuous variables. As the application, the conservation laws of the Kundu equation are firstly obtained. Its conserved quantities are deduced for comparing by Fokas' method different from the method used in the existed literature. The integrability of the equation is proved through taking the conservation laws as a starting point.

KEYWORDS:
Lagrangian multiplier, constraint variational derivative, Conservation law, multi-Hamiltonian structure

1 | INTRODUCTION

It is well known that the research of the infinite dimensional integrable system is based on the finite dimensional complete one presented by V.I. Arnold and J. Moser. According to the classical Liouville completely integrability theory on a finite dimensional symplectic manifold, if a $2N$-dimensional Hamiltonian structure possesses N involutive independent conserved integrals, it is completely integrable. On soliton equations, there are two different type ways to prove their integrability.

The first one is nonlinearization of Lax pairs[1,2]. From Lax pairs of soliton equations, a class of finite-dimensional Hamiltonian systems is obtained under a constraint between the potential and the eigenfunctions. But for some

†This is an example for title footnote.
[0]**Abbreviations:**CLs, conservation laws

spectral problems, it is not easy to find these constrains. To solve this problem, the constraints called binary non-linearlization were presented from spectral problems and their conjugate ones[3,4]. To solve the resulting Hamiltonian structure, Zhou introduced algebraic geometric method[5] and found finite-band solutions of these systems. Through the given constraint, solutions of the corresponding soliton equation can be obtained, too. Recent research shows that the solutions of these Hamiltonian systems can also be obtained by the bilinear direct method and the Wronskian technique[6-8].

Another method presented by Fokas et al. is through generalizing the concept of finite dimensional integrability to infinite dimensional integrability to prove the integrality of soliton hierarchies[9-13]. In this method, recursion operators of soliton hierarchies play a central role. Usually, these operators must be a hereditary and strong symmetrical. This property ensure that each equation in the isospectral hierarchy is Liouville integrable. When we prove integrability of soliton equations, recursion operators often are factorized into an implectic operator and a symplectic operator. Finally, by the properties of the operator and evolution equations the Hamiltonian structures can be set up. At the same time, the corresponding infinite conserved quantities named Hamiltonian functionals are obtained from the resulting Hamiltonian structures. But it is not very easy to obtain the infinite conserved quantities by this method. Tu presented the trace identities to obtain conserved quantities more easily base on the constraint variational principle[14,15].

There are many approaches to find their conservation laws (CLs), such as the approach in the non-semisimple Lie algebras framework to find generating functions for conserved densities[16,17] by the variational identities, through adjoint symmetries[18-20] and the expansion technique of ratios of eigenfunctions of spectral problems[21,22]. Among them, the most popular one is generating the CLs from Lax pairs[21-23].

Notice that the conserved quantities obtained from the Hamiltonian structures are similar to the conserved densities of the CLs. In this paper, we will find a relationship between CLs and Hamiltonian structures and build a method to prove the infinite-dimensional Liouville integrability of soliton equations with continuous variables. First, from a Riccati equation that a ratio of two eigenfunctions need to satisfy we will derive out the CLs of the Kundu equation. But its conserved quantities have already been obtained through the Tu scheme[25]. For comparison, we will re-derive them by the Fokas' method. Finally, the general method will be constructed to prove the integrability of evolution equations from its conserved densities of the CLs. As its application, the Kundu equation will be considered. We show that the integrability of soliton hierarchy can be proved by either getting the infinite conserved quantities from Hamiltonian structures or establishing the Hamiltonian structures from conserved quantities.

We organize the paper as follows. In section 2, we will recall some basic notions and notations. In section 3, we will re-derive the Kundu hierarchy and present its CLs. In section 4, conserved quantities will be given by the Fokes' method. In section 5, a method to prove the integrability of evolution equations will be constructed and the Kundu equation will be taken as an example. We conclude the paper in section 6.

2 | BASIC NOTIONS

In this section, we recall some notions, notations and propositions used in this paper[9] (see also[12,13]).

Let u be a manifold variable and $\mathbf{M} = \mathbf{M}(u)$ is a suitable manifold, where u is a column vector. Denote the tangent bundles and cotangent bundles on $\mathbf{M}$ by $\mathbf{T}(\mathbf{M})$ and $\mathbf{T}^*(\mathbf{M})$ respectively. $C^\infty(\mathbf{M})$ expresses the spaces of smooth functions on $\mathbf{M}$.

Now let us introduce the conception of the Gateaux derivative for it is a powerful tool to study all kinds of tensor fields. The Gateaux derivative of a tangent field $X \in \mathbf{T}(\mathbf{M})$ at direction $S \in \mathbf{T}(\mathbf{M})$ is defined as

$$X'(u)[S] = \frac{\partial X(u + \varepsilon S)}{\partial \varepsilon}|_{\varepsilon=0}$$

which is usually written as $X'[S]$ or $X'S$ for short when there is no confusion. X' is actually a linear operator of ∂ and ∂^{-1}. If X is a tensor functions of variables $x, t, \partial^{-l}u, \cdots, u, \cdots \partial^m u$, then X' can be calculated as follows

$$X' = \sum_{j=-l}^{m} \frac{\partial X}{\partial(\partial^j u)} \partial^j,$$

where l and m are positive integers. The Gateaux derivatives of operators can be defined similarly or with the aid of their tensor fields. Let $K, S \in \mathbf{T}(\mathbf{M})$, then

$$[K,S] = K'[S] - S'[K], \quad \mathrm{ad}_K S = [K,S]$$

represent the commutator of them and the adjoint map ad_K. According to the duality between cotangent and tangent vectors we can work out the conjugate operator of an operator. For example, through

$$< \Phi^*\alpha, K > = < \alpha, \Phi K >, \quad \alpha \in \mathbf{T}^*(\mathbf{M}), \quad K \in \mathbf{T}(\mathbf{M}),$$

we can compute the conjugate operator $\Phi^* : \mathbf{T}^*(\mathbf{M}) \to \mathbf{T}^*(\mathbf{M})$ of an operator $\Phi : \mathbf{T}(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$. An operator $J : \mathbf{T}^*(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ (or $\theta : \mathbf{T}(\mathbf{M}) \to \mathbf{T}^*(\mathbf{M})$) is called skew-symmetric, if it equals its negative conjugate operator.

