Analysis of fractional COVID-19 epidemic model under Caputo operator

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Abstract

The dynamic of fractional covid-19 epidemic model with a convex incidence rate is studied in this article. Under Caputo operator, existence and uniqueness for the solutions of the fractional covid-19 epidemic model have been analyzed using fixed point theorems. We study all the basic properties and results including local and global stability. We show the global stability of disease free equilibrium using the method of Lyapunov function theory while for disease endemic, we use the method of geometrical approach. Moreover, sensitivity analysis complemented by simulations are performed to determine how changes in parameters affect the dynamical behavior of the system.

Keywords: Epidemic model, stability analysis, sensitivity analysis, numerical simulations.

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1 Introduction

In December 2019, the capital of Hubei Province i.e. Wuhan city gains attention worldwide, at the end of 2019 as an unknown virus attack people and start killing. Later on, the causative agent was identified as a novel corona virus and now named as severe acute respiratory syndrome corona virus 2 (SARS-CoV-2). Chinese government did its best to minimize the spread but unluckily the virus spreads to other countries. The latency period of covid-19 is from 11 to 14 days whereas myalgia, diarrhea, cough, fever, and shortness of breath are the common symptoms. Aged people are highly exposed to covid-19 as well as patients having comorbid conditions. Covid-19 transmission rate is very high and the basic reproduction number lies between 2.2 and 3.58. That's why it spread throughout the world and affected 213 countries. Hence, the world health organization (WHO) declared covid-19 as one of the global pandemic on January 30, 2020. The two main routes of transmission is direct physical contact with the infected person and respiratory droplet coming out from covid-19 patient [1].

At first, Iran and Italy were the highly exposed countries to covid-19 after China. Iran is the neighboring country of Pakistan and every year thousands of people came from Pakistan to Iran for religious ceremonies. While coming back from Iran some of these people were infected and they become the cause for spreading covid-19 in Pakistan. Although the government closed the Pakistan-Iran border but yet the first case was confirmed officially on February 26, 2020 from Karachi. Avoiding the spread and causalities government decided to quarantine most of the people at home [1].

The primary step taken by most of the governmental agencies to control COVID-19 is the implementation of lockdown to maintain social distance. This procedure is an excellent measure to control the spreading of the disease. Still, from an economic point of view, the complete lockdown may be the cause of a significant financial crisis for the near future. In particular, lockdown in high dense countries may reduce the disease transmission rate, although complete control may not be achievable. Hence to alive the economic status of a country, a full lockdown for an indefinite period is not desirable at all in any circumstances. Therefore there should be a suitable balance between the two different characteristics of governmental policies complete lockdown and healthy free conditions.

Mathematical modeling is considered as an effective tool for describing the dynamical behavior of infections [2–6]. Mathematicians frequently using the tools of mathematical modeling from the last century. More recently, the field of mathematical modeling got considerable attention and a number of authors put their valuable contributions in this area. For realizing and controlling the outbreak of transmissible diseases in a group, many researchers have formulated models. The second leading source of death around the globe is infectious diseases. The application of mathematical modeling has been in vogue for the study of transmissible infectious diseases. Many scholars have been discussing over the last few years about infectious diseases and their dynamics using the various approaches. To study the dynamics of various infectious diseases, mathematical modeling is considered as one of the best techniques to formulate the phenomenon in the system of equations. Several researchers have worked on different infectious diseases. They have developed different mathematical models for these epidemic diseases, and then studied the stability analysis and optimal control of these epidemic models (see e.g., [7–12]), which not only helps in the control/spread of infectious diseases but also helps in prevention of these diseases in daily life. Many researchers have worked on epidemic models to analyze and control different diseases for

example hepatitis B, avian influenza, leishmaniasis, tuberculosis etc. Modeling of epidemic models is helpful to academia as well as to daily life. Mathematical modeling has a rich literature on the transmission dynamics of the infection diseases.

Since the nature and destruction of COVID-19 depends on various parameters (namely personal immunity, history of visiting into a COVID-19 pandemic country, maintaining the required hygiene, etc.) of the affected system, using a single model we cannot de- scribe the whole disease system throughout the globe. Addition of detail and complexity can make models more accurate but this also complicates their mathematics. However, even this kind of simple model is very helpful when formal vaccination or proper treatment control is not available. Motivated by this, in the present paper, we propose a mathematical model introducing a quarantine class and governmental intervention measures like lockdown, media coverage on social distancing, and improvement of public hygiene, etc to mitigate disease transmission.

2 Model Formulation

In this section, we formulate a mathematical model on COVID-19 based on some realistic assumptions. At any time instant t, the human populations are subdivided into five time-dependent classes, namely Susceptible $S(t)$, Exposed $E(t)$, Hospitalized infected $I(t)$, Quarantine $Q(t)$ and Recovered or Removed $R(t)$. Based on those five state variables, we aim to form an autonomous system using first-order differential equations. Let A be the constant recruitment rate to the susceptible population, and β be the disease transmission rate. However the disease transmission from vulnerable to infected persons (here the class E) depends on several parameters, namely, precautions (use of face mask, social distancing, not rubbing face and nose using hand, etc.) and hygienic environment (use of soap and sanitizer, hand washing, cleaning, etc.) taken by both susceptible as well as infected persons. Since here, we have assumed that the virus of COVID-19 is spreading when a vulnerable person comes into contact with an exposed person; therefore we think that $\rho_1(0 < \rho_1 < 1)$ portion of susceptible human would maintain proper precaution measure and $\rho_2(0 < \rho_2 < 1)$ portion of the exposed class would take proper precaution measure for disease transmission (i.e., use of face mask, social distancing and implementing hygiene). Therefore the disease can only be transmitted to the $(1 - \rho_1)S$ portion of susceptible individuals due to the contact of $(1 - \rho_2)E$ portion of exposed individuals with a bi-linear disease transmission rate β. We know that a person is whether infected by the SARS-CoV-2 virus or not can be clinically detected using RT-PCR examination and a person with negative results in the RT-PCR test may still be COVID-19 positive a sit may take some days (from 7 to 21 days) to express infection. Therefore, the portion with positive COVID-19 of the class of population E is considered as infected, and they are hospitalized. Letα and b_2 be the portions of the exposed class goes to the infected class and quarantine class, respectively. It should be noted that $0 < \alpha + b_2 < 1$ since it would take quite along time to get the out put of the RT-PCR test, and sometimes it requires more than one RT-PCR analysis for a single person for confirmation of COVID-19. Let among the quarantine classes of populations, cQ portion of communities move to infected level, and the b_1Q part would become susceptible to the disease after the quarantine period. Let η and σ be respectively recovery rate of the hospitalized infected populations I and exposed class E. Let d be the natural death rate, which is common to all classes of communities and δ be the COVID-19 induced death rate. Also, it is statistically observed a person once recovered from the disease COVID-19 has very little chance to become infected again for the same disease. Hence, we assume that no portion of the recovered population moves to the susceptible class back. Also, in the formulation of a mathematical model, we understand that the people of the hospitalized infected class (i.e., $I(t)$) would not be spreading the disease or spreading an eligible amount of disease since they are kept completely isolated from the susceptible individuals. However, to control the pandemic COVID-19, suitable governmental measures (like complete or semi lock-down, rationing system, continue media coverage on social isolation and improvement of public hygiene, home deliveries of essential commodities, to make an alternate source of income for job-losers during the lockdown, etc.) have been implemented by the various governmental and non-governmental agencies. Thus this policy may be considered as one of the effective control tools, and mainly the susceptible population of COVID-19 cases would be benefited due to this policy. Mathematically we represent this policy by the parameter M and let this policy be implemented at a rate p. Therefore due to the implementation of this policy, the portion, pSM moves from susceptible to the recovered or removed class ($R(t)$). It should be noted that usually, the parameter M should be time-dependent as for the optimum result, the policy should be applied according to the necessity of the situation and keeping a proper balance between two different states, namely lockdown and complete free state. However, in the qualitative analysis of this model, we assume the parameter M as time-independent, but in formulating an optimal control problem to keep the infected individuals at the minimum level, M is assumed as a time-dependent control $M(t)$, where indeed $0 \leq M(t) \leq 1$. Further, based on the above-stated assumptions, we reformulated

an autonomous dynamical system consisting of five first-order differential equations shown as below [26]:

$$\begin{cases} C_{\mathbb{D}^{\chi}_{0,t}} S(t) = A^{\chi} - \beta^{\chi}(1-\rho_1)(1-\rho_2)SE + b_1^{\chi}Q - d^{\chi}S - p^{\chi}SM^{\chi}, \\ C_{\mathbb{D}^{\chi}_{0,t}} E(t) = \beta^{\chi}(1-\rho_1)(1-\rho_2)SE - b_2^{\chi}E - \alpha^{\chi}E - \sigma^{\chi}E - d^{\chi}E, \\ C_{\mathbb{D}^{\chi}_{0,t}} Q(t) = b_2^{\chi}E - b^{\chi}Q - c^{\chi}Q - d^{\chi}Q, \\ C_{\mathbb{D}^{\chi}_{0,t}} I(t) = \alpha^{\chi}E + c^{\chi}Q - (\eta^{\chi} + d^{\chi} + \delta^{\chi})I, \\ C_{\mathbb{D}^{\chi}_{0,t}} R(t) = \eta^{\chi}I + \sigma^{\chi}E - d^{\chi}R + p^{\chi}SM^{\chi}. \end{cases} \tag{1}$$

