Expansion of a compressible non-barotropic fluid in vacuum

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Abstract: In this paper, we consider a region occupied by viscous or inviscid compressible magnetohydrodynamic fluids, and surrounded by vacuum. It is shown that the fluid region will expand at least linearly in time as soon as there are no singularities. The expanding rate is proportional to initial total energy and is inversely proportional to initial mass. The result indicates an interesting fact that the expansion of the viscous monatomic fluids seems similar to the inviscid fluids.

Key Words: Free boundary problem, Compressible magnetohydrodynamic equations (MHD), Navier-Stokes equations, Diameter, Vacuum.

1 Introduction

The motions of electrically conducting fluids (e.g. plasmas) surrounded by vacuum in multi-dimensional space can be modeled by the free boundary problem of the full magnetohydrodynamic system (MHD):

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \quad \text{in } \Omega(t), \tag{1.1a}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \operatorname{div}(\delta\mathcal{T}(\mathbf{u})) + \operatorname{curl}\mathbf{H}\times\mathbf{H} \quad \text{in } \Omega(t), \tag{1.1b}$$

$$c_v[(\rho\theta)_t + \operatorname{div}(\rho\mathbf{u}\theta)] + P\operatorname{div}\mathbf{u} = \delta\mathcal{T}(\mathbf{u}) : \nabla\mathbf{u} + \operatorname{div}(\delta\kappa\nabla\theta) + \nu|\operatorname{curl}\mathbf{H}|^2 \quad \text{in } \Omega(t), \tag{1.1c}$$

$$\mathbf{H}_t - \operatorname{curl}(\mathbf{u}\times\mathbf{H}) = -\operatorname{curl}(\nu\operatorname{curl}\mathbf{H}),\ \operatorname{div}\mathbf{H} = 0 \quad \text{in } \Omega(t), \tag{1.1d}$$

$$\rho = 0,\ \mathbf{H}\cdot\mathbf{n} = 0,\ \delta\frac{\partial\theta}{\partial\mathbf{n}} = 0,\ (\delta\mathcal{T}(\mathbf{u}) - PI_d - \frac{1}{2}|\mathbf{H}|^2 I_d)\mathbf{n} = 0 \quad \text{on } \Gamma(t), \tag{1.1e}$$

$$\mathcal{V}(\Gamma(t)) = \mathbf{u}\cdot\mathbf{n} \quad \text{on } \Gamma(t), \tag{1.1f}$$

$$(\rho,\mathbf{u},\theta,\mathbf{H})(x,0) = (\rho_0,\mathbf{u}_0,\theta_0,\mathbf{H}_0)(x) \quad \text{in } \Omega(0) = \Omega_0. \tag{1.1g}$$

Here $x\in\mathbb{R}^d (d\geq 2)$ is the spatial coordinate, $t\geq 0$ is time. The unknowns are the fluid density ρ, velocity $\mathbf{u}$, absolute temperature θ, magnetic field $\mathbf{H}$, and the free boundary $\Gamma(t)$, which represents the fluid-vacuum interface. $\Omega(t)$ is the occupied domain by the fluids. $\mathcal{V}(\Gamma(t))$ denotes the normal velocity of $\Gamma(t)$ and $\mathbf{n}$ is the outward unit normal vector on $\Gamma(t)$. $\mathcal{T}(\mathbf{u})$ is the shear stress tensor with the form

$$\mathcal{T}(\mathbf{u}) = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + \lambda(\operatorname{div}\mathbf{u})I_d,$$

with I_d being the $d\times d$ identity matrix, and $\nabla\mathbf{u}^\top$ being the transpose of the matrix $\nabla\mathbf{u}$. The functions $\mu=\mu(\theta)$ and $\lambda=\lambda(\theta)$ are the viscosity coefficients satisfying the following constraints

$$\mu\geq 0,\quad 2\mu + d\lambda\geq 0.$$

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$\kappa = \kappa(\theta) \geq 0$ is the coefficient of heat conductivity, and $\nu = \nu(\mathbf{H}) \geq 0$ is the coefficient of magnetic diffusion. The pressure P is given by the following equation of state

$$P = c_v(\gamma - 1)\rho\theta, \tag{1.2}$$

where $c_v > 0$ is the specific heat at constant volume, and $\gamma > 1$ is the ratio of specific heat. The constant δ is assumed to be 0 or 1. If $\delta = 0$, we ignore the effects of the viscosity and the heat conductivity, which implies that the fluids are inviscid; if $\delta = 1$, then we are considering the viscous fluids.