Definition 1. $\gamma \in \mathbf{T}^*(\mathbf{M})$ is called a gradient field or the variational derivative if there exists $H \in C^\infty(\mathbf{M})$, so that

$$H'[K] = < \gamma, K >, \quad for \quad all \quad K \in \mathbf{T}(\mathbf{M}).$$

It is often denoted by $\gamma = \delta H/\delta u$.

As one can easily confirm, a cotangent vector field $\gamma \in \mathbf{T}^*(\mathbf{M})$ is a gradient field if and only if $\gamma' = \gamma'^*$. Its corresponding potential $H \in C^\infty(\mathbf{M})$ can be calculated as follows

$$H(u) = \int_0^1 < \gamma(\lambda u), u > d\lambda. \tag{1}$$

Setting

$$u_t = K(u), \quad K \in \mathbf{T}(\mathbf{M}), \tag{2}$$

be an evolution equation, then

$$\tau_t = K'\tau, \quad \tau \in \mathbf{T}(\mathbf{M}), \tag{3}$$

$$\gamma_t = -K'^*\gamma, \quad \gamma \in \mathbf{T}^*(\mathbf{M}), \tag{4}$$

are its linearized and adjoint linearized equation respectively. As mentioned, K' expresses the Gateaux derivative operator of $K(u)$ with respect to u, K'^* is its adjoint operator and f_t is the total derivative of f on the variable t.

Definition 2. Supposing that a linear operator $\Phi : \mathbf{T}(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ satisfies

$$\frac{\partial \Phi}{\partial t} + \Phi'[K] - [K, \Phi] = 0,$$

then it is a strong symmetry operator of Eq.(2).

Evidently a strong symmetry operator $\Phi : \mathbf{T}(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ maps symmetries into new symmetries of Eq.(2).

Definition 3. If a linear operator $\Phi : \mathbf{T}(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ meet

$$\Phi'[\Phi f]g - \Phi'[\Phi g]f = \Phi(\Phi'[f]g - \Phi'[g]f)$$

for all vector fields $f, g \in \mathbf{T}(\mathbf{M})$, it is a hereditary symmetry operator.

Obviously, if Φ is a strong and hereditary symmetry operator of Eq.(2), Φ is also the strong operator of $u_t = \Phi K(u)$, i.e., Φ is a strong operator of $u_t = \Phi^n K(u)$, where n is a natural number.

Definition 4. A tangent vector $\tau \in \mathbf{T}(\mathbf{M})$ is a symmetry of Eq.(2) if it solves Eq.(3) when u is a solution of Eq.(2). A cotangent vector $\gamma \in \mathbf{T}^*(\mathbf{M})$ is called an adjoint symmetry of Eq.(2), if it solves Eq.(4) when u satisfies (2).

Through the following proposition, we can see that symmetries and adjoint symmetries are related closely to each other[9].

Proposition 1. [18,19] Let $\tau(x,u)$ be a symmetry of Eq.(2) while $\gamma(x,u)$ be an adjoint symmetry, then $I = < \tau(x,u), \gamma(x,u) >$ is a conserved quantity of Eq.(2).

Proof. The total derivative of I can be computed as follows

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &=< \frac{\mathrm{d}\tau}{\mathrm{d}t}, \gamma > + < \tau, \frac{\mathrm{d}\gamma}{\mathrm{d}t} > \\ &=< K'[\tau], \gamma > + < \tau, -K'^*[\gamma] > \\ &=< K'[\tau], \gamma > + < K'[\tau], -\gamma > \\ &= 0.\end{aligned}$$

□

Definition 5. A linear operator $\Omega : \mathbf{T}^*(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ satisfies

$$\Omega'[K] - \Omega K'^* - K'\Omega = 0,$$

then Ω is a Noether operator of Eq.(2).

A linear inverse Noether operator $\Lambda : \mathbf{T}(\mathbf{M}) \to \mathbf{T}^*(\mathbf{M})$ of Eq. (2) satisfies

$$\Lambda'[K] + K^{'*}\Lambda + \Lambda K' = 0.$$

By simple calculation, we find that the Noether operator maps adjoint symmetries of (2) to its symmetries. Accordingly, the inverse of the Noether operator (if exists) maps symmetries of (2) to its adjoint symmetries.

Let $\Phi = \Omega\Lambda$ be a strong symmetry operator of Eq.(2) and Ω be its Noether operator, it is easy to verify that Λ is an inverse Noether operator of Eq.(2) if the inverse operator of Ω exists.

Definition 6. If $\theta : \mathbf{T}^*(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ is a linear skew-symmetric operator and satisfies the Jacobi identity

$$< f, \theta'[\theta g]h > +\mathrm{cycle}(f, g, h) = 0, \quad \forall f, g, h \in \mathbf{T}^*(\mathbf{M}),$$

θ is an implectic operator (also known as Hamiltonian operator or Poisson tensor (see [9,13]). The Poisson bracket can be defined as

$$\{H_1, H_2\}_\theta =< \frac{\delta H_1}{\delta u}, \theta\frac{\delta H_2}{\delta u} >, \quad H_1, H_2 \in C^\infty(\mathbf{M}).$$

If $\{H_1, H_2\}_\theta = 0$, they are considered as involutive with respect to θ, where $H_1, H_2 \in C^\infty(\mathbf{M})$.