with

$$S(t), E(t), Q(t), I(t), R(t) \geq 0.$$

3 Existence and uniqueness of solutions for the Caputo model

The existence and uniqueness of the solution with regard to Caputo will be provided herein for the (1). Assume that a continuous real-valued function denoted by $B(J)$ containing the sup norm property is a Banach space on $J = [0, b]$ and $P = B(J) \times B(J) \times B(J) \times B(J) \times B(J)$ with norm $\|(S, E, Q, I, R)\| = \|S\| + \|E\| + \|Q\| + \|I\| + \|R\|$, where $\|S\| = \sup_{t\in J} |S(t)|, \|E\| = \sup_{t\in j} |E(t)|, \|Q\| = \sup_{t\in j} |Q(t)|, \|I\| = \sup_{t\in j} |I(t)|, \|R\| = \sup_{t\in j} |R(t)|$, Applying the Caputo fractional integral operator to the both sides of Eq.(1), we obtain

$$\begin{cases} S(t) - S(0) = C_{\mathbb{D}^{\chi}_{0,t}} S(t) \left\{A^{\chi} - \beta^{\chi}(1-\rho_1)(1-\rho_2)SE + b_1^{\chi}Q - d^{\chi}S - p^{\chi}SM^{\chi}\right\}, \\ E(t) - E(0) = C_{\mathbb{D}^{\chi}_{0,t}} E(t) \left\{\beta^{\chi}(1-\rho_1)(1-\rho_2)SE - b_2^{\chi}E - \alpha^{\chi}E - \sigma^{\chi}E - d^{\chi}E\right\}, \\ Q(t) - Q(0) = C_{\mathbb{D}^{\chi}_{0,t}} Q(t) \left\{b_2^{\chi}E - b^{\chi}Q - c^{\chi}Q - d^{\chi}Q\right\}, \\ I(t) - I(0) = C_{\mathbb{D}^{\chi}_{0,t}} I(t) \left\{\alpha^{\chi}E + c^{\chi}Q - (\eta^{\chi} + d^{\chi} + \delta^{\chi})I\right\}, \\ R(t) - R(0) = C_{\mathbb{D}^{\chi}_{0,t}} R(t) \left\{\eta^{\chi}I + \sigma^{\chi}E - d^{\chi}R + p^{\chi}SM^{\chi}\right\}. \end{cases} \tag{2}$$

And this leads to the following:

$$\begin{array}{l} S(t) - S(0) = M(\chi) \int_0^t (t-\vartheta)^{-\chi} K_1(\chi, \vartheta, S(\vartheta)) d\vartheta, \\ E(t) - E(0) = M(\chi) \int_0^t (t-\vartheta)^{-\chi} K_2(\chi, \vartheta, E(\vartheta)) d\vartheta, \\ Q(t) - Q(0) = M(\chi) \int_0^t (t-\vartheta)^{-\chi} K_3(\chi, \vartheta, Q(\vartheta)) d\vartheta, \\ I(t) - I(0) = M(\chi) \int_0^t (t-\vartheta)^{-\chi} K_4(\chi, \vartheta, I(\vartheta)) d\vartheta, \\ R(t) - R(0) = M(\chi) \int_0^t (t-\vartheta)^{-\chi} K_5(\chi, \vartheta, R(\vartheta)) d\vartheta, \end{array} \tag{3}$$

where

$$\begin{array}{l} K_1(\chi, t, S(t)) = A^{\chi} - \beta^{\chi}(1-\rho_1)(1-\rho_2)SE + b_1^{\chi}Q - d^{\chi}S - p^{\chi}SM^{\chi}, \\ K_2(\chi, t, E(t)) = \beta^{\chi}(1-\rho_1)(1-\rho_2)SE - b_2^{\chi}E - \alpha^{\chi}E - \sigma^{\chi}E - d^{\chi}E, \\ K_3(\chi, t, Q(t)) = b_2^{\chi}E - b^{\chi}Q - c^{\chi}Q - d^{\chi}Q, \\ K_4(\chi, t, I(t)) = \alpha^{\chi}E + c^{\chi}Q - (\eta^{\chi} + d^{\chi} + \delta^{\chi})I, \\ K_5(\chi, t, R(t)) = \eta^{\chi}I + \sigma^{\chi}E - d^{\chi}R + p^{\chi}SM^{\chi}. \end{array} \tag{4}$$

The symbols K_1, K_2, K_3, K_4 and K_5 have to hold for the Lipschitz condition only if $S(t), E(t), Q(t), I(t)$ and $R(t)$ possess an upper bound. Surmising that $S(t)$ and $S^*(t)$ are couple functions, we reach

$$\|K_1(\chi, t, S(t)) - K_1(\chi, t, S^*(t))\| = \|-(\beta^{\chi}(1-\rho_1)(1-\rho_2)E + d^{\chi} + p^{\chi}M^{\chi})(S(t) - S^*(t))\|. \tag{5}$$

Taking into account

$$\eta_1 := \|-(\beta^{\chi}(1-\rho_1)(1-\rho_2)E + d^{\chi} + p^{\chi}M^{\chi})\|,$$

one reaches

$$\|K_1(\chi, t, S(t)) - K_1(\chi, t, S^*(t))\| \leq \eta_1 \|S(t) - S^*(t)\|. \tag{6}$$

Continuing in the same way, one gets

$$\begin{aligned}
&\|K_2(\chi,t,E(t))-K_2(\chi,t,E^*(t))\|\leq \eta_2\|E(t)-E^*(t)\|,\\
&\|K_3(\chi,t,Q(t))-K_3(\chi,t,Q^*(t))\|\leq \eta_3\|Q(t)-Q^*(t)\|,\\
&\|K_4(\chi,t,I(t))-K_4(\chi,t,I^*(t))\|\leq \eta_4\|I(t)-I^*(t)\|,\\
&\|K_5(\chi,t,R(t))-K_5(\chi,t,R^*(t))\|\leq \eta_5\|R(t)-R^*(t)\|.
\end{aligned} \tag{7}$$

Where

$$\begin{aligned}
\eta_2 &= \|-(\beta^\chi(1-\rho_1)(1-\rho_2)S-(b_2^\chi+\alpha^\chi+\sigma^\chi+d^\chi))\|,\\
\eta_3 &= \|-(b^\chi+c^\chi+d^\chi)\|,\\
\eta_4 &= \|-(\eta^\chi+d^\chi+\delta^\chi)\|,\\
\eta_5 &= \|-d^\chi\|.
\end{aligned}$$

This implies that the Lipschitz condition has held for all the five functions. Going in a recursive manner, the expressions in (3) yields

$$\begin{aligned}
S_n(t) &= M(\chi)\int_0^t(t-\vartheta)^{-\chi}K_1(\chi,\vartheta,S_{n-1}(\vartheta))\,d\vartheta,\\
E_n(t) &= M(\chi)\int_0^t(t-\vartheta)^{-\chi}K_2(\chi,\vartheta,E_{n-1}(\vartheta))\,d\vartheta,\\
Q_n(t) &= M(\chi)\int_0^t(t-\vartheta)^{-\chi}K_3(\chi,\vartheta,Q_{n-1}(\vartheta))\,d\vartheta,\\
I_n(t) &= M(\chi)\int_0^t(t-\vartheta)^{-\chi}K_4(\chi,\vartheta,I_{n-1}(\vartheta))\,d\vartheta,\\
R_n(t) &= M(\chi)\int_0^t(t-\vartheta)^{-\chi}K_5(\chi,\vartheta,R_{n-1}(\vartheta))\,d\vartheta,
\end{aligned} \tag{8}$$

together with $S_0(t)=S(0), E_0(t)=E(0), Q_0(t)=Q(0), I_0(t)=I(0)$ and $R_0(t)=R(0)$. When the successive terms difference is taken, we get

$$\begin{aligned}
\Xi_{S,n}(t) =& S_n(t)-S_{n-1}(t)\\
=& M(\chi)\int_0^t(t-\vartheta)^{-\chi}\left(K_1(\chi,\vartheta,S_{n-1}(\vartheta))-K_1(\chi,\vartheta,S_{n-2}(\vartheta))\right)d\vartheta,\\
\Xi_{E,n}(t) =& E_n(t)-E_{n-1}(t)\\
=& M(\chi)\int_0^l(t-\vartheta)^{-\chi}\left(K_2(\chi,\vartheta,E_{n-1}(\vartheta))-K_2(\chi,\vartheta,E_{n-2}(\vartheta))\right)d\vartheta,\\
\Xi_{Q,n}(t) =& I_{1n}(t)-Q_{n-1}(t)\\
=& M(\chi)\int_0^t(t-\vartheta)^{-\chi}\left(K_3(\chi,\vartheta,Q_{n-1}(\vartheta))-K_3(\chi,\vartheta,Q_{n-2}(\vartheta))\right)d\vartheta,\\
\Xi_{I,n}(t) =& I_{2n}(t)-I_{n-1}(t)\\
=& M(\chi)\int_0^t(t-\vartheta)^{-\chi}\left(K_4(\chi,\vartheta,I_{n-1}(\vartheta))-K_4(\chi,\vartheta,I_{n-2}(\vartheta))\right)d\vartheta,\\
\Xi_{R,n}(t) =& F_n(t)-R_{n-1}(t)\\
=& M(\chi)\int_0^t(t-\vartheta)^{-\chi}\left(K_5(\chi,\vartheta,R_{n-1}(\vartheta))-K_5(\chi,\vartheta,R_{n-2}(\vartheta))\right)d\vartheta.
\end{aligned} \tag{9}$$