Free boundary problems in fluid mechanics have important physical background and it attracts many research interests. However, the study for the MHD system seems far from being complete. For the compressible MHD equations, Chen and Wang [1] investigated a free boundary problem for nonlinear magnetohydrodynamics with large initial data. Qin and Yao [13] proved the existence and uniqueness of the global classical solutions for the planar magnetohydrodynamic equations with radiation. When the initial density connects to vacuum smoothly, Ou and Shi [12] proved the global existence and showed that the expanding rate of the free interface was not faster than $(1 + t)$. For the incompressible MHD flows, we refer readers to [3, 4, 15].

In this paper, we consider the expanding rate of the monatomic fluids in vacuum. The kinetic theory indicates that the monatomic fluids satisfy the following relations

$$\lambda + \frac{2}{d}\mu = 0, \qquad \gamma = \frac{d+2}{d}, \tag{1.3}$$

which implies that the viscous fluids are affected only by the shear viscosity, but not the bulk viscosity. In fact, the relations hold true in practice for most fluids and gases (see [2, 8]).

2 Main results

Before we state the main results, we need to introduce some physical quantities. Denote by $m(t), \bar{x}(t)$ and $\bar{\mathbf{u}}(t)$ the mass, centroid and average velocity of the fluid region respectively, that is,

$$m(t) = \int_{\Omega(t)} \rho \mathrm{d}x; \qquad \bar{x}(t) = \frac{1}{m(t)} \int_{\Omega(t)} \rho x \mathrm{d}x; \qquad \bar{\mathbf{u}}(t) = \frac{1}{m(t)} \int_{\Omega(t)} \rho \mathbf{u} \mathrm{d}x. \tag{2.1}$$

Set

$$m_1(t) = \int_{\Omega(t)} \rho \mathbf{u} \mathrm{d}x; \qquad m_2(t) = \int_{\Omega(t)} \rho(\mathbf{u} - \bar{\mathbf{u}}(t)) \cdot (x - \bar{x}(t)) \mathrm{d}x; \tag{2.2}$$

$$m_3(t) = \int_{\Omega(t)} \rho |x - \bar{x}(t)|^2 \mathrm{d}x; \quad \mathcal{E}(t) = \int_{\Omega(t)} \left(\rho \left(\frac{|\mathbf{u} - \bar{\mathbf{u}}(t)|^2}{2} + c_v\theta \right) + \frac{|\mathbf{H}|^2}{2} \right) \mathrm{d}x. \tag{2.3}$$

The classical solutions to the free boundary problem (1.1) are defined as follows.

Definition 2.1. *For $T > 0$, $(\rho, \mathbf{u}, \theta, \mathbf{H}, \Gamma(t))$ is called a classical solution to (1.1) if*

$$\begin{cases} (\rho, \theta) \in C^0(\overline{\mathfrak{D}_T}) \cap C^1(\mathfrak{D}_T), \mathbf{u} \in C^1(\overline{\mathfrak{D}_T}), \mathbf{H} \in C^1(\overline{\mathfrak{D}_T}) \cap C^2(\mathfrak{D}_T), \delta = 0; \\ \rho \in C^0(\overline{\mathfrak{D}_T}) \cap C^1(\mathfrak{D}_T), (\mathbf{u}, \theta, \mathbf{H}) \in C^1(\overline{\mathfrak{D}_T}) \cap C^2(\mathfrak{D}_T), \delta = 1, \end{cases}$$

and $\Gamma(t)$ is regular, where $\mathfrak{D}_T = \{(x, t) : x \in \Omega(t), t \in (0, T)\}$ is the space-time region.

Next, we state the result of this paper.

Theorem 2.2. *Let* $(\rho, \mathbf{u}, \theta, \mathbf{H}, \Gamma(t))$ *be a classical solution to the free boundary problem* (1.1)*. Suppose that* $m(0) > 0$ *and* $\mathcal{E}(0) > 0$*, then it holds that*

$$diam\Omega(t) \geq \left[\frac{\mathcal{E}(0)t^2 + 2m_2(0)t + m_3(0)}{m(0)}\right]^{\frac{1}{2}} \tag{2.4}$$

for any $t \in [0, T)$*. Here* $diam\Omega(t)$ *denotes the diameter of the fluid region* $\Omega(t)$*.*

Remark 2.3. *In [16], the magnetic field* $\mathbf{H}$ *need to be 0 on* $\Gamma(t)$*, however, in this paper, it is relaxed to "*$\mathbf{H} \cdot \mathbf{n} = 0$ *on* $\Gamma(t)$*".*

