LONG TIME WELL-POSEDNESS OF 2-D MHD BOUNDARY LAYER EQUATION WITHOUT RESISTIVITY

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Abstract. This paper investigates the long time well-posedness of 2-D MHD boundary layer equation without resistivity. It is proved that if the initial data satisfies

$$\|(u_0,h_0-1)\|_{H^{3,0}_\mu}+\|(u_0,h_0-1)\|_{H^{1,2}_\mu}+\|(u_0,h_0-1)\|_{H^{2,1}_\mu}\le\varepsilon,$$

then the life span of the solution is at least of order $\varepsilon^{-\frac{4}{3}}$.

Keywords: 2D MHD boundary layer, long time well-posedness, Sobolev space, lifespan
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1. Introduction

The purpose of this paper is to understand the long time well-posedness of the MHD boundary layer equation without resistivity in $\Omega=\mathbf{T}\times\mathbf{R}_+$:

$$\begin{cases}\partial_t u+u\partial_x u+v\partial_y u-h\partial_x h-g\partial_y h-\nu\partial_y^2 u=\partial_x P,\\ \partial_t h+\partial_y(vh-ug)=0,\\ \partial_x u+\partial_y v=0, \quad ,\partial_x h+\partial_y g=0,\\ (u,v,h,g)|_{y=0}=0, \quad \lim\limits_{y\to\infty}(u,h)=(U(t,x),H(t,x))\\ (u,h)_{t=0}=(u_0,h_0).\end{cases}\tag{1.1}$$

where (u,v) denotes the velocity field of the boundary layer flow, (h,g) is the magnetic field and $(U(t,x),H(t,x),P(t,x))$ is the outflow of velocity, magnetic and pressure, which verifies the Bernouli law:

$$\partial_t U+U\partial_x U-H\partial_x H+\partial_x P=0,\quad \partial_t H+U\partial_x H-H\partial_x H=0.$$

It is well-known that electrically conducting fluid such as plasmas and liquid metals, the system of magnetohydrodynamics(denoted by MHD) is a fundamental system to describe the movement of fluid under the influence of electro-magnetic field. The study on the MHD was initiated by Alfvén who showed that the magnetic field can induce current in a moving conductive fluid with a new propagation mechanism along the magnetic field, called Alfven waves. The system (1.1) can be derived from the fundamental MHD system and they are more complicated than the classical Prandtl system because of the coupling of the magnetic field with velocity field through the Maxwell equations. It is also a boundary layer model, which describes the behavior of the solution to the viscous MHD equations when the viscosity and the resistivity tend to zero (see [8],[13]).

For simplicity, we consider a uniform outflow $(U, H) = (0, 1)$ and take $\nu = 1$. Let $h(t, x, y) = 1 + \tilde{h}(t, x, y)$. Then $(u, \tilde{h})$ obeys the following system

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - h\partial_x \tilde{h} - g\partial_y \tilde{h} - \partial_y^2 u = 0, \\ \partial_t \tilde{h} + u\partial_x \tilde{h} + v\partial_y \tilde{h} - h\partial_x u - g\partial_y u = 0, \\ \partial_x u + \partial_y v = 0, \quad \partial_x \tilde{h} + \partial_y g = 0, \\ (u, v, \tilde{h}, g)|_{y=0} = 0, \quad \lim_{y\to\infty} (u, \tilde{h}) = (0, 0), \\ (u, \tilde{h})_{t=0} = (u_0, \tilde{h}_0). \end{cases} \tag{1.2}$$