It is vital to observe that $S_n(t)=\sum_{i=0}^n\Xi_{S,i}(t)$, $E_n(t)=\sum_{i=0}^n\Xi_{E,i}(t)$, $Q_n(t)=\sum_{i=0}^n\Xi_{Q,i}(t)$, $I_n(t)=\sum_{i=0}^n\Xi_{I,i}(t)$, $R_n(t)=\sum_{i=0}^n\Xi_{R,i}(t)$. Additionally, by using Eqs. (6)-(7) and considering that $\Xi_{S,n-1}(t)=S_{n-1}(t)-S_{n-2}(t)$, $\Xi_{E,n-1}(t)=E_{n-1}(t)-E_{n-2}(t)$, $\Xi_{Q,n-1}(t)=Q_{n-1}(t)-Q_{n-2}(t)$, $\Xi_{I,n-1}(t)=I_{n-1}(t)-I_{n-2}(t)$, $\Xi_{R,n-1}(t)=R_{n-1}(t)-R_{n-2}(t)$, we reach

$$\begin{aligned}
&\|\Xi_{S,n}(t)\|\leq M(\chi)\eta_1\int_0^t(t-\vartheta)^{-\chi}\|\Xi_{S,n-1}(\vartheta)\|\,d\vartheta,\\
&\|\Xi_{E,n}(t)\|\leq M(\chi)\eta_2\int_0^t(t-\vartheta)^{-\chi}\|\Xi_{E,n-1}(\vartheta)\|\,d\vartheta,\\
&\|\Xi_{Q,n}(t)\|\leq M(\chi)\eta_3\int_0^t(t-\vartheta)^{-\chi}\|\Xi_{Q,n-1}(\vartheta)\|\,d\vartheta,\\
&\|\Xi_{I,n}(t)\|\leq M(\chi)\eta_4\int_0^t(t-\vartheta)^{-\chi}\|\Xi_{I,n-1}(\vartheta)\|\,d\vartheta,\\
&\|\Xi_{R,n}(t)\|\leq M(\chi)\eta_5\int_0^t(t-\vartheta)^{-\chi}\|\Xi_{R,n-1}(\vartheta)\|\,d\vartheta.
\end{aligned} \tag{10}$$

Now, the following theorem will be proved.

Theorem 1. *Surmising that the following condition holds*

$$\frac{M(\chi)}{\chi}b^{\chi}\eta_i < 1, i = 1, 2, \ldots, 5. \tag{11}$$

Then, (1) has a unique solution for $t \in [0, b]$.

Proof It is shown $S(t), E(t), Q(t), I(t)$ and $R(t)$ are bounded functions. In Addition, as can be seen from Eqs. (6) and (7), the symbols K_1, K_2, K_3, K_4 and K_5 hold for Lipchitz condition. Therefore, utilizing Eq. (10) together with a recursive hypothesis, we arrive at

$$\begin{aligned}
\|\Xi_{S,n}(t)\| &\leq \|S_0(t)\| \left(\tfrac{M(\chi)}{\chi}b^{\chi}\eta_1\right)^n,\\
\|\Xi_{E,n}(t)\| &\leq \|E_0(t)\| \left(\tfrac{M(\chi)}{\chi}b^{\chi}\eta_2\right)^n,\\
\|\Xi_{Q,n}(t)\| &\leq \|Q_0(t)\| \left(\tfrac{M(\chi)}{\chi}b^{\chi}\eta_3\right)^n,\\
\|\Xi_{I,n}(t)\| &\leq \|I_0(t)\| \left(\tfrac{M(\chi)}{\chi}b^{\chi}\eta_4\right)^n,\\
\|\Xi_{R,n}(t)\| &\leq \|R_0(t)\| \left(\tfrac{M(\chi)}{\chi}b^{\chi}\eta_5\right)^n.
\end{aligned} \tag{12}$$

Thus, one can see that sequences satisfy and exist $\|\Xi_{S,n}(t)\| \to 0$, $\|\Xi_{E,n}(t)\| \to 0$, $\|\Xi_{Q,n}(t)\| \to 0$, $\|\Xi_{I,n}(t)\| \to 0$, $\|\Xi_{R,n}(t)\| \to 0$ as $n \to \infty$. Moreover, from Eq. (12) and imposing the triangle inequality, for any k, we have

$$\begin{aligned}
\|S_{n+k}(t) - S_n(t)\| &\leq \textstyle\sum_{j=n+1}^{n+k} r_1^j = \frac{r_1^{n+1} - r_1^{n+k+1}}{1-r_1},\\
\|E_{n+k}(t) - E_n(t)\| &\leq \textstyle\sum_{j=n+1}^{n+k} r_2^j = \frac{r_2^{n+1} - r_2^{n+k+1}}{1-r_2},\\
\|Q_{n+k}(t) - Q_n(t)\| &\leq \textstyle\sum_{j=n+1}^{n+k} r_3^j = \frac{r_3^{n+1} - r_3^{n+k+1}}{1-r_3},\\
\|I_{n+k}(t) - I_n(t)\| &\leq \textstyle\sum_{j=n+1}^{n+k} r_4^j = \frac{r_4^{n+1} - r_4^{n+k+1}}{1-r_4},\\
\|R_{n+k}(t) - R_n(t)\| &\leq \textstyle\sum_{i=n+1}^{n+k} r_5^j = \frac{r_5^{n+1} - r_5^{n+k+1}}{1-r_5},
\end{aligned} \tag{13}$$

with $r_i = \frac{M(\chi)}{\chi}b^{\chi}\eta_i < 1$ by hypothesis. Therefore, S_n, E_n, Q_n, I_n and R_n can be seen as a Cauchy sequences in the Banach space $B(J)$. This has shown that they are uniformly convergent [22]. Imposing the limit theorem in Eq. (9) as $n \to \infty$ affirms that the limit of these sequences is the unique solution of (1). This guarantee the existence of a unique solution for Eq. (1) under the condition (11).

3.1 Basic reproductive number R_0

The disease free equilibrium point of system (1) is denoted by E^0, i.e

$$E^0 = (S^0, E^0, Q^0, I^0, R^0) = \left(\frac{A^{\chi}}{d^{\chi} + p^{\chi}M^{\chi}}, 0, 0, 0, \frac{A^{\chi}p^{\chi}M^{\chi}}{d^{\chi}(d^{\chi} + p^{\chi}M^{\chi})}\right). \tag{14}$$

The spread and control of the disease is basically linked with the basic reproduction number. This quantity shows the disease spread and control. If this threshold quantity $R_0 < 1$, the disease dies out from the population and the disease-free equilibrium exists which is stable locally as well as globally. It helps to control the outbreak of an epidemic through preventive measures. But if vice versa i.e. $R_0 > 1$ then the endemic equilibria are stable locally as well as globally under certain conditions. The disease takes the epidemic shape and permanently resides in the society. To find the basic reproduction number for our model.

Let (E, Q, I) is our infected compartment, then it follows from system (1):

$$\begin{cases}
C_{\mathbb{D}_{0,t}^{\chi}} E(t) = \beta^{\chi}(1-\rho_1)(1-\rho_2)SE - b_2^{\chi}E - \alpha^{\chi}E - \sigma^{\chi}E - d^{\chi}E,\\
C_{\mathbb{D}_{0,t}^{\chi}} Q(t) = b_2^{\chi}E - b_1^{\chi}Q - c^{\chi}Q - d^{\chi}Q,\\
C_{\mathbb{D}_{0,t}^{\chi}} I(t) = \alpha^{\chi}E + c^{\chi}Q - (\eta^{\chi} + d^{\chi} + \delta^{\chi})I.
\end{cases} \tag{15}$$

Using the next generation matrix approach, the Jacobian matrix J for the above system at the disease free equilibrium point E^0 is given by

$$J = \begin{pmatrix}
\beta^{\chi}(1-\rho_1)(1-\rho_2)S^0 - (b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi}) & 0 & 0\\
b_2^{\chi} & -(b^{\chi} + c^{\chi} + d^{\chi}) & 0\\
\alpha^{\chi} & c^{\chi} & -(\eta^{\chi} + d^{\chi} + \delta^{\chi})
\end{pmatrix}.$$

Now decomposing the matrix J in terms of F and V i.e $J = F - V$ we get

$$F = \begin{pmatrix} \beta^{\chi}(1-\rho_1)(1-\rho_2)S^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} (b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi}) & 0 & 0 \\ -b_2^{\chi} & (b^{\chi} + c^{\chi} + d^{\chi}) & 0 \\ -\alpha^{\chi} & -c^{\chi} & +(\eta^{\chi} + d^{\chi} + \delta^{\chi}) \end{pmatrix}.$$

The basic reproduction number (R_0) is the spectral radius of the matrix (FV^{-1}) and for the present model it is given by

$$R_0 = \frac{A^{\chi}\beta^{\chi}(1-\rho_1)(1-\rho_2)}{(d^{\chi} + p^{\chi}M^{\chi})(b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})}.$$

3.2 Invariant Region

Let the total population at time t is represented by $N(t)$, which satisfied,

$$N(t) = S(t) + E(t) + Q(t) + I(t) + R(t).$$

The equation (1) gives

$$\frac{dN}{dt} = b^{\chi} - \mu^{\chi}N - \delta^{\chi}I \leq b^{\chi} - \mu^{\chi}N.$$

Now as $t \to \infty$, we obtain $N \leq \frac{b^{\chi}}{\mu^{\chi}}$. We study (1) in the following closed set

$$\Gamma = \left\{ (S, E, Q, I, R) \in R_+^5 : 0 < S, E, Q, I, R \leq \frac{b^{\chi}}{\mu^{\chi}} \right\}.$$

3.2.1 Endemic equilibrium point

The endemic equilibrium point for the system (1) is denoted by $E^* = (S^*, E^*, Q^*, I^*, R^*)$ for which the disease is endemic in the population (i.e. at least one of E^*, Q^* and I^* is nonzero), Equations of system (1) are rearranged to get S^*, E^*, Q^*, I^* and R^*. This gives

$$\begin{cases} S^* = \dfrac{(b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})}{\beta^{\chi}(1-\rho_1)(1-\rho_2)}, \\ E^* = \dfrac{(b_1^{\chi} + c^{\chi} + d^{\chi})[A^{\chi}\beta^{\chi}(1-\rho_1)(1-\rho_2) - (d^{\chi} + p^{\chi}M^{\chi})(b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})]}{\beta^{\chi}(1-\rho_1)(1-\rho_2)[b_2^{\chi}(c^{\chi} + d^{\chi}) + (b_1^{\chi} + c^{\chi} + d^{\chi})(\alpha^{\chi} + \sigma^{\chi} + d^{\chi})]}, \\ Q^* = \dfrac{b_2^{\chi}[A^{\chi}\beta^{\chi}(1-\rho_1)(1-\rho_2) - (d^{\chi} + p^{\chi}M^{\chi})(b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})]}{\beta^{\chi}(1-\rho_1)(1-\rho_2)[b_2^{\chi}(c^{\chi} + d^{\chi}) + (b_1^{\chi} + c^{\chi} + d^{\chi})(\alpha^{\chi} + \sigma^{\chi} + d^{\chi})]}, \\ I^* = \dfrac{[\alpha^{\chi}(b_1^{\chi} + c^{\chi} + d^{\chi}) + b_2^{\chi}c^{\chi}][(d^{\chi} + p^{\chi}M^{\chi})(b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})(R_0 - 1)]}{\beta^{\chi}(1-\rho_1)(1-\rho_2)[b_2^{\chi}(c^{\chi} + d^{\chi}) + (b_1^{\chi} + c^{\chi} + d^{\chi})(\alpha^{\chi} + \sigma^{\chi} + d^{\chi})]}, \\ R^* = \dfrac{\eta^{\chi}I^* + \sigma^{\chi}E^* + p^{\chi}M^{\chi}S^*}{d^{\chi}}. \end{cases} \tag{16}$$

For $R_0 > 1$ the positivity of the above equilibrium point (16) is assured.