Definition 7. If $J : \mathbf{T}(\mathbf{M}) \to \mathbf{T}^*(\mathbf{M})$ is a linear skew-symmetric operator and satisfies the Jacobi identity

$$< f, J'[g]h > +\mathrm{cycle}(f, g, h) = 0, \quad \forall f, g, h \in \mathbf{T}(\mathbf{M}),$$

J is a symplectic operator.

It is easy to verify that the inverse of a symplectic operator is implectic if it exists and vice versa.

Definition 8. Eq.(2) is named a Hamiltonian equation if it can be rewritten as

$$u_t = K(u) = \theta\frac{\delta H}{\delta u},$$

where $H \in C^\infty(\mathbf{M})$ and θ is an implectirc operator. Furthermore, if Eq.(2) has following format

$$u_t = K(u) = \theta\frac{\delta H_1}{\delta u} = \vartheta\frac{\delta H_2}{\delta u},$$

it is called bi-Hamiltonian equation, where $H_1, H_2 \in C^\infty(\mathbf{M})$, $\theta, \vartheta : \mathbf{T}^*(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$ are Hamiltonian operators and form a Hamiltonian pair.

Proposition 2. [18,20] Let $I(u) = I(x, u)$ be a functional , then $I(u)$ is a conserved quantity of Eq.(2) if and only if its variational derivative $\frac{\delta I(u)}{\delta u}$ is an adjoint symmetry, where $I(u)$ does not depend explicitly on time t.

Proof. Let γ be the variational derivative of $I(u)$, i.e., $I(u)'[g] =< \gamma, g >$ for any $g \in \mathbf{T}(\mathbf{M})$, we have

$$\begin{aligned}\partial_t < \gamma, g > &= \partial_t(I(u)'[g]) \\ &= (\partial_t I(u))'[g] + I(u)'[\partial_t g] \\ &= (\partial_t I(u))'[g] + < \gamma, \partial_t g > .\end{aligned}$$

It gives rise to

$$(\partial_t I(u))'[g] =< \partial_t\gamma, g > .$$

For u satisfying Eq.(2) and noting that

$$\frac{\mathrm{d}I(u)}{\mathrm{d}t} = \partial_t I(u) + I(u)'[u_t] = \partial_t I(u) + I(u)'[K] = \partial_t I(u) + <\gamma, K>,$$

we have

$$\begin{aligned}(\frac{\mathrm{d}I(u)}{\mathrm{d}t})'[g] &= (\partial_t I(u))'[g] + <\gamma, K>'[g] \\ &= <\partial_t\gamma, g> + <\gamma'[g], K> + <\gamma, K'[g]> \\ &= <\partial_t\gamma, g> + <\gamma'^* K, g> + <K'^*\gamma, g> \\ &= <\partial_t\gamma + \gamma' K + K'^*\gamma, g>,\end{aligned}$$

where $\gamma' = \gamma'^*$ has been used. Obviously, $I(u)$ is a conserved quantity of Eq.(2) if and only if γ is an adjoint symmetry of (2). □

Hamiltonian structures, gradient fields and Noether operators are related as follows.

Proposition 3. [12] Let $\theta = \theta(u) : \mathbf{T}^*(\mathbf{M}) \to \mathbf{TM}$ is an implectic operator and Eq.(2) is rewritten as

$$u_t = K(u) = \theta\gamma, \quad \gamma \in \mathbf{T}^*(\mathbf{M}). \tag{5}$$

Then θ is a Noether operator of Eq.(2) iff γ is a gradient field, in other words, (5) is a Hamiltonian structure.

Proof. For arbitrary $f, g \in \mathbf{T}^*(\mathbf{M})$, we have

$$\begin{aligned}&< f, (\theta'[K] - K'\theta - \theta K'^*)g > \\ &= <\gamma, \theta'[\theta f]g> + <f, \theta'[\theta g]\gamma> + <g, \theta'[\theta\gamma]f> + <\theta f, (\gamma' - \gamma'^*)[\theta g]> .\end{aligned}$$

From the above equation, it is easy to see that θ is a Noether operator iff $\gamma' = \gamma^{'*}$, i.e., γ is a gradient field for θ is an implectic operator. □

Now let us consider an evolution equation

$$u_t = K_n(u) = \Phi^n K(u), \tag{6}$$

where $K \in \mathbf{T}(\mathbf{M})$ and Φ is a hereditary and strong symmetry operator.

Proposition 4. [12] Suppose that the recursion operator Φ of Eq.(6) can be factorized into the product of an implectic operator θ and a symplectic operator J and the first equation in (6) has a Hamiltonian structure

$$u_t = K(u) = \theta f, \tag{7}$$

then every equation in (6) is Hamiltonian equation

$$u_t = \theta\Phi^{*n} f = \theta\frac{\delta H_n}{\delta u} \quad (n = 0, 1, 2, \cdots),$$

where the inverse operator of θ exists, Φ^* is the conjugate operator of Φ and $\Phi\theta = \theta\Phi^*$. Furthermoer the functional H_n can be expressed as

$$H_n = \int_0^1 < (\Phi^{*n} f)(\lambda u), u > d\lambda.$$

Proof. Since (7) is a Hamiltonian equation, we know that θ is a Noether operator of Eq.(7) by Proposition 3. Next we will prove that θ is also a Noether operator of Eq.(6) in the case of $n = 1$.