Remark 2.4. *When we consider the viscous monatomic fluids without eletro-magnetic effect, System* (1.1) *can be reduced to the following free boundary problem for compressible Navier-Stokes system:*

$$\begin{aligned}
&\rho_t + div(\rho\mathbf{u}) = 0 && in\ \Omega(t), && \text{(2.5a)}\\
&(\rho\mathbf{u})_t + div(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla P = div(\mathcal{T}(\mathbf{u})) && in\ \Omega(t), && \text{(2.5b)}\\
&c_v[(\rho\theta)_t + div(\rho\mathbf{u}\theta)] + P div\mathbf{u} = \mathcal{T}(\mathbf{u}) : \nabla\mathbf{u} + div(\kappa\nabla\theta) && in\ \Omega(t), && \text{(2.5c)}\\
&\rho = 0,\ \frac{\partial\theta}{\partial\mathbf{n}} = 0,\ (\mathcal{T}(\mathbf{u}) - PI_d)\mathbf{n} = 0 && on\ \Gamma(t), && \text{(2.5d)}\\
&\mathcal{V}(\Gamma(t)) = \mathbf{u} \cdot \mathbf{n} && on\ \Gamma(t), && \text{(2.5e)}\\
&(\rho, \mathbf{u}, \theta)(x, 0) = (\rho_0, \mathbf{u}_0, \theta_0)(x) \qquad in\ \Omega(0) = \Omega_0. && && \text{(2.5f)}
\end{aligned}$$

A direct application of Theorem 2.2 to System (2.5) *indicates that the diameter of the fluid region satisfies*

$$diam\Omega(t) \geq \left[\frac{E(0)t^2 + 2m_2(0)t + m_3(0)}{m(0)}\right]^{\frac{1}{2}},$$

where $E(t) = \int_{\Omega(t)} \rho\left(\frac{|\mathbf{u}-\bar{\mathbf{u}}(t)|^2}{2} + c_v\theta\right) dx$*. Liu and Yuan [7] showed the local existence and uniquess of strong solutions to the free boundary problem of the full compressibe Navier-Stokes equations in three dimensional space. Liu [6] studied the stability of the expanding configurations of radiation gaseous stars. It should be noted that this paper was motivated by Sideris [14], where he had proved a similar result for Euler system in multi-dimensional s-pace. Later, Hadžić and Jang [5] constructed global solutions to three dimensional isentropic compressible Euler equations without any symmetry assumptions on the initial data. In one dimensional case, Luo, Xin and Yang [9] proved that the expanding rate of the free interface is* $(1+t)^{\frac{1}{\gamma}}$ *for compressible isentropic Navier-Stokes system, while Luo and Zeng [10] showed that the expanding rate was* $(1+t)^{\frac{1}{\gamma+1}}$ *for compressible Euler system with damping.*

3 Proof of the main results

First, let us introduce the Transport Formula, which will be frequently used in this section.

Lemma 3.1 (Transport Formula [11]). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with a smooth boundary, and define $\Omega(t) = \{X(\alpha,t) : \alpha \in \Omega\}$ with X being a given particle-trajectory mapping of a smooth velocity field u. Then for any smooth function $F(x,t)$, we have*

$$\frac{d}{dt}\int_{\Omega(t)} F\,dx = \int_{\Omega(t)} \{F_t + div_x(F\mathbf{u})\}\,dx.$$

In particular, for any smooth function $f(x,t)$, it holds that

$$\frac{d}{dt}\int_{\Omega(t)} \rho f\,dx = \int_{\Omega(t)} \rho\frac{Df}{Dt}\,dx,$$

where ρ satisfies (1.1a) *and* $\frac{D}{Dt} \triangleq \partial_t + \mathbf{u}\cdot\nabla$ *is the material derivative.*

Lemma 3.2. *Suppose the conditions in Theorem 2.2 hold. Then for any $t \in [0,T)$, we have*

$$m(t) = m(0); \tag{3.1}$$

$$\bar{\mathbf{u}}(t) = \bar{\mathbf{u}}(0); \tag{3.2}$$

$$\bar{x}(t) = \bar{x}(0) + \bar{\mathbf{u}}(0)t; \tag{3.3}$$

$$m_3'(t) = 2m_2(t); \tag{3.4}$$

$$\sup_{x\in\Omega(t)} |x - \bar{x}(t)| \leq diam\Omega(t). \tag{3.5}$$

Proof: Equation (3.1) follows immediately from the transport formula. This together with (2.1), (2.2), (1.1b) and the boundary condition (1.1e) leads to