When $\tilde{h} = 1$, this system is the classical Prandtl equation which was first introduced by L. Prandtl in [19] to understand the structure if incompressible fluid with high Rynolds number and physical boundaries. The well posedness of the two dimensional Prandtl equation was well understand. When the tangential initial data satisfies the monotonicity condition, Oleinik [17]-[18] proved the local existence and uniqueness of classical solutions. Xin and Zhang [23] achieved the global existence of weak solutions to the Prandtl equation under an additional favourable pressure. Sammartino and Calfinish [20] showed the local well-posedness of the Prandtl equation with analytic initial data. Recently, Alexandre [1] e tal. and Masmoudi-Wong [16] established the well-posedness of the Prandtl equation for monotonic initial data in Sobolev space with energy methods independently. Without monotonicity, Gérard, Dormy obtained the ill-posedness of the Prandtl equation in Sobolev spaces in [10]. On the other hand, the Prandtl equation is well-posedness in Gevrey class 2 for a class of non-monotone data with non-degenerate critiacl points, we refer the reads to [5], [15]. For small analytic initial data, Zhang Ping and Zhang Zhifei [25] proved if the initiao data satisfies

$$\|e^{\frac{1+y^2}{8}} e^{|D_x|} u_0\|_{B^{\frac{1}{2}}} \leq \varepsilon,$$

then the lifespan of the solution of the Prandtl equation is greater than $\varepsilon^{-\frac{4}{3}}$. Ignatova and Vicol [12] achieved a bigger lifespan $\exp\{\frac{\varepsilon^{-1}}{log\varepsilon^{-1}}\}$ under the assumption the analytic initial data is small of size $\mathcal{O}(\varepsilon)$. Now it is time to give some background on MHD bounder layer equations. Liu, Xie and Yang [13] proved that the tangetial magnetic field has stabilization effect on the boundary layer of the fluid and they also obtained the well-posedness of the system (1.1) with resistivity for the initial data without monotonicity under an uniform tangential magnetic field. The long time existence of solutions to the MHD boundary layer equations in analytic setting for two different physical regimes were also investigated by Xie and Yang in [21] and [22]. When the magnetic Reynolds is much larger than the hydrodynamic Reynolds number, the resistivity terms can be ignored in MHD equations. Consequently, there is no partial viscous effect in normal variable y for the second equation in (1.1). Liu et al [14] obtained the local well posedness of the solution of (1.1) in Sobolev spaces and they also proved if the tangential magnetic field of shear layer system around at one point, then the linearized MHD boundary layer system around the shear layer profile is ill-posed in the Gevrey function space. Motivated by [12], [14] and [25], the purpose of this paper is to study the long time well-posedness of the system (1.2) and we will give the explicit lifespan of the solution of the system (1.2).

Our result is stated as follows.

Theorem 1.1. *Let* $\mu = \exp\{\frac{y^2}{8\langle t\rangle}\}$ *with* $\langle t\rangle = 1+t$. *There exists a constant* $\varepsilon > 0$ *such that if the initial data* $(u_0, \tilde{h}_0)$ *satisfies*

$$\|(u_0, \tilde{h}_0)\|_{H^{3,0}_\mu} + \|(u_0, \tilde{h}_0)\|_{H^{1,2}_\mu} + \|(u_0, \tilde{h}_0)\|_{H^{2,1}_\mu} \le \varepsilon, \tag{1.3}$$

then there exists a time $T_\varepsilon > \varepsilon^{-\frac{4}{3}}$ *such that the equations* (1.2) *has a unique solution* $(u, \tilde{h})$ *on* $[0, T_\varepsilon]$ *satisfying*

$$\begin{aligned} u \in & L^\infty([0,T_\varepsilon]; H^{3,0}_\mu \cap H^{1,2}_\mu(\Omega) \cap H^{2,1}_\mu(\Omega)) \cap L^2([0,T_\varepsilon]; H^{3,1}_\mu \cap H^{1,3}_\mu(\Omega) \cap H^{2,2}_\mu(\Omega)); \\ \tilde{h} \in & L^\infty([0,T_\varepsilon]; H^{3,0}_\mu \cap H^{1,2}_\mu(\Omega) \cap H^{2,1}_\mu(\Omega)) \cap L^2([0,T_\varepsilon]; H^{2,1}_\mu(\Omega)). \end{aligned}$$

Remark 1.2. *It is unclear whether the lifespan of the solution obtained in Theorem 1.1 is sharp. It remains open whether the solution is global in time for small data.*

The rest of this paper is organized as follows. Some crucial lemmas are presented in Section 2. Paralinearization and symmetrizing can be found in Section 3. Section 4 discusses the tangential estimates and vertical estimate is given in Section 5. Theorem 1.1 is obtained by a bootstrap argument in Section 6.