If Φ is a strong symmetry operator and θ is a Noether operator of Eq.(7), then J is an inverse Noether operator since the inverse operator of θ exists. By the definition of inverse Noether operator, we have

$$< \theta h, (J'[K] + K^{'*} + JK')\theta g >= 0 \quad h, g \in \mathbf{T}(\mathbf{M}).$$

On the other hand, we have

$$< g, \theta'[\theta JK]h > + \mathrm{cycle}(g, JK, h) = 0$$

for $\theta(u)$ is an implectic operator. Noticing that $J(u)$ is a symplectic operator, we easily obtain

$$< g, \{\theta'[\Phi K] - (\Phi K)'\theta - \theta(\Phi K)^{'*}\}h >=< \theta g, J'[\theta h]K > + \mathrm{cycle}(\theta g, \theta h, K) = 0.$$

That is to say θ is a Noether operator for Eq.(6) in the case of $n=1$. According to Proposition 3, this equation is a Hamiltonian equation and its Hamiltonian functional is

$$H_1=\int_0^1 <(\Phi^* f)(\lambda u),u> d\lambda$$

by (1).

Similarly, we can prove this proposition for the case of $n\geq 2$, based on the case of $n-1$. This completes the proof by the mathematical induction. □

3 | THE ISOSPECTRAL KUNDU HIERARCHY AND ITS CONSERVATION LAWS

In this section, we briefly recall the hierarchy of the isospectral Kundu equation and obtain its CLs by expanding the ratios of the corresponding eigenfunctions into Laurent series.

Let

$$\sigma=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\delta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and assume that T denotes the transpose of a matrix. Let us consider the following spectral problem[24,25]

$$\phi_x=M\phi,\quad M=\begin{pmatrix}-\frac{1}{2}(\eta^2-\beta qr) & \eta q\\ \eta r & \frac{1}{2}(\eta^2-\beta qr)\end{pmatrix},\quad \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\tag{8}$$

and its time evolution

$$\phi_t=N\phi,\quad N=\begin{pmatrix}A & B\\ C & -A\end{pmatrix},$$

where $q=q(t,x),r=r(t,x)$ are the smooth functions and η is a spectral parameter. Supposing that the derivatives of any order with respect to x of q and r vanish rapidly as $x\to\infty$, the compatibility condition reads

$$M_t-N_x+[M,N]=0,$$

which yields

$$\eta\hat{u}_t=L_1L_2\begin{pmatrix}B\\ -C\end{pmatrix}+\eta^2L_3\begin{pmatrix}B\\ -C\end{pmatrix}-2\eta A_0\sigma\hat{u}-\eta_tL_1\hat{u}+2\eta^2\eta_tx\hat{u},\tag{9}$$

where $\hat{u}=(q,r)^T$ and

$$L_1=e-\beta\sigma\hat{u}\partial^{-1}\hat{u}^T\delta,\quad L_2=-(\sigma\partial+\beta qre),\quad L_3=e+(2-\beta)\sigma\hat{u}\partial^{-1}(r,q).$$

Expanding $(B,C)^T$ in (9) as

$$\begin{pmatrix}B\\ C\end{pmatrix}=\sum_{j=1}^{n}(-1)^{n-j}\begin{pmatrix}b_j\\ c_j\end{pmatrix}\eta^{2(n-j)+1},$$

and comparing the coefficients of the same power of η, we can obtain the related hierarchy of isospectral flow $(\eta_t=0,A_0=\frac{1}{2}(-1)^n\eta^{2n})$

$$\hat{u}_t=\hat{K}_n=\hat{\Phi}^n\hat{K}_0=-\hat{\Phi}^n\sigma\hat{u},\tag{10}$$

where n is a positive integer and

$$\begin{aligned}\hat{\Phi}=&L_1L_2L_3^{-1}\\ =&-\sigma\partial+\frac{1}{2}\beta\hat{u}^T\delta\hat{u}e+(2-\beta)\hat{u}_x\partial^{-1}\hat{u}^T\delta+\beta\sigma\hat{u}\partial^{-1}\hat{u}_x^T\sigma\delta+2(1-\beta)\hat{u}\hat{u}^T\delta\\ &+\beta(\beta-1)\sigma\hat{u}\partial^{-1}\hat{u}^T\delta\hat{u}^T\delta\hat{u}.\end{aligned}$$

From[26], we know that the operator $\hat{\Phi}$ is hereditary and strong symmetrical and the isospectral hierarchy and non-isospectral hierarchy form an infinite-dimensional τ-symmetry Lie algebra. Furthermore, the hierarchy (10) includes the Kaup-Newell equation[27], the Chen-Lee-Liu equation[28], the Gerdjikov-Ivanov equation[29,30], the modified Korteweg-de Vries equation, the Sharma-Tasso-Olever equation[31] and a new equation as special reductions.

Now, let us work out the CLs from the Lax pairs. Considering the ratio of the two eigenfunctions

$$\omega(x,\eta)=\eta\frac{\phi_2(x,\eta)}{\phi_1(x,\eta)},$$

obviously the ratio $\omega(x,\eta)$ satisfies the Riccati equation

$$q\omega_x(x,\eta)=-q^2\omega^2(x,\eta)+(\eta^2-\beta qr)q\omega(x,\eta)+\eta^2qr. \tag{11}$$

By the spectral problem (8), the following CLs relation holds

$$[-\frac{1}{2}(\eta^2-\beta qr)+q\omega(x,\eta)]_t=(A+B\frac{\omega(x,\eta)}{\eta})_x. \tag{12}$$

Expanding $q\omega(x,\eta)$ into a Laurent series

$$q\omega(x,\eta)=\sum_{n=0}^{\infty}\omega_n(x)\eta^{-2n},$$

we get a recursion relation for defining ω_n:

$$\omega_0(x)=-qr,\qquad \omega_1(x)=-qr_x+(1-\beta)q^2r^2,$$
$$\omega_{n+1}(x)=q(\frac{\omega_n(x)}{q})_x+\sum_{j=0}^{n}\omega_j(x)\omega_{n-j}(x)+\beta qr\omega_n(x),\quad (n=0,1,2,\cdots),$$

from the above Riccati equation (11).