$$\begin{aligned}
\bar{\mathbf{u}}'(t) &= \frac{m_1'(t)}{m(0)} \\
&= \frac{1}{m(0)}\int_{\Omega(t)} \rho\ \frac{D\mathbf{u}}{Dt}\mathrm{d}x \\
&= \frac{1}{m(0)}\int_{\Omega(t)} [-\nabla P + \mathrm{div}(\delta\mathcal{T}(\mathbf{u})) + (\mathrm{curl}\mathbf{H}\times\mathbf{H})]\mathrm{d}x \\
&= \frac{1}{m(0)}\int_{\Gamma(t)} \left(\delta\mathcal{T}(\mathbf{u}) - PI_d + \mathbf{H}\otimes\mathbf{H} - \frac{|\mathbf{H}|^2}{2}I_d\right)\cdot\mathbf{n}\mathrm{d}S \\
&= 0,
\end{aligned}$$

which gives (3.2), where we used the fact that $\mathrm{curl}\mathbf{H}\times\mathbf{H} = \mathrm{div}(\mathbf{H}\otimes\mathbf{H}) - \nabla\left(\frac{|\mathbf{H}|^2}{2}\right)$.
Using (2.1) and (3.1), one obtains

$$\bar{x}'(t) = \frac{1}{m(0)}\int_{\Omega(t)} \rho\ \frac{Dx}{Dt}\mathrm{d}x = \frac{1}{m(0)}\int_{\Omega(t)} \rho\mathbf{u}\mathrm{d}x = \frac{m_1(t)}{m(t)} = \bar{\mathbf{u}}(t),$$

which together with (3.3) yields (3.3).
Next, the equation (3.4) is a directly consequence of the following identity

$$\frac{D}{Dt}(|x-\bar{x}(t)|^2) = 2(x-\bar{x}(t))\cdot(\mathbf{u}-\bar{\mathbf{u}}(t)).$$

Finally, we turn to prove (3.5). For any $y \in \Omega(t)$, we have

$$|y-\bar{x}(t)| = \left| y - \frac{1}{m(0)} \int_{\Omega(t)} \rho x \mathrm{d}x \right| \leq \frac{1}{m(0)} \int_{\Omega(t)} \rho \, |y-x| \, \mathrm{d}x \leq \frac{\mathrm{diam}\Omega(t)}{m(0)} \int_{\Omega(t)} \rho \mathrm{d}x = \mathrm{diam}\Omega(t).$$

Now taking the supremum over all $y \in \Omega(t)$ gives (3.5). $\square$

Lemma 3.3. *Suppose that the conditions in Theorem 2.2 hold. Then for any $t \in [0, T)$, we have*

$$\mathcal{E}(0)t^2 + 2m_2(0)t + m_3(0) \leq m_3(t) \leq 2\mathcal{E}(0)t^2 + 2m_2(0)t + m_3(0); \tag{3.6}$$

Proof: We claim that

$$\mathcal{E}(t) = \mathcal{E}(0), \;\; t \in [0, T). \tag{3.7}$$

In fact, using (3.2), one can rewrite (1.1b) as

$$\partial_t[\rho(\mathbf{u}-\bar{\mathbf{u}})] + \mathrm{div}[\rho\mathbf{u} \otimes (\mathbf{u}-\bar{\mathbf{u}})] + \nabla P = \mathrm{div}(\delta\mathcal{T}(\mathbf{u})) + \mathrm{curl}\mathbf{H} \times \mathbf{H}. \tag{3.8}$$

Multiplying (3.8) by $\mathbf{u}-\bar{\mathbf{u}}$, multiplying (1.1d) by $\mathbf{H}$, and adding them to (1.1c), one finds that

$$\begin{aligned} \partial_t \left(\rho E + \frac{|\mathbf{H}|^2}{2} \right) + \mathrm{div}(\rho\mathbf{u}E) &= \mathrm{div}((\mathbf{u}\times\mathbf{H})\times\mathbf{H}) + \mathrm{div}(\nu\mathbf{H}\times\mathrm{curl}\mathbf{H}) \\ &+ \mathrm{div}(\delta\kappa\nabla\theta) + \mathrm{div}[(\delta\mathcal{T}(\mathbf{u}) - PI_d)(\mathbf{u}-\bar{\mathbf{u}})] - (\mathrm{curl}\mathbf{H}\times\mathbf{H})\cdot\bar{\mathbf{u}}, \end{aligned} \tag{3.9}$$

where $E = \frac{|\mathbf{u}-\bar{\mathbf{u}}|^2}{2} + c_v\theta$.