2. Preliminaries

In this section, we first introduce the Littlewood-Paley decomposition in the horizontal direction $x \in \mathbf{R}$.

Choose two smooth functions $\chi(\tau)$ and $\varphi(\tau)$, which satisfy

$$supp\varphi \subset \{\tau \in \frac{3}{4} \le |\tau| \le \frac{8}{3}\}, \quad supp\chi \subset \{\tau \in \mathbf{R} : |\tau| \le \frac{4}{3}\},$$

and for any $\tau \in \mathbf{R}$,

$$\chi(\tau) + \sum_{j\ge 0} \varphi(2^{-j}\tau) = 1.$$

Then we define

$$\begin{aligned} \Delta_j f = \mathcal{F}(\varphi(2^{-j\xi})\hat{f}), \quad S_j f = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\hat{f}) \quad for \quad j \ge 0, \\ \Delta_{-1} f = S_0 f, \quad S_j f = S_0 f, \quad for \quad j < 0. \end{aligned}$$

The Bony's decomposition $T_f g$ is defined by

$$T_f g = \sum_{j\ge -1} S_{j-1} f \Delta_j g.$$

Then we have the following Bony's decomposition

$$fg = T_f g + R_g f, \tag{2.1}$$

where the remainder term $R_g f$ is defined by

$$R_f g = \sum_{j\ge 0} \Delta_j f S_1 g + \sum_{j\ge 1, j'\ge j-1} \Delta_{j'} \Delta_j g.$$

Let $W^{s,p}$ be the usual Sobolev spaces in $\mathbf{R}$ and denote $W^{s,2}$ by H^s. We now recall classical paraproduct estimates and paraproduct calculus.

Lemma 2.1. *Let $s \in \mathbf{R}$. It holds that*

$$\|T_f g\|_{H^s} \leq C\|f\|_{L^\infty}\|g\|_{H^s}.$$

If $s > 0$, then it admits

$$\|R(f,g)\|_{H^s} \leq C\min\{\|f\|_{L^\infty}\|g\|_{H^s}, \|g\|_{L^\infty}\|f\|_{H^s}\}.$$

Lemma 2.2. *Let $s \in \mathbf{R}$ and $\sigma \in (0,1]$. It holds that*

$$\|(T_aT_b - T_{ab})f\|_{H^s} \leq C(\|b\|_{L^\infty}\|a\|_{W^{\sigma,\infty}} + \|a\|_{L^\infty}\|b\|_{W^{\sigma,\infty}})\|f\|_{H^{s-1}},$$

Especially, we have

$$\begin{aligned}&\|[T_a, T_b]f\|_{H^s} \leq\leq C(\|b\|_{L^\infty}\|a\|_{W^{\sigma,\infty}} + \|a\|_{L^\infty}\|b\|_{W^{\sigma,\infty}})\|f\|_{H^{s-1}},\\ &\|(T_a - T_{a^*})f\|_{H^s} \leq C\|a\|_{W^{\sigma,\infty}}\|f\|_{H^{s-\sigma}},\end{aligned}$$

Here T_a^* is the adjoint of T_a.

The above lemmas are referred to [2].

3. Paralinearization and symmetrization

Thanks to Bony's decomposition, we can rewrite the system (1.2) as

$$\begin{cases}\partial_t u + T_u\partial_x u + T_{\partial_y u}v - T_h\partial_x\tilde{h} - T_{\partial_y\tilde{h}}g - \partial_y^2 u = W_1,\\ \partial_t\tilde{h} + T_u\partial_x\tilde{h} + T_{\partial_y\tilde{h}}v - T_h\partial_x u - T_{\partial_y u}g = W_2,\end{cases} \tag{3.1}$$

where

$$\begin{aligned}W_1 &= -R_{\partial_x u}u - R_v\partial_y u + R_{\partial_x\tilde{h}}h + R_g\partial_y\tilde{h},\\ W_2 &= R_v\partial_y\tilde{h} - R_{\partial_x h}u + R_{\partial_x u}\tilde{h} + R_g\partial_y u.\end{aligned}$$