Therefore, for example, letting[26]

$$\begin{cases}A=\frac{1}{2}\eta^4-qr\eta^2+\frac{1}{2}\beta(3-2\beta)q^2r^2+\frac{1}{2}\beta(rq_x-qr_x),\\ B=-q\eta^3+q_x\eta+(2-\beta)q^2r\eta,\end{cases}$$

we have

$$\begin{aligned}[\frac{1}{2}\beta(qr)+\sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n}}]_t=&[\frac{1}{2}\eta^4-qr\eta^2+\frac{1}{2}\beta(3-2\beta)q^2r^2+\frac{1}{2}\beta(rq_x-qr_x)\\&-\sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n-2}}+\frac{q_x}{q}\sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n}}+(2-\beta)qr\sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n}}]_x.\end{aligned} \tag{13}$$

The first several conservation laws in (13) are listed as follows:

$$(qr)_t=[(3-2\beta)q^2r^2+q_xr-qr_x]_x, \tag{14a}$$

$$[qr_x-(1-\beta)q^2r^2]_t=[2(3-2\beta)q^2rr_x+q_xr_x-qr_{xx}-2(1-\beta)(2-\beta)q^3r^3]_x, \tag{14b}$$

$$\begin{aligned}&[-qr_{xx}+(1-\beta)qq_xr^2+(4-3\beta)q^2rr_x-(\beta-1)(\beta-2)q^3r^3]_t=[qr_{xxx}-q_xr_{xx}\\&-(1-\beta)qq_{xx}r^2-(5-3\beta)q^2r_x^2+(1-\beta)q_x^2r^2-2(1-\beta)qq_xrr_x+(5\beta-8)q^2rr_{xx}\\&+3(1-\beta)(2-\beta)q^2q_xr^3+(9\beta^2-31\beta+24)q^3r^2r_x+(1-\beta)(3-\beta)(2\beta-3)q^4r^4]_x.\end{aligned} \tag{14c}$$

4 | THE CONSERVED QUANTITIES OF THE KUNDU HIERARCHY

In this section, we will get the conserved quantities to prove the Liouville integrability of the Kundu hierarchy by establishing the Hamiltonian structure from the recursion relation (10). Actually, the conserved quantities have already been derived out through the Tu scheme[25]. To illuminate the relationship between Hamiltonian structures and conservation laws, we re-derive them by the Fokas' method again.

Let us rewrite the system (10) as

$$\hat{u}_t=-\hat{\Phi}^n\sigma\hat{u}=\hat{\theta}(\hat{\theta}^{-1}\hat{\Phi}\hat{\theta})^n\delta\hat{u}, \tag{15}$$

where

$$\hat{\theta}=\delta\sigma+2(1-\beta)\sigma\hat{u}\partial^{-1}\hat{u}^T\sigma.$$

To show that the Eq.(15) is a Hamiltonian equation, we first prove that $\hat{\theta}$ is an implectic operator. It is easy to verify that

$$\hat{\theta}^* = -\hat{\theta}, \quad \hat{\theta}^{-1}\hat{\Phi}\hat{\theta} = \hat{\Phi}^*,$$

thus $\hat{\theta}$ is skew-symmetric operator and $\hat{\Phi}^*$ is similar to the operator $\hat{\Phi}$, where $\hat{\Phi}^*$ is the conjugate operator of the recursion operator $\hat{\Phi}$.

Lemma 1. $\hat{\theta}$ is an implectic operator.

Proof. For arbitrary three cotangent vectors $f, g, h \in \mathbf{T}^*(\mathbf{M})$, we have

$$\begin{aligned}
&(f, \hat{\theta}'[\hat{\theta}g]h) + (g, \hat{\theta}'[\hat{\theta}h]f) + (h, \hat{\theta}'[\hat{\theta}f]g),\\
=&2(1-\beta)[(f, \sigma\delta\sigma g\partial^{-1}\hat{u}^T\sigma h + \sigma\hat{u}\partial^{-1}g^T\sigma\delta\sigma h)] + (g, \sigma\delta\sigma h\partial^{-1}\hat{u}^T\sigma f + \sigma\hat{u}\partial^{-1}h^T\sigma\delta\sigma f)\\
&+ (h, \sigma\delta\sigma f\partial^{-1}\hat{u}^T\sigma g + \sigma\hat{u}\partial^{-1}f^T\sigma\delta\sigma g)] + 4(1-\beta)^2[(f, \hat{u}\partial^{-1}\hat{u}^T\sigma g\partial^{-1}\hat{u}^T\sigma h)\\
&+ (f, \sigma\hat{u}\partial^{-1}\hat{u}^T h\partial^{-1}\hat{u}^T\sigma g) + (g, \hat{u}\partial^{-1}\hat{u}^T\sigma h\partial^{-1}\hat{u}^T\sigma f) + (g, \sigma\hat{u}\partial^{-1}\hat{u}^T f\partial^{-1}\hat{u}^T\sigma h)\\
&+ (h, \hat{u}\partial^{-1}\hat{u}^T\sigma f\partial^{-1}\hat{u}^T\sigma g) + (h, \sigma\hat{u}\partial^{-1}\hat{u}^T g\partial^{-1}\hat{u}^T\sigma f).
\end{aligned}$$