It follows from (2.3), (3.9), (1.1a) and the boundary condition (1.1e) that

$$\begin{aligned} \mathcal{E}'(t) &= \int_{\Omega(t)} \rho \, \frac{DE}{Dt} \mathrm{d}x + \int_{\Omega(t)} \partial_t \left(\frac{|\mathbf{H}|^2}{2} \right) \mathrm{d}x + \int_{\Gamma(t)} \frac{|\mathbf{H}|^2}{2} (\mathbf{u}(x,t)\cdot\mathbf{n}) \mathrm{d}S \\ &= \int_{\Omega(t)} \rho\partial_t E + \rho\mathbf{u}\cdot\nabla E + \partial_t \left(\frac{|\mathbf{H}|^2}{2} \right) \mathrm{d}x \\ &= \int_{\Omega(t)} \mathrm{div}((\mathbf{u}\times\mathbf{H})\times\mathbf{H}) + \mathrm{div}(\nu\mathbf{H}\times\mathrm{curl}\mathbf{H}) + \mathrm{div}(\delta\kappa\nabla\theta) \\ &\qquad + \mathrm{div}[(\delta\mathcal{T}(\mathbf{u}) - PI_d)(\mathbf{u}-\bar{\mathbf{u}})] - (\mathrm{curl}\mathbf{H}\times\mathbf{H})\cdot\bar{\mathbf{u}} \mathrm{d}x \\ &= \int_{\Gamma(t)} [(\mathbf{u}\times\mathbf{H})\times\mathbf{H} + \nu\mathbf{H}\times\mathrm{curl}\mathbf{H} + \delta\kappa\nabla\theta]\cdot\mathbf{n}\mathrm{d}S \\ &\quad + \int_{\Gamma(t)} \left[(\mathbf{u}-\bar{\mathbf{u}})^\top (\delta\mathcal{T}(\mathbf{u}) - PI_d)\mathbf{n} - \bar{\mathbf{u}}^\top \left(\mathbf{H}\otimes\mathbf{H} - \frac{|\mathbf{H}|^2}{2} I_d \right) \mathbf{n} \right] \mathrm{d}S \\ &= 0, \end{aligned}$$

which implies that the claim holds. Here we used the fact that the matrix $\mathcal{T}(\mathbf{u})$ is symmetric.

By (2.2), (1.1b), (1.1e) and (3.7), we obtain that

$$\begin{aligned}
m_2'(t) &= \int_{\Omega(t)} \left[\rho(\mathbf{u}-\bar{\mathbf{u}})\cdot\frac{D}{Dt}(x-\bar{x}(t)) + \rho(x-\bar{x}(t))\cdot\frac{D}{Dt}(\mathbf{u}-\bar{\mathbf{u}})\right]\mathrm{d}x \\
&= \int_{\Omega(t)} \rho|\mathbf{u}-\bar{\mathbf{u}}|^2 + \rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})\cdot(x-\bar{x}(t))\mathrm{d}x \\
&= \int_{\Omega(t)} \rho|\mathbf{u}-\bar{\mathbf{u}}|^2 + [\mathrm{div}(\delta\mathcal{T}(\mathbf{u}))-\nabla P+(\mathrm{curl}\mathbf{H}\times\mathbf{H})]\cdot(x-\bar{x}(t))\mathrm{d}x \\
&= \int_{\Omega(t)} \left(\rho|\mathbf{u}-\bar{\mathbf{u}}|^2 + 2c_v\rho\theta + \frac{|\mathbf{H}|^2}{2}\right)\mathrm{d}x, \\
&= \mathcal{E}(0) + \int_{\Omega(t)} \left(\rho\frac{|\mathbf{u}-\bar{\mathbf{u}}|^2}{2} + c_v\rho\theta\right)\mathrm{d}x, && (3.10) \\
&= 2\mathcal{E}(0) - \int_{\Omega(t)} \frac{|\mathbf{H}|^2}{2}\mathrm{d}x, && (3.11)
\end{aligned}$$

where we used the fact that

$$tr(\mathcal{T}(\mathbf{u})) = (2\mu+d\lambda)\mathrm{div}\mathbf{u} = 0.$$

Therefore, (3.6) follows immediately from (3.10), (3.11) and (3.4). □

Thanks to Lemma 3.2 and Lemma 3.3, we are now in a position to prove Theorem 2.2.
Proof of Theorem 2.2: It follows from (2.3) and (3.5) that

$$m_3(t) = \int_{\Omega(t)} \rho|x-\bar{x}(t)|^2\mathrm{d}x \le (\mathrm{diam}\Omega(t))^2\int_{\Omega_t}\rho\mathrm{d}x = m(0)(\mathrm{diam}\Omega(t))^2.$$

This together with (3.6) immediately gives the result. □

Acknowledgements

The research was partially supported by Fundamental Research Funds for the Central Universities (Grants No.19lgpy237).

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