4 Local Stability

We establish the local stability of the system (1) in this section at corona free point E^0 as well as at corona present equilibrium point E^*.

4.1 Corona free equilibrium local stability

Theorem 2. *The corona free equilibrium (CFE) point E^0 of the system (1) is locally asymptotically stable if $R_0 < 1$.*

Proof. Jacobian matrix of the system (1) at E^0 is

$$J(E^0) = \begin{pmatrix} -(d^\chi + p^\chi M^\chi) & \beta^\chi(1-\rho_1)(1-\rho_2)S^0 & b_1^\chi & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & b_2^\chi & -(b_1^\chi + c^\chi + d^\chi) & 0 & 0 \\ 0 & \alpha^\chi & c^\chi & -(\eta^\chi + d^\chi + \delta^\chi) & 0 \\ p^\chi M^\chi & \sigma^\chi & 0 & \eta^\chi & -d^\chi \end{pmatrix}, \tag{17}$$

where $a_{22} = \beta^\chi(1-\rho_1)(1-\rho_2)S^0 - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)$.

The characteristic equation of $J(E^0)$ takes the following form:

$$[\omega_1 + d^\chi][\omega_2 + (\eta^\chi + d^\chi + \delta^\chi)][\omega_3 + (d^\chi + p^\chi M^\chi)](A\omega^2 + B\omega + C) = 0,$$

where

$$\begin{aligned} A =& 1, \\ B =& (b_1^\chi + c^\chi + d^\chi) - \beta^\chi(1-\rho_1)(1-\rho_2)S^0 + (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi), \\ =& (b_1^\chi + c^\chi + d^\chi) + (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)(1 - R_0), \\ C =& -(b_1^\chi + c^\chi + d^\chi)[\beta^\chi(1-\rho_1)(1-\rho_2)S^0 - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)], \\ =& (b_1^\chi + c^\chi + d^\chi)(b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)[1 - R_0]. \end{aligned} \tag{18}$$

It is clear from the characteristic equation that the first three eigenvalues are negative whereas for the remaining factors we need to find condition under which the real parts of the two eigenvalues are negative.
We see that $A > 0$, $B > 0$, and the value of C can either be positive or negative. Therefore, considering the value of C we have the following cases.
1: When $C < 0$, then the nature of the roots is real with one root positive and the other is negative. Thus CFE is unstable.
2: When $C = 0$, then the root are real, negative and zero, CFE is stable (not asymptotically), some trajectories will not be heading to the equilibrium point of CFE for $t \to \infty$.
3: When $B^2 - 4AC = 0$ ($C = \frac{B^2}{4A}$) then the roots are real, negative and repeated, CFE asymptotically stable.
4: When $B^2 - 4AC > 0$ ($0 < C < \frac{B^2}{4A}$) we obtain two real and negative roots, hence the CFE asymptotically stable.
5: When $B^2 - 4AC < 0$ ($C > \frac{B^2}{4A}$) then the roots are complex conjugate with negative real part, hence CFE asymptotically stable.
Again from the five cases we can decide that when $C < 0$ then CFE is stable, when $C = 0$, the CFE is stable (not asymptotically), and if $C > 0$ (**cases** 3, 4, 5) then the CFE is asymptotically stable. All the trajectory towards the CFE. So the stability of the the CFE point depends only on the value of C.
Furthermore, the value of $C > 0$ if

$$(b_1^\chi + c^\chi + d^\chi)(b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)[1 - R_0] > 0.$$

This shows that for $R_0 < 1$ the corona free equilibrium point is asymptotically stable.

4.2 At corona present equilibrium point

Now we study the local asymptotic stability of the endemic equilibrium E^*.

Theorem 3. *If $R_0 > 1$, then the corona present equilibrium E^* of the system (1) is locally asymptotically stable..*

Proof. Suppose $R_0 > 1$; then the existence of the endemic equilibrium point is assured. The Jacobian matrix of the system (1) at E^* is

$$J^{|*|} = \begin{pmatrix} a_{11} & -\beta^\chi(1-\rho_1)(1-\rho_2)S^* & b_1^\chi & 0 & 0 \\ \beta^\chi(1-\rho_1)(1-\rho_2)E^* & a_{22} & 0 & 0 & 0 \\ 0 & b_2^\chi & -(b_1^\chi + c^\chi + d^\chi) & 0 & 0 \\ 0 & \alpha^\chi & c^\chi & -(\eta^\chi + d^\chi + \delta^\chi) & 0 \\ p^\chi M^\chi & \sigma^\chi & 0 & \eta^\chi & -d^\chi \end{pmatrix}. \tag{19}$$

where $a_{11} = -\beta^{\chi}(1-\rho_1)(1-\rho_2)E^* - (d^{\chi} + p^{\chi}M^{\chi}), a_{22} = \beta^{\chi}(1-\rho_1)(1-\rho_2)S^* - (b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})$.

Clearly two eigenvalues of the Jacobian matrix $J^{|*|}$ are negative, i.e $\omega_1 = -d^{\chi}$ and $\omega_2 = -(\eta^{\chi} + d^{\chi} + \delta^{\chi})$. For more three eigenvalues we have the following reduced matrix:

$$J_1^{|*|} = \begin{pmatrix} a_{11} & -\beta^{\chi}(1-\rho_1)(1-\rho_2)S^* & b_1^{\chi} \\ \beta^{\chi}(1-\rho_1)(1-\rho_2)E^* & a_{22} & 0 \\ 0 & b_2^{\chi} & -(b_1^{\chi} + c^{\chi} + d^{\chi}) \end{pmatrix}.$$

The characteristic equation of $J_1^{|*|}$ takes the following form:

$$(\omega^3 + A_1\omega^2 + A_2\omega + A_3) = 0,$$

where

$$\begin{aligned} A_1 =& 2d^{\chi} + b_1^{\chi} + c + p^{\chi}M^{\chi} + \beta^{\chi}(1-\rho_1)(1-\rho_2), \\ A_2 =& ((b_1^{\chi} + c^{\chi} + d^{\chi})(d^{\chi} + p^{\chi}M^{\chi} + \beta^{\chi}(1-\rho_1)(1-\rho_2)E^*) + (b_2^{\chi} + \alpha^{\chi} \\ & + \sigma^{\chi} + d^{\chi})\beta^{\chi}(1-\rho_1)(1-\rho_2)), \\ A_3 =& [(b_2^{\chi} + \alpha^{\chi} + \sigma^{\chi} + d^{\chi})(b_1^{\chi} + c^{\chi} + d^{\chi}) - b_1^{\chi}b_2^{\chi}]\beta^{\chi}(1-\rho_1)(1-\rho_2)E^*. \end{aligned} \tag{20}$$

It is observe that here A_1, A_2, A_3 and $A_1A_2 - A_3$ all are positive for any parametric value. Hence following the Routh Hurwitz criterion we may conclude that the system (1) is locally asymptotically stable around its endemic equilibrium point E^*.

5 Global Asymptotic Stability

Here we discuss the global stability analysis of model (1) for both disease-free and endemic equilibrium. We use the method of Castillo Chavez [19] to establish the global stability for disease-free equilibrium, whereas, for the global stability of endemic equilibrium, the generalization of Lyapunov theory [20] is used.

5.1 Corona free equilibrium global stability

We decompose (1) into two subsystems

$$\begin{aligned} \frac{d\chi_1}{dt} &= G(\chi_1, \chi_2), \\ \frac{d\chi_2}{dt} &= H(\chi_1, \chi_2), \end{aligned} \tag{21}$$

where χ_1 and χ_2 are the number of uninfected and infected individuals respectively. Thus $\chi_1 = (S, R) \in R^2$ and $\chi_2 = (E, Q, I) \in R^3$. Let us denote the disease-free equilibrium by E^0 and define as $E^0 = (\chi_1^0, 0)$. For disease-free equilibrium the existence of global stability depends on

1. If $\frac{d\chi_1}{dt} = G(\chi_1, 0)$, χ_1^0 is globally asymptotically stable.
2. $H(\chi_1, \chi_2) = B\chi_2 - \bar{H}(\chi_1, \chi_2)$, where $\bar{H}(\chi_1, \chi_2) \geq 0$ for $(\chi_1, \chi_2) \in \Delta$,

where $B = D_{\chi_2}H(\chi_1^0, 0)$ is an M-matrix having the positive off-diagonal entries and Δ represents the feasible region. Thus the following statement holds.