It is easy to verify the following relations

$$\begin{aligned}
\tilde{g}^T\sigma\delta\sigma\tilde{f} &= \tilde{f}^T\sigma\delta\sigma\tilde{g},\\
(\tilde{f}, \sigma\delta\sigma\tilde{g}\partial^{-1}\hat{u}^T\sigma\tilde{h}) &= -(\partial^{-1}\tilde{g}^T\sigma\delta\sigma\tilde{f}, \hat{u}^T\sigma\tilde{h}),\\
(\tilde{f}, \sigma\hat{u}\partial^{-1}\tilde{g}^T\sigma\delta\sigma\tilde{h}) &= (\hat{u}^T\sigma\tilde{f}, \partial^{-1}\tilde{g}^T\sigma\delta\sigma\tilde{h})
\end{aligned}$$

hold for arbitrary cotangent vectors $\tilde{f}, \tilde{g}, \tilde{h} \in \mathbf{T}^*(\mathbf{M})$. Thus, we have

$$(f, \hat{\theta}'[\hat{\theta}g]h) + (g, \hat{\theta}'[\hat{\theta}h]f) + (h, \hat{\theta}'[\hat{\theta}f]g) = 0.$$

□

Since the operator $\hat{\theta}$ is an implectic operator, it is possible to rewrite the Eq.(10) into a Hamiltonian equation. Now let us find the corresponding symplectic operator. Upon setting

$$\begin{aligned}
\hat{J} =&\hat{\Phi}^*\hat{\theta}^{-1}\\
=&\delta\partial + (\beta-2)\delta\hat{u}\partial^{-1}\hat{u}_x^T\sigma\delta + (2-\frac{3}{2}\beta)\hat{u}^T\delta\hat{u}\sigma\delta + (2-\beta)\sigma\delta\hat{u}_x\partial^{-1}\hat{u}^T\delta +\\
&(1-\beta)(2-\beta)\delta\hat{u}\hat{u}^T\delta\hat{u}\partial^{-1}\hat{u}^T\delta + (1-\beta)(2-\beta)\delta\hat{u}\partial^{-1}\hat{u}^T\delta\hat{u}\hat{u}^T\delta,
\end{aligned}$$

then the operator $\hat{\Phi}$ is factorized into the product of the operator $\hat{\theta}$ and $\hat{J}$. Obviously, $\hat{J}$ is a skew-symmetric operator. Next, we prove that it is a symplectic operator.

Lemma 2. $\hat{J}$ is a symplectic operator.

Proof. For arbitrary three tangent vectors $f, g, h \in \mathbf{T}(\mathbf{M})$, let

$$\begin{aligned}
w_1 &= (f, g^T\delta\hat{u}\sigma\delta h) + cycle(f, g, h),\\
w_2 &= (f, \delta\hat{u}\partial^{-1}g_x^T\sigma\delta h - \sigma\delta g_x\partial^{-1}\hat{u}^T\delta h) + cycle(f, g, h),\\
w_3 &= (f, \delta g\partial^{-1}\hat{u}_x^T\sigma\delta h - \sigma\delta\hat{u}_x\partial^{-1}g^T\delta h) + cycle(f, g, h),\\
w_4 &= (f, \delta g\partial^{-1}\hat{u}^T\delta\hat{u}\hat{u}^T\delta h + \delta\hat{u}\hat{u}^T\delta\hat{u}\partial^{-1}g^T\delta h) + cycle(f, g, h),\\
w_5 &= (f, \delta\hat{u}\partial^{-1}\hat{u}^T\delta\hat{u}g^T\delta h + \delta g\hat{u}^T\delta\hat{u}\partial^{-1}\hat{u}^T\delta h) + cycle(f, g, h),\\
w_6 &= (f, \delta\hat{u}\partial^{-1}g^T\delta\hat{u}\hat{u}^T\delta h + \delta\hat{u}\hat{u}^T\delta g\partial^{-1}\hat{u}^T\delta h) + cycle(f, g, h),
\end{aligned}$$

and then we have $w_i = 0, \quad i \in \{1, 2, 3, 4, 5, 6\}$.

Here we only prove $w_2 = 0$ and $w_3 = 0$, and the others can be proved similarly.

It is easy to see that

$$(f, \delta g\partial^{-1}\hat{u}_x^T\sigma\delta h) = (h, \sigma\delta\hat{u}_x\partial^{-1}g^T\delta f),$$

and thus $w_3 = 0$.

Trough simple calculation, we find that

$$\delta\hat{u}\partial^{-1}g_x^T\sigma\delta h+\delta\hat{u}\partial^{-1}g^T\sigma\delta h_x=\delta\hat{u}g^T\sigma\delta h,$$
$$(f,\delta\hat{u}\partial^{-1}g^T\sigma\delta h_x)=-(g,\sigma\delta h_x\partial^{-1}\hat{u}^T\delta f),$$

and

$$(f,\delta\hat{u}g^T\sigma\delta h)+\mathrm{cycle}(f,g,h)=0,$$

which leads to $w_2=0$.

Now let us prove the operator $\hat{J}$ is a symplectic operator. With massive complex computations, we have

$$\begin{aligned}&(f,\hat{J}'[g]h)+(g,\hat{J}'[h]f)+(h,\hat{J}'[f]g)\\=&(4-3\beta)w_1+(\beta-2)(w_2+w_3)+(1-\beta)(2-\beta)(w_4+w_5+2w_6)\\=&0.\end{aligned}$$

So, $\hat{J}$ is a symplectic operator. □

We have shown that the recursion operator $\hat{\Phi}$ can be factorized into the product of the implectic operator $\hat{\theta}$ and the symplectic operator $\hat{J}$. Now, let us obtain the conserved quantities and complete the proof of the integrability according to Proposition 4.