Now we introduce

$$h_1 = \int_0^y \tilde{h}(t,x,z)dz.$$

Then we infer from the second equation of (3.1)

$$\partial_t h_1 + T_h v - T_u g = \int_0^y W_2 dz.$$

Let us introduce the following good unknowns

$$\begin{cases}u_\alpha = u - T_{\frac{\partial_y u}{h}}h_1,\\ \tilde{h}_\alpha = \tilde{h} - T_{\frac{\partial_y h}{h}}h_1\end{cases} \tag{3.2}$$

It is easy to verify

$$\begin{cases}\partial_t u_\alpha + T_u\partial_x u_\alpha - \partial_x\tilde{h}_\alpha - T_{\tilde{h}}\partial_x\tilde{h}_\alpha - \partial_y^2 u_\alpha = G_1,\\ \partial_t\tilde{h}_\alpha - \partial_x u_\alpha - T_{\tilde{h}}\partial_x u_\alpha + T_u\partial_x\tilde{h}_\alpha = G_2,\end{cases} \tag{3.3}$$

where

$$
\begin{aligned}
G_1 =&[T_{\frac{\partial_y u}{h}}T_h - T_{\partial_y h}]v - [T_{\frac{\partial_y u}{h}}, T_u]g - [T_h T_{\frac{\partial_y h}{h}} - T_{\partial_y h}]g - T_{(\partial_t - \partial_y^2)\frac{\partial_y u}{h}} h_1 \\
&+ 2T_{\partial_y \frac{\partial_y u}{h}} \tilde{h} - T_u T_{\partial_x \frac{\partial_y h}{h}} h_1 - T_{\frac{\partial_y u}{h}} \int_0^y W_2 dz + W_1 + T_{\frac{\partial_y u}{h}} \partial_y \tilde{h} \\
=& \sum_{i=1}^{9} G_{1i},
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
G_2 =&[T_{\frac{\partial_y h}{h}}T_h - T_{\partial_y h}]v - [T_h T_{\frac{\partial_y u}{h}} - T_{\partial_y u}]g - [T_{\frac{\partial_y h}{h}}, T_u]g - T_{\partial_t \frac{\partial_y h}{h}} h_1 \\
&+ T_h T_{\partial_x \frac{\partial_y u}{h}} h_1 - T_u T_{\partial_x \frac{\partial_y h}{h}} h_1 - T_{\frac{\partial_y h}{h}} \int_0^y W_2 dz + W_2 \\
=& \sum_{i=1}^{8} G_{2i}.
\end{aligned} \tag{3.5}
$$

Moreover, it is easy to check that $(u_\alpha, \tilde{h}_\alpha)$ satisfies the condition below

$$
(u_\alpha, \tilde{h}_\alpha)|_{y=0} = (0,0), \quad \lim_{y\to\infty}(u_\alpha, \tilde{h}_\alpha) = (0,0).
$$

4. Tangential estimate

We first introduce the energy functional

$$
E(t) = (\|(u_\alpha, \tilde{h}_\alpha)\|^2_{H^{3,0}_\mu} + \|(u,\tilde{h})\|^2_{H^{1,2}_\mu} + \|(\partial_t u, \partial_t \tilde{h})\|^2_{H^{1,0}_\mu} + \|(u,\tilde{h})\|^2_{H^{2,1}_\mu}),
$$

$$
D(t) = (\|\partial_y u_\alpha\|_{H^{3,0}_\mu} + \|\partial_y u\|_{H^{1,2}_\mu} + \|\partial_y u\|^2_{H^{2,1}_\mu} + \|\partial_y \tilde{h}\|^2_{H^{2,0}_\mu} + \|\partial_y \tilde{h}\|^2_{H^{2,0}_\mu} + \|\partial_y \partial_t u\|^2_{H^{1,0}_\mu}).
$$

In this section, we always assume that $(u, \tilde{h})$ is a smooth solution of (1.2) on $[0, T]$ and

$$
\sup_{t\le T} E(t) \le C_1 \varepsilon^2, \quad T \le C_1 \varepsilon^{-2} \tag{4.1}
$$

for some $C_1 > 0$.