Lemma 1. *The equilibrium point $E^0 = (\chi_1^0, 0)$ of the system (1) is globally asymptotically stable, if the above conditions are satisfied and $R_0 < 1$.*

Now for proving the global stability of (1) at disease-free equilibrium, we apply the above technique. Therefore, we have the following result.

Theorem 4. *At corona free equilibrium E^0, the model (1) is globally asymptotically stable if $R_0 < 1$ and unstable otherwise.*

Proof: Consider $\chi_1 = (S, R)$, $\chi_2 = (E, I)$ and define $E^0 = (\chi_1^0, 0)$, where

$$\chi_1^0 = \left(\frac{A^\chi}{d^\chi + p^\chi M^\chi}, \frac{A^\chi p^\chi M^\chi}{d^\chi (d^\chi + p^\chi M^\chi)} \right). \tag{22}$$

From the model (1), we have

$$\begin{aligned} \frac{d\chi_1}{dt} &= G(\chi_1, \chi_2), \\ \frac{d\chi_1}{dt} &= \begin{pmatrix} A^\chi - \beta^\chi(1-\rho_1)(1-\rho_2)SE + b_1^\chi Q - d^\chi S - p^\chi S M^\chi \\ \eta^\chi I + \sigma^\chi E - d^\chi R + p^\chi S M^\chi. \end{pmatrix}. \end{aligned} \tag{23}$$

For $S = S^0$, $R = R^0$, and $G(\chi_1, 0) = 0$, we get

$$G(\chi_1, 0) = \begin{pmatrix} A^\chi - (d^\chi + p^\chi M^\chi) S^0 \\ -d^\chi R^0 + p^\chi M^\chi S^0. \end{pmatrix} = 0. \tag{24}$$

From equation (24) as $t \to \infty$, $\chi_1 \to \chi_1^0$. Thus $\chi_1 = \chi_1^0$ is globally asymptotically stable.

To show the second condition, i.e.
$H(\chi_1, \chi_2) = B\chi_2 - \bar{H}(\chi_1, \chi_2)$, where $\bar{H}(\chi_1, \chi_2) \geq 0$ for $(\chi_1, \chi_2) \in \Delta$,
we have

$$B\chi_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ b_2^\chi & -(b^\chi + c^\chi + d^\chi) & 0 \\ \alpha^\chi & c^\chi & -(\eta^\chi + d^\chi + \delta^\chi) \end{pmatrix} \begin{pmatrix} E \\ Q \\ I \end{pmatrix}, \tag{25}$$

where $a_{11} = \beta^\chi(1-\rho_1)(1-\rho_2)S^0 - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)$. Since from second condition
$H(\chi_1, \chi_2) = B\chi_1 - \bar{H}(\chi_1, \chi_2)$, or $\bar{H}(\chi_1, \chi_2) = B\chi_2 - H(\chi_1, \chi_2)$, and

$$H(\chi_1, \chi_2) = \begin{pmatrix} \beta^\chi(1-\rho_1)(1-\rho_2)SE - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)E \\ b_2^\chi E - b^\chi Q - c^\chi Q - d^\chi Q \\ \alpha^\chi E + c^\chi Q - (\eta^\chi + d^\chi + \delta^\chi) I \end{pmatrix}. \tag{26}$$

Then, we can calculate $\bar{H}(\chi_1, \chi_2) = B\chi_2 - H(\chi_1, \chi_2)$,

$$\bar{H}(\chi_1, \chi_2) = \begin{pmatrix} \beta^\chi(1-\rho_1)(1-\rho_2)S^0 E - \beta^\chi(1-\rho_1)(1-\rho_2)SE \\ 0 \\ 0 \end{pmatrix} \geq 0. \tag{27}$$

Thus $\bar{H}(\chi_1, \chi_2)$ is positive definite.
The matrix B is given by

$$B = \begin{pmatrix} a_{11} & 0 & 0 \\ b_2^\chi & -(b^\chi + c^\chi + d^\chi) & 0 \\ \alpha^\chi & c^\chi & -(\eta^\chi + d^\chi + \delta^\chi) \end{pmatrix}. \tag{28}$$

As from the model (1), the total population is bounded by S_0, that is $S, E, Q, I, R \leq S_0$, so $\beta^\chi(1-\rho_1)(1-\rho_2)SE \leq \beta^\chi(1-\rho_1)(1-\rho_2)S^0 E$ which implies that $\bar{H}(\chi_1, \chi_2)$ is positive definite. Also from equation (28), it is clear that the matrix B is M-matrix that is the off diagonal element are non-negative. Thus condition 1 and 2 are satisfied, so by Lemma 1, the disease free equilibrium point E^0 is globally asymptotically stable.

5.2 Endemic equilibrium (global stability)

For the global stability of (1) at endemic equilibrium E^*, we use the geometrical approach [20]. Thus we investigate the sufficient condition through which the E^* is globally asymptotically stable. Therefore, consider the differential equation

$$\dot{x} = f(x), \tag{29}$$

where the open set $U \subset R^n$ is simply connected and $f : U \to R^n$ is a function such that $f \in C^1(U)$. Assuming that $f(x^*) = 0$ is the solution of equation (29) and for $x(t, x_0)$, the following are true.

3. There exist a compact absorbing set $K \in U$.

4. System (29) has a unique equilibrium.

The solution x^* is said to be globally asymptotically stable in U, if it is locally asymptotically stable and all trajectories in U converges to the equilibrium x^*. For $n \geq 2$, a condition satisfied for f, which precludes the existence of non-constant periodic solution of equation (29) known is Bendixson criteria. The classical Bendixson criteria $divf(x) < 0$ for $n = 2$ is robust under C^1 [20]. Furthermore a point $x_0 \in U$ is wandering for equation (29), if there exist a neighborhood N of x_0 and $\tau > 0$, such that $N \cap x(t, N)$ is empty for all $t > \tau$. Thus the following global stability principle established for autonomous system in any finite dimension.

Lemma 2. *If the conditions* $(3) - (4)$ *and Bendixson criterion are satisfied for equation (29)* $\big($*i.e. robust under* C^1 *local perturbation of* f *at all non equilibrium, non wandering point for equation (29)*$\big)$*, then* x^* *is globally asymptotically stable in* U *provided it is stable.*

Define a matrix valued function P on U by

$$P(x) = \binom{n}{2} \times \binom{n}{2}. \tag{30}$$

Equation (30) is a matrix valued function on U. Further assume that P^{-1} exist and is continuous for $x \in K$. Now define a quantity define, such that

$$\bar{q} = lim_{t\to\infty}\ sup\ sup \frac{1}{t}\int_0^t [\mu(B(x(s, x_0)))]ds, \tag{31}$$

where $J^{[2]}$ is the second additive compound matrix of J i.e. $J(x) = Uf(x)$ and $B = P_f P^{-1} + PJ^{[2]}P^{-1}$. Let $\ell(B)$ be the Lozinski measure of the matrix B with respect to the norm $\|.\|$ in R^n [21] defined by

$$\ell(B) = lim_{x\to 0}\frac{|I + Bx| - 1}{x}. \tag{32}$$

Hence if $\bar{q} < 0$, which shows that the presence of any orbit that give rise to a simple closed rectifiable curve, such as periodic orbits and heterocyclic cycles.

Lemma 3. *Let* U *is simply connected and the condition* $(3)-(4)$ *are satisfied, then the unique equilibrium* x^* *of equation (29) is globally asymptotically stable in* U*, if* $\bar{q} < 0$.

Now we apply the above techniques to prove the global stability of model (1) at endemic equilibrium. Thus we have the following stability

Theorem 5. *If* $R_0 > 1$*, then the model (33) is globally asymptotically stable at endemic equilibrium* E^* *and unstable otherwise.*

Proof: Now examine the sub-system of (1), that is

$$\begin{aligned}
C_{\mathbb{D}^\chi_{0,t}} S(t) &= A^\chi - \beta^\chi(1-\rho_1)(1-\rho_2)SE + b_1^\chi Q - d^\chi S - p^\chi S M^\chi,\\
C_{\mathbb{D}^\chi_{0,t}} E(t) &= \beta^\chi(1-\rho_1)(1-\rho_2)SE - b_2^\chi E - \alpha^\chi E - \sigma^\chi E - d^\chi E,\\
C_{\mathbb{D}^\chi_{0,t}} Q(t) &= b_2^\chi E - b^\chi Q - c^\chi Q - d^\chi Q.
\end{aligned} \tag{33}$$

The Jacobian matrix of system (33) is

$$J = \begin{pmatrix} -k_{11} & -\beta^\chi(1-\rho_1)(1-\rho_2)S^* & b_1^\chi \\ \beta^\chi(1-\rho_1)(1-\rho_2)E^* & k_{22} & 0 \\ 0 & b_2^\chi & -(b^\chi + c^\chi + d^\chi) \end{pmatrix},$$

where $k_{11} = (\beta^\chi(1-\rho_1)(1-\rho_2)E^* + d^\chi + p^\chi M^\chi)$, $\quad k_{22} = \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)$. The second additive compound matrix is

$$J^{[2]} = \begin{pmatrix} -k_{11}+k_{22} & 0 & -b_1^\chi \\ b_2^\chi & -k_{11}-(b^\chi+c^\chi+d^\chi) & -\beta^\chi(1-\rho_1)(1-\rho_2)S^* \\ 0 & \beta^\chi(1-\rho_1)(1-\rho_2)E^* & k_{22}-(b^\chi+c^\chi+d^\chi) \end{pmatrix},$$