Theorem 1. Every equation in the Kundu hierarchy (10) possesses infinite conserved quantities and is integrable in Liouville sense.

Proof. For $\hat{\Phi}$ is a hereditary and strong symmetry operator and it can be decomposed into the product of the implectic operator $\hat{\theta}$ and the symplectic operator $\hat{J}$, every equation in (10) has a Hamiltonian structure

$$\hat{u}_t=\hat{\theta}(\hat{\Phi}^*)^n\hat{f}(t,x,\hat{u})=\hat{\theta}\frac{\delta\hat{H}_n}{\delta\hat{u}},\quad(n=0,1,2,\cdots),$$

where $\hat{f}(t,x,\hat{u})=-\hat{\theta}\sigma\hat{u}$ and the conserved quantity $\hat{H}_n$ is expressed as

$$\hat{H}_n=\int_0^1<(\hat{\Phi}^{*n}\hat{f})(\lambda\hat{u}),\hat{u}>d\lambda.$$

The first three conserved quantities are

$$\hat{H}_0=\int_{-\infty}^{\infty}qrdx$$

$$\hat{H}_1=\int_{-\infty}^{\infty}(\frac{1}{2}q_xr-\frac{1}{2}r_xq+(1-\beta)q^2r^2)dx,$$

$$\hat{H}_2=\int_{-\infty}^{\infty}(\frac{1}{2}q_{xx}r+\frac{1}{2}r_{xx}q+(1-\beta)(2-\beta)q^3r^3+(\frac{3}{2}-\beta)q_xqr^2-(\frac{3}{2}-\beta)q^2rr_x)dx.$$

□

According to classical Liouville integrable theory, an equation is integrable if it possesses a Hamiltonian structure and sufficiently many conserved quantities, i.e., Hamiltonian functionals which are involutive. In theorem 1, we have derived out infinite conserved quantities of the Kundu equation. From the first few conserved quantities, we can see that these conserved quantities of the Kundu equations are similar to the conserved densities of its CLs.

5 | CONSTRUCTING HAMILTONIAN STRUCTURES FROM THE CLS

In section 3, we have already obtained infinite CLs of the Kundu Equation. In this section, we will develop a method to prove the integrability of evolution equations. Thus, the Kundu equation can also be proved to be integrable in Liouville sense through a way different from the one in Section 4.

From the equation (12), we know

$$[\int_{-\infty}^{\infty}(\frac{1}{2}\beta qr+q\omega)dx]_t=0, \tag{17}$$

for q, r and their derivatives of any order with respect to x vanish rapidly as $x\to\infty$. So the left side of the CLs (12) is conserved densities of equation (10) and it can be written in the following integral type functional form

$$\hat{H}=\int_{-\infty}^{\infty}Fdx,\quad F=\frac{1}{2}\beta qr+\hat{v},\quad \hat{v}=q\omega.$$

Through simple calculation, we have

$$q(\frac{\hat{v}}{q})_x+\hat{v}^2-(\eta^2-\beta qr)\hat{v}-\eta^2qr=0 \tag{18}$$

by the Riccati equation (11).

In order to prove the integrability of the Kundu equation, we first find the relation between the Hamiltonian structure and conserved quantities.

Lemma 3. The variational derivative on u of the integrable type functional

$$H(u,v)=\int_{-\infty}^{\infty}F(u,v)dx$$

under the condition

$$G(u,v,u_x,v_x)=0 \tag{19}$$

is

$$\frac{\delta H(u,v)}{\delta u}=\frac{\partial F(u,v)}{\partial u}+\rho\frac{\partial G(u,v,u_x,v_x)}{\partial u}-\frac{\partial}{\partial x}\rho\frac{\partial G(u,v,u_x,v_x)}{\partial u_x},$$

where ρ is a lagrangian multiplier which is determined by (19) and the following relation

$$\frac{\partial F(u,v)}{\partial v}+\rho\frac{\partial G(u,v,u_x,v_x)}{\partial v}-\frac{\partial}{\partial x}\rho\frac{\partial G(u,v,u_x,v_x)}{\partial v_x}=0. \tag{20}$$

Proof. Given a Lagrangian multiplier ρ, let

$$H(u,v)=\int_{-\infty}^{\infty}[F(u,v)+\rho G(u,v,u_x,v_x)]dx.$$

Then its Gateaux derivative on u at a direction of h is

$$H(u,v)'[h]=\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\int_{-\infty}^{\infty}F(u+\varepsilon h,v)+\rho G(u+\varepsilon h,v,(u+\varepsilon h)_x,v_x)dx.$$

Under assumption $[h_x\partial^{-1}(\rho\frac{\partial G}{\partial u_x})]|_{x=-\infty}^{\infty}=0$, we have

$$\begin{aligned}H(u,v)'[h]&=\int_{-\infty}^{\infty}\frac{\partial F(u,v)}{\partial u}h+\rho\frac{\partial G(u,v,z_x,v_x)}{\partial u}h-h\frac{\partial}{\partial x}\rho\frac{\partial G(u,v,z_x,v_x)}{\partial u_x}dx\\&=<\frac{\partial F(u,v)}{\partial u}+\rho\frac{\partial G(u,v,u_x,v_x)}{\partial u}-\frac{\partial}{\partial x}\rho\frac{\partial G(u,v,z_x,v_x)}{\partial u_x},h>.\end{aligned}$$

Similarly, we can obtain the Gateaux derivative of $H(u,v)$ on v at a direction h

$$H(u,v)'[h]=<\frac{\partial F(u,v)}{\partial v}+\rho\frac{\partial G(u,v,u_x,v_x)}{\partial v}-\frac{\partial}{\partial x}\rho\frac{\partial G(u,v,u_x,v_x)}{\partial v_x},h>. \tag{21}$$

So the functional derivative of $H(u,v)$ on u under the condition (19) is

$$\frac{\delta H(u,v)}{\delta u} = \frac{\partial F(u,v)}{\partial u} + \rho\frac{\partial G(u,v,u_x,v_x)}{\partial u} - \frac{\partial}{\partial x}\rho\frac{\partial G(u,v,u_x,v_x)}{\partial u_x},$$

and the lagrangian multiplier ρ can be defined by (19) and (20). □

Lemma 3 provides us a method to calculate the variational derivative under a certain constraint condition.