Let us give some crucial lemmas that could be found in [6].

Lemma 4.1. *There exists $\epsilon_0 > 0$ such that if $\varepsilon \in (0, \epsilon_0)$, then*

$$
h(t,x,y) \ge \frac{1}{2}, \quad for \quad (t,x,y) \in [0,T]\times R^2_+.
$$

Lemma 4.2. *It holds that*

$$
\| \int_0^y f dz \|_{L^\infty_y} \le C\langle t\rangle^{\frac{1}{4}} \|f\|_{L^2_{y,\mu}}.
$$

In particular, thanks to $\partial_x u + \partial_y v = 0$ and $\partial_x \tilde{h} + \partial_y g = 0$, it holds that for any $k \in \mathbf{N}$

$$
\|v\|_{H^k_x L^\infty_y} \le C\langle t\rangle^{\frac{1}{4}} \|u\|_{H^{k+1,0}_\mu}, \quad \|g\|_{H^k_x L^\infty_y} \le C\langle t\rangle^{\frac{1}{4}} \|\tilde{h}\|_{H^{k+1,0}_\mu}.
$$

Now we would like to build the bridge of norm between good unknown $(u_\alpha, \tilde{h}_\alpha)$ and $(u, \tilde{h})$.

Lemma 4.3. *There exists a positive constant ϵ_0 such that if $(0, \epsilon_0)$, then for any $t \in [0, T]$*

$$
\|u\|_{H^{3,0}_\mu} + \|\tilde{h}\|_{H^{3,0}_\mu} \le CE(t)^{\frac{1}{2}}, \quad \|\partial_y u\|_{H^{3,0}_\mu} \le CD(t)^{\frac{1}{2}}.
$$

In order to state our main result, we shall establish the following Lemma.

Lemma 4.4. *It holds that*

$$\begin{aligned}\|\partial_t u(0,\cdot)\|_{H^{1,0}_\mu} \le \|\partial_y^2 u\|_{H^{1,0}_\mu} + \|\tilde h_0\|_{H^{2,0}_\mu} + \|u_0\|_{H^{1,1}_\mu}\|u_0\|_{H^{2,0}_\mu} + \|u_0\|_{H^{1,0}_\mu}\|\partial_y u_0\|_{H^{1,0}_\mu}\\ + \|\tilde h\|_{H^{1,1}_\mu}\|\tilde h\|_{H^{2,0}_\mu} + \|\tilde h_0\|_{H^{1,0}_\mu}\|\partial_y \tilde h_0\|_{H^{1,0}_\mu}.\end{aligned}$$

and

$$\begin{aligned}\|\partial_t \tilde h(0,\cdot)\|_{H^{1,0}_\mu} \le \|u_0\|_{H^{2,0}_\mu} + C\|u_0\|_{H^{1,1}_\mu}\|\tilde h_0\|_{H^{3,0}_\mu} + C\|u_0\|_{H^{1,0}_\mu}\|\partial_y \tilde h_0\|_{H^{2,0}_\mu}\\ + \|\tilde h_0\|_{H^{1,1}_\mu}\|u_0\|_{H^{2,0}_\mu} + C\|\tilde h_0\|_{H^{1,0}_\mu}\|\partial_y u_0\|_{H^{2,0}_\mu}.\end{aligned}$$

Proof. A direct calculation yields

$$\begin{aligned}\|\partial_t u\|_{H^{1,0}_\mu} \le \|\partial_y^2 u\|_{H^{1,0}_\mu} + \|\tilde h\|_{H^{2,0}_\mu} + \|u\|_{H^{1,1}_\mu}\|u\|_{H^{2,0}_\mu} + \langle t\rangle^{\frac14}\|u\|_{H^{1,0}_\mu}\|\partial_y u\|_{H^{1,0}_\mu}\\ + \|\tilde h\|_{H^{1,1}_\mu}\|\tilde h\|_{H^{2,0}_\mu} + \langle t\rangle^{\frac14}