The function $P(\chi) = P(S,E,Q) = diag\left(1, \frac{E}{Q}, \frac{E}{Q}\right)$, then

$$P_f = diag\left(0, \frac{E'Q - Q'E}{Q^2}, \frac{E'Q - Q'E}{Q^2}\right).$$

So, it follows

$$P_f P^{-1} = diag\left(0, \frac{E'}{E} - \frac{Q'}{Q}, \frac{E'}{E} - \frac{Q'}{Q}\right),$$

$$PJ^{[2]}P^{-1} = \begin{pmatrix} -k_{11}+k_{22} & 0 & -\frac{b_1^\chi Q}{E} \\ \frac{b_2^\chi E}{Q} & -k_{11}-(b^\chi+c^\chi+d^\chi) & -\beta^\chi(1-\rho_1)(1-\rho_2)S^* \\ 0 & \beta^\chi(1-\rho_1)(1-\rho_2)E^* & k_{22}-(b^\chi+c^\chi+d^\chi) \end{pmatrix}$$

The matrix $B = PJ^{[2]}P^{-1} + P_f P^{-1}$, can be express in matrix form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$\begin{aligned} B_{11} &= -K_{11} + K_{22}, \\ B_{12} &= \begin{pmatrix} 0 & -\frac{b_1^\chi Q}{E} \end{pmatrix}, \\ B_{21} &= \begin{pmatrix} \frac{b_2^\chi E}{Q} \\ 0 \end{pmatrix}, \\ B_{22} &= \begin{pmatrix} \frac{E'}{E} - \frac{Q'}{Q} - k_{11} - (b^\chi+c^\chi+d^\chi) & -\beta^\chi(1-\rho_1)(1-\rho_2)S^* \\ \beta^\chi(1-\rho_1)(1-\rho_2)E^* & \frac{E'}{E} - \frac{Q'}{Q} + k_{22} - (b^\chi+c^\chi+d^\chi) \end{pmatrix}. \end{aligned}$$

$k_{11} = (\beta^\chi(1-\rho_1)(1-\rho_2)E^* + d^\chi + p^\chi M^\chi)$, $\quad k_{22} = \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)$.
Suppose (k_1, k_2, k_3) be a vector in $\mathbb{R}^3$, with the norm $\|.\|$ defined by

$$max\{|k_1|, |k_2|, |k_3|\} = \|(k_1, k_2, k_3)\|.$$

Let $\mu(B)$ be the Lozinski measure with respect to this norm, we choose

$$\mu(B) \leq sup\{g_1, g_2\},$$

where

$$g_1 = |B_{12}| + \mu_1(B_{11}), \quad \mu_1(B_{22}) + |B_{21}| = g_2,$$

$|B_{12}|, |B_{21}|$ are matrix norms with respect to the l_1 vector norm, and μ_1 refers the Lozinski measure with respect to this l_1 norm, then

$$\mu_1(B_{11}) = -K_{11} + K_{22}, \qquad |B_{21}| = \frac{b_2^\chi E}{Q},$$

$$|B_{12}| = max\left\{0, \quad -\frac{b_1^\chi Q}{E}\right\} = 0,$$

and

$$\mu_1(B_{22}) = \left\{ \frac{E'}{E} - \frac{Q'}{Q} + k_{22} - (b^\chi + c^\chi + d^\chi) \right\}.$$

Hence, we have

$$\begin{array}{rcl} g_1 & = & -K_{11} + K_{22}, \\ g_2 & = & \frac{b_2^\chi E}{Q} + \frac{E'}{E} - \frac{Q'}{Q} + k_{22} - (b^\chi + c^\chi + d^\chi). \end{array} \tag{34}$$

From (1), we get

$$\frac{E'}{E} = \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi),$$

$$\frac{Q'}{Q} = \frac{b_2^\chi E}{Q} - (b^\chi + c^\chi + d^\chi)$$

Thus, we have

$$\begin{array}{rcl} g_1 & = & -(\beta^\chi(1-\rho_1)(1-\rho_2)E^* + d^\chi + p^\chi M^\chi) + \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi) \\ g_2 & = & \frac{E'}{E} + \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi) \end{array} \tag{35}$$

By using $S^* = \frac{(b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi)}{\beta^\chi(1-\rho_1)(1-\rho_2)}$, we get

$$\begin{array}{rcl} g_1 & = & -(\beta^\chi(1-\rho_1)(1-\rho_2)E^* + d^\chi + p^\chi M^\chi) + \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi) \\ & \leq & -(\beta^\chi(1-\rho_1)(1-\rho_2)E^* + d^\chi + p^\chi M^\chi), \\ g_2 & = & \frac{E'}{E} + \beta^\chi(1-\rho_1)(1-\rho_2)S^* - (b_2^\chi + \alpha^\chi + \sigma^\chi + d^\chi) \\ & \leq & \frac{E'}{E}. \end{array} \tag{36}$$

Moreover, we get

$$\mu(B) \leq sup\{g_1, g_2\} \leq \frac{E'}{E},$$

then

$$\frac{1}{t}\int_0^t \mu(B)ds \leq \frac{1}{t}\int_0^t (\frac{E'}{E})ds = \frac{1}{t} ln \frac{E(t)}{E(0)},$$

which implies $q = 0$, Therefore, the Bendixson criterion is verified. We prove that positive equilibrium (S^*, E^*, Q^*) is globally asymptotically stable.

Examine the sub system of system (1)

$$\begin{array}{l} C_{\mathbb{D}_{0,t}^\chi} I(t) = \alpha^\chi E + c^\chi Q - (\eta^\chi + d^\chi + \delta^\chi)I, \\ C_{\mathbb{D}_{0,t}^\chi} R(t) = \eta^\chi I + \sigma^\chi E - d^\chi R + p^\chi S M^\chi, \end{array} \tag{37}$$

Now rewrite the system of the form

$$\begin{array}{l} C_{\mathbb{D}_{0,t}^\chi} I(t) + (\eta^\chi + d^\chi + \delta^\chi)I = \alpha^\chi E^* + c^\chi Q^*, \\ C_{\mathbb{D}_{0,t}^\chi} R(t) + d^\chi R = \eta^\chi I^* + \sigma^\chi E^* + p^\chi S^* M^\chi. \end{array} \tag{38}$$

The integrating factors for the system is $e^{t(\eta^\chi + d^\chi + \delta^\chi)}$ and $e^{t(d^\chi)}$.
Using the integrating factors and solve the system .
So for large time t that is $t \to \infty$, $I \to I^*$ and $R \to R^*$, which is sufficient to prove that the endemic equilibrium point E^* is globally asymptotically stable.

6 Sensitivity Analysis

Parameters	S.Index	Value	Parameters	S.Index	Value
β	S_β	1.00000000	d	S_d	-0.4999885937
σ	S_σ	-0.7142914285	b_2	S_{b_2}	-0.3711929652
A	S_A	1.00000000	ρ_1	S_{ρ_1}	-0.7213098196
ρ_2	S_{ρ_1}	-0.4285717142	α	S_α	0.31632629344
M	S_M	0.1245906249	p	S_p	-0.2446571428

Table 1: Sensitivity indices of the reproduction number R_0 against mentioned parameters.

Determining the parameters which are helpful in decreasing the spread of infectious disease is carried out by sensitivity analysis. Forward sensitivity analysis is considered as vital component of disease modeling although its computation become tedious for complex biological model. Sensitivity analysis of R_0 have received much attention from the ecologist and epidemiologist. The basic reproduction number R_0 is the normalized forward sensitivity index that depends differentiability on a parameter ω is defined as

$$S_\omega = \frac{\omega}{R_0}\frac{\partial R_0}{\partial \omega} \tag{39}$$

Three methods are normally used to calculate the sensitivity indices, (i) by direct differentiation, (ii) by a Latin hypercube sampling method (iii) by linearizing system (1) and then solving the obtain set of linear algebraic equations. We will apply the direct differentiation method as it gives analytical expressions for the indices. The indices not only shows us the influence of various aspects associated with the spreading of infectious disease but also gives us important information regarding the comparative change between R_0 and different parameter. Consequently, it helps in developing the control strategies.

Table 1 shows that the parameters β, A, M and α have a positive influence on the reproduction number R_0, which describe that the growth or decay of these parameters say by 10 percent will increase or decrease the reproduction number by 10 percent, 10 percent, 1.2 percent and 3.1 percent, respectively. But on the other hand, the index for parameters d, b_2, σ, ρ_1, ρ_2 and p illustrates that increasing their values by 10 percent will decrease the values of reproduction number R_0 by 4.9 percent, 3.7 percent, 7.4 percent, 7.2 percent, 4.2 percent and 2.4 percent, respectively. The sensitivity of various parameters with R_0 is highlighted in Fig 1, Fig 2, Fig 3, Fig 4 and Fig 5.

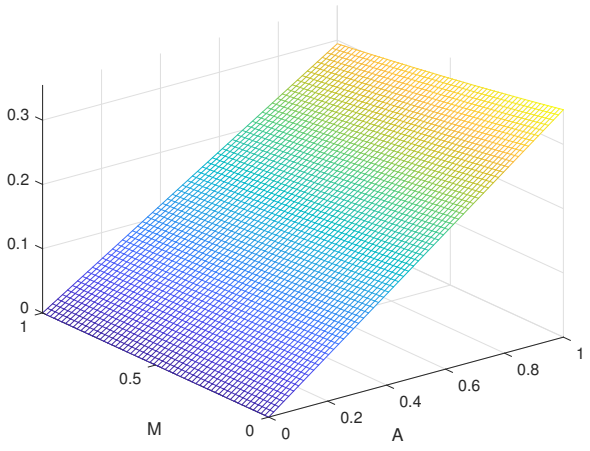


Figure 1: The plot demonstrates the variation of R_0 against M and A.