Theorem 2. Consider the nonlinear evolution equation hierarchy (6). Its recusion operator Φ and the corresponding conjugate operator Φ^* satisfy

$$\Phi\theta = \theta\Phi^*, \tag{22}$$

where θ is an implectic operator. If the integrable type functional

$$H(u,v,\lambda) = \int_{-\infty}^{\infty} f(u,v,\lambda)dx$$

is the conserved quantity of (6) and $\Gamma = \frac{\delta H(u,v,\lambda)}{\delta u}$ satisfies

$$\Phi^*\Gamma = \lambda\Gamma - \theta^{-1}K(u),$$

then every equation in the nonlinear evolution equation hierarchy (6) has a Hamiltonian structure and it is integrable in the Liouville sense, where λ is a parameter.

Proof. Substituting the Laurent series of Γ on λ

$$\Gamma = \sum_{j=0}^{\infty}\Gamma_j\lambda^{-j-1}$$

into (24) and comparing the coefficient of the same power of λ, we obtain

$$\Gamma_j = \Phi^{*j}\theta^{-1}K(u). \tag{23}$$

For $\Phi\theta = \theta\Phi^*$, $u_t = \Phi^m K(u)$ can be rewritten as $u_t = \theta\Phi^{*m}\theta^{-1}K(u)$. Expanding H as the following Laurent series

$$H = \sum_{j=0}^{\infty} H_j\lambda^{-j-1}, \tag{24}$$

we can find that every equation in the nonlinear evolution equation hierarchy (6) has a Hamiltonian structure

$$u_t = \theta\Gamma_m = \theta\frac{\delta H_m}{\delta u}$$

by comparing the coefficient of the same power of λ.

It is easy to verify that

$$\begin{aligned}\{H_l, H_m\}_\theta &=< \frac{\delta H_l}{\delta u}, \theta\frac{\delta H_m}{\delta u} >=< \Gamma_l, \theta\Gamma_m >=< \theta^{-1}K(u), \Phi^l\theta\Phi^{*m}\theta^{-1}K(u) > \\ &=< \theta^{-1}K(u), L^{m+n}K(u) >=< L^{m+n}K(u), \theta^{-1}K(u) >= 0\end{aligned}$$

by using (22), namely, H_l and H_m are involutive. So every equation in the nonlinear evolution equation hierarchy (6) is integrable in the Liouville sense. □

As an application of Theorem 2, we have the following corollary:

Corollary 1. Every equation in the Kundu hierarchy (10) is integrable in Liouville sense.

Proof. Through the Lemma 3, we have

$$G = \frac{\delta\hat{H}}{\delta\hat{u}} = \begin{pmatrix}\frac{\delta\hat{H}}{\delta q} \\ \frac{\delta\hat{H}}{\delta r}\end{pmatrix} = \begin{pmatrix}\frac{1}{q}(\rho v)_x + \beta\rho r v - \rho\eta^2 r + \frac{1}{2}\beta r, \\ \beta\rho q v - \rho\eta^2 q + \frac{1}{2}\beta q,\end{pmatrix}$$

where $\hat{u} = (q,r)^T$ and ρ can be defined by (18) and

$$\rho_x = 1 + \rho(-\frac{q_x}{q} + 2v - \eta^2 + \beta q r).$$

By complicated calculation, we have

$$\hat{\Phi}^* G = -\eta^2 \begin{pmatrix} B \\ C \end{pmatrix} - \hat{\theta}^{-1} \begin{pmatrix} q \\ -r \end{pmatrix},$$

where

$$\hat{\theta}^{-1} = \sigma\delta + 2(1-\beta)\delta\hat{u}\partial^{-1}\hat{u}^T\delta.$$

So every equation in the Kundu hierarchy (10) is integrable in Liouville sense. □

Through the corollary 1, we prove the integrability of the Kundu equations in a way different from the previous method. The main difference between the two methods is that they have different starting points.

6 | CONCLUSIONS

In general, we developed a method to prove the Liouville integrability of evolution equations with continuous variable through the connection between CLs and Hamiltonian structures. From the Lax pair of the Kundu hierarchy, we firstly deduced its CLs through expanding the ratio of two eigenfunctions into a Laurent series. To obtain the conserved quantities of the Kundu hierarchy from its Hamiltonian structure, we decomposed the recursion operator $\hat{\Phi}$ into the product of an implectic operator $\hat{\theta}$ and a symplectic operator $\hat{J}$. Although the conserved quantities have already be deduced by the Tu scheme in[25], we re-derived them through the Fokas' method to demonstrate the connection between the Cls and Hamiltonian structures. By the Largrange multiplier and the functional derivatives under certain constraint conditions, the connection between the conserved quantities and Hamiltonian structures was found. Then a method was built to prove the integrability of evolution equations from their conserved quantities. Finally, as an application, the integrability of the Kundu equation was proved by the Cls resulted in section 3.

ACKNOWLEDGMENTS

The work was supported in part by NSFC under the grants 11101350, 11671177 and 11771186, the Jiangsu Qing Lan Project (2014), Six Talent Peaks Project of Jiangsu Province (2016-JY-08).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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