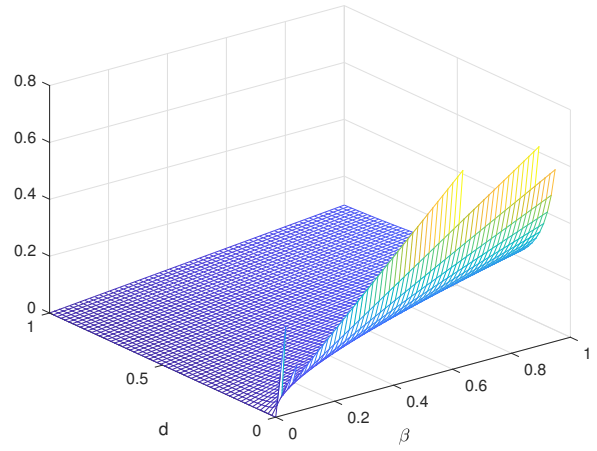


Figure 2: The plot demonstrates the variation of R_0 against β and d.

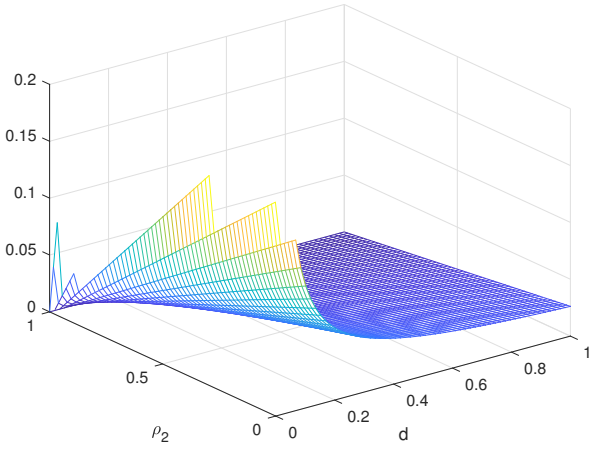


Figure 3: The plot demonstrates the variation of R_0 against ρ_2 and d.

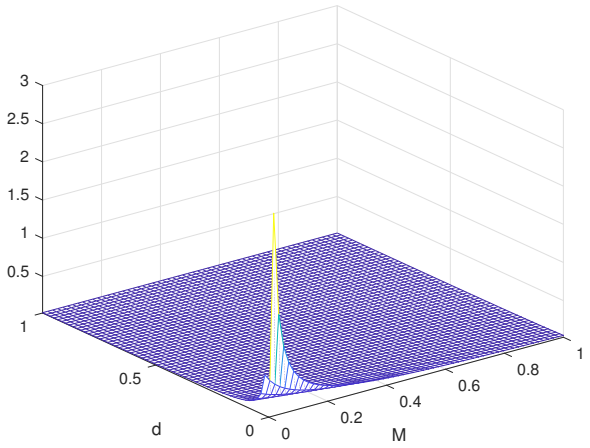


Figure 4: The plot demonstrates the variation of R_0 against M and d.

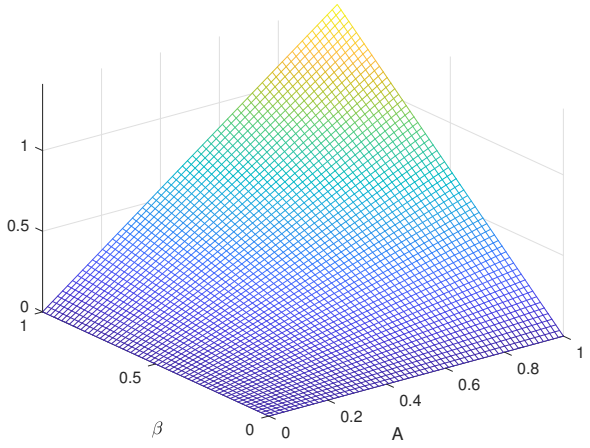


Figure 5: The plot demonstrates the variation of R_0 against β and A.

7 Numerical Simulations and Discussion

Herein, the fractional variant of the model under consideration via Caputo fractional operator is numerically simulated via first order convergent numerical techniques as proposed in [23–25]. These numerical techniques are accurate, conditionally stable, and convergent for solving fractional-order both linear and nonlinear system of ordinary differential equations.

Consider a general Cauchy problem of fractional order having autonomous nature

$$^{\star}D^{\chi}_{0,t}\Big(y(t)\Big) = g\Big(y(t)\Big), \chi \in (0,1], t \in [0,T], y(0) = y_0, \tag{40}$$

where $y = (a, b, c, w) \in \mathbb{R}^4_+$ is a real-valued continuous vector function which satisfies the Lipchitz criterion given as

$$||g(y_1(t)) - g(y_2(t))|| \leq M||y_1(t) - y_2(t)||, \tag{41}$$

where M is a positive real Lipchitz constant.

Parameters	Description	Values/Ranges
A	Total recruitment	50
β	Disease transmission rate	[0.5,2.3]
ρ_1	Portion of S contact with E	(0,1)
ρ_2	Portion of E contact with S	(0,1)
d	Natural death rate	0.2
b_1	The rate that Q becomes S	0.25
b_2	The rate that E becomes quarantine	0.8
α	The rate that E becomes I	0.3
η	The rate that I becomes R naturally	0.25
σ	The rate that E becomes R naturally	0.2
c	The rate that Q becomes I	0.12
δ	The mortality rate for I	0.25
M	Policy parameter	0.8
p	Implementation rate of policy	0.78

Using the fractional-order integral operators, one obtains

$$y(t) = y_0 + J^{\chi}_{0,t} g(y(t)),\ t \in [0, T], \tag{42}$$

where $J^{\Omega}_{0,t}$ is the fractional-order integral operator in Riemann-Liouville. Consider an equi-spaced integration intervals over $[0, T]$ with the fixed step size $h\,(= 10^{-2}$ for simulation$) = \dfrac{T}{n}$, $n \in \mathbb{N}$. Suppose that y_p be the approximation of $y(t)$ at $t = t_p$ for $p = 0, 1, \ldots n$. The numerical technique for the governing model under Caputo fractional derivative operator takes the form

$$\begin{aligned}
{}^{c}S_{p+1} &= a_0 + \frac{h^{\chi}}{\Gamma(\chi+1)} \sum_{k=0}^{p} \Big((p-k+1)^{\chi} - (p-k)^{\chi}\Big)\Big(A^{\chi} - \beta^{\chi}(1-\rho_1)(1-\rho_2)SE + b_1^{\chi}Q - d^{\chi}S - p^{\chi}SM^{\chi}\Big),\\
{}^{c}E_{p+1} &= b_0 + \frac{h^{\chi}}{\Gamma(\chi+1)} \sum_{k=0}^{p} \Big((p-k+1)^{\chi} - (p-k)^{\chi}\Big)\Big(\beta^{\chi}(1-\rho_1)(1-\rho_2)SE - b_2^{\chi}E - \alpha^{\chi}E - \sigma^{\chi}E - d^{\chi}E\Big),\\
{}^{c}Q_{p+1} &= d_0 + \frac{h^{\chi}}{\Gamma(\chi+1)} \sum_{k=0}^{p} \Big((p-k+1)^{\chi} - (p-k)^{\chi}\Big)\Big(b_2^{\chi}E - b^{\chi}Q - c^{\chi}Q - d^{\chi}Q\Big),\\
{}^{c}I_{p+1} &= e_0 + \frac{h^{\chi}}{\Gamma(\chi+1)} \sum_{k=0}^{p} \Big((p-k+1)^{\chi} - (p-k)^{\chi}\Big)\Big(\alpha^{\chi}E + c^{\chi}Q - (\eta^{\chi} + d^{\chi} + \delta^{\chi})I\Big),\\
{}^{c}R_{p+1} &= f_0 + \frac{h^{\chi}}{\Gamma(\chi+1)} \sum_{k=0}^{p} \Big((p-k+1)^{\chi} - (p-k)^{\chi}\Big)\Big(\eta^{\chi}I + \sigma^{\chi}E - d^{\chi}R + p^{\chi}SM^{\chi}\Big).
\end{aligned} \tag{43}$$

Now we discuss the obtained numerical outcomes of the governing model in respect of the approximate solutions. To this aim, we employed the effective Euler method under the Caputo fractional operator to do the job. The initial conditions and the parameters values are used as described in the table above. The initial conditions as well as the values of the parameters that are used in carrying out the simulating results are as described in subsection 2.1. The physical perspective of the model's individual state variables under the Caputo fractional operator is shown in Figure 6. In figure 7, it can be noticed that the fractional order $\chi =$ is varied for $1, 0.888, 0.666$. One can easily see the robust nature of the Caputo operator than the integer variant of the model. For a decreasing varying values of β (desease transmission rate) as shown in figures 8, $I(t)$ is also increasing. Similarly, in figures 9, the effect of mortality rate δ on $I(t)$ has been shown. For an increasing values of δ as in figure 9(a), an increasing patterns in $I(t)$ is noticed. Similarly for a decreasing values of δ, a decreasing patterns in $I(t)$ is noticed. An increasing-decreasing patterns are shown in this case.

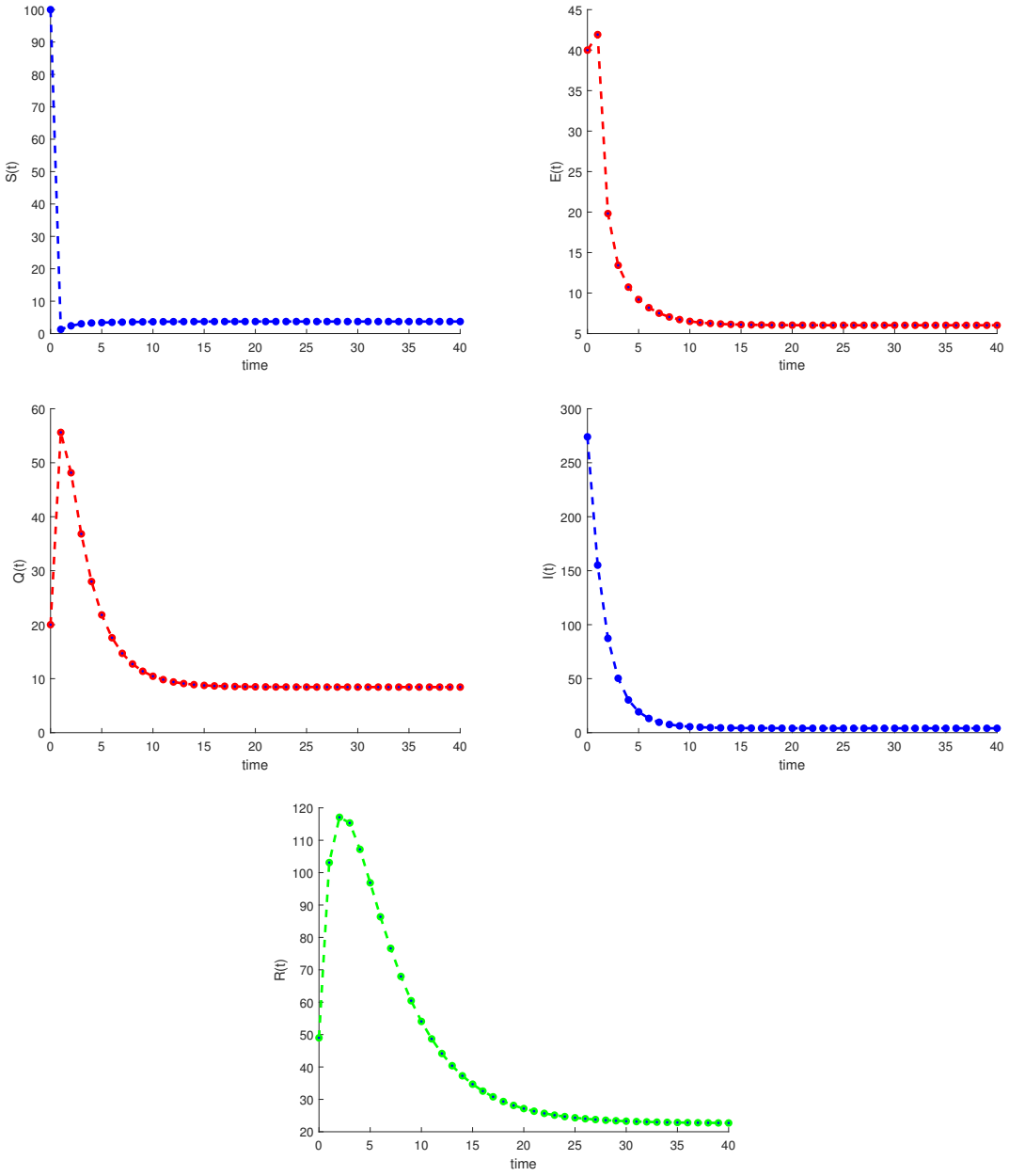


Figure 6: Profiles for behavior of each state variable for the Caputo version of the fractional model using the values of the parameters.

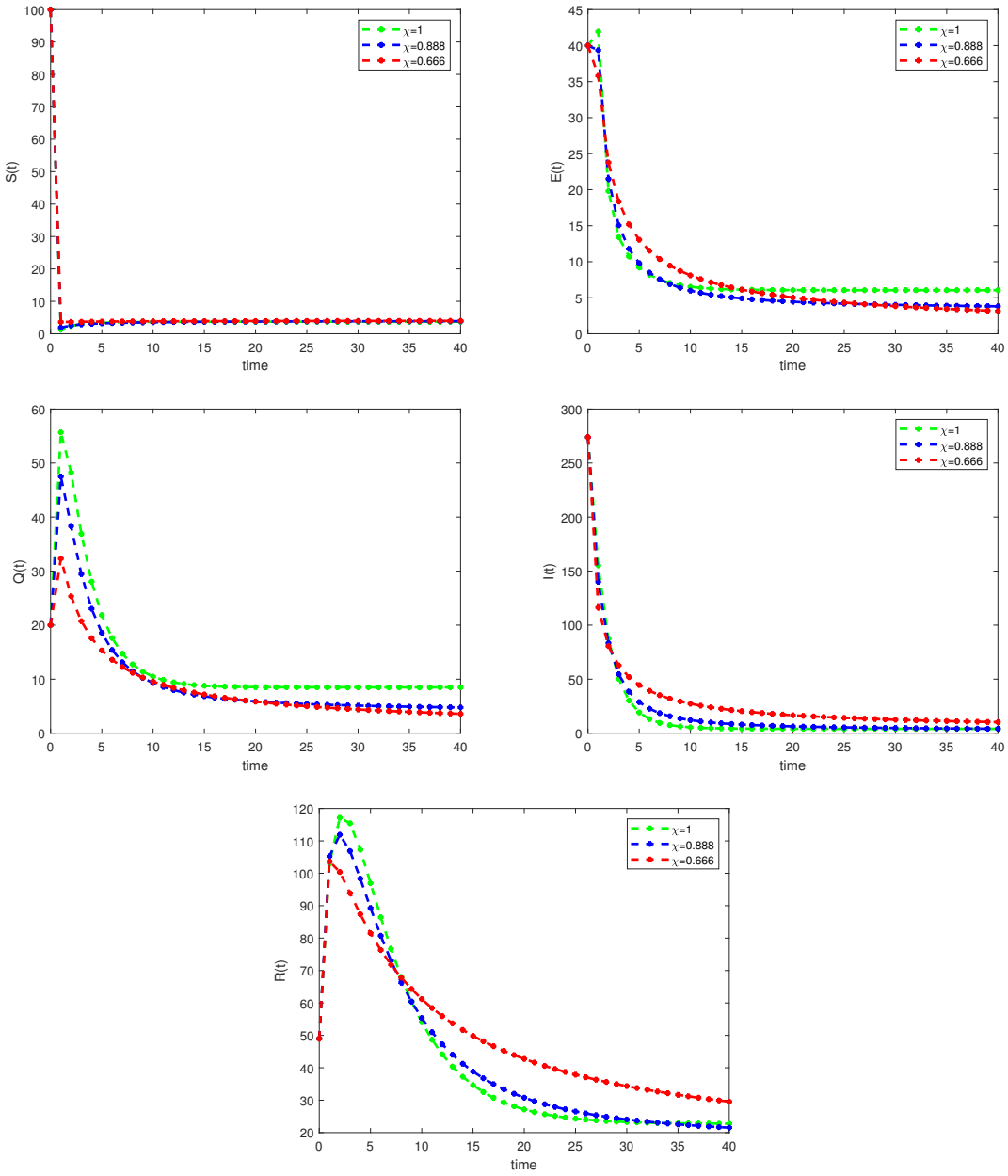
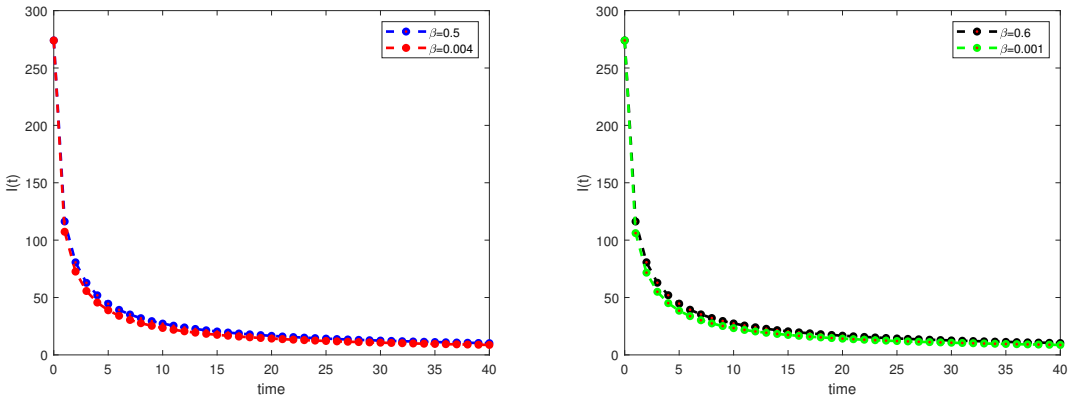
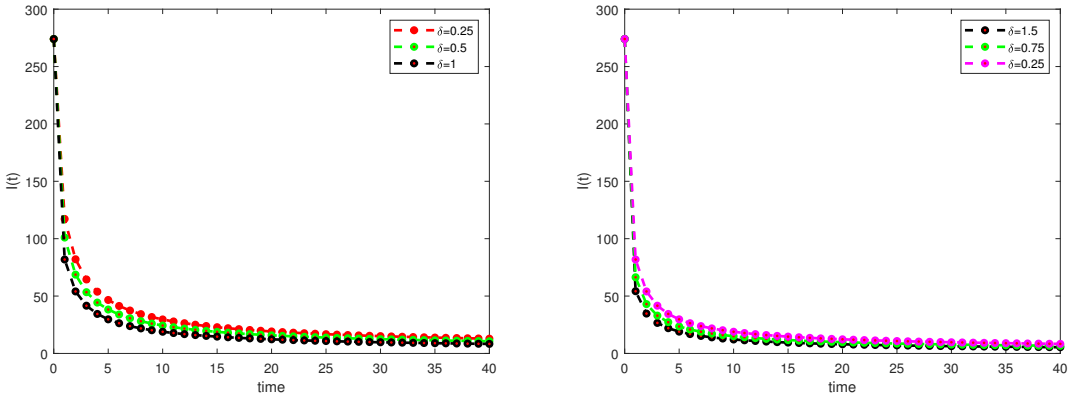


Figure 7: The dynamics of each state variable for different χ values.



(a) (b)

Figure 8: Behavior of the infectious class $I(t)$ for (a) decreasing values of β (disease transmission rate) and (b) increasing values of β (disease).



(a) (b)

Figure 9: Behavior of the infectious class $I(t)$ for (a) increasing values of δ (mortality rate) and (b) decreasing values of δ (mortality rate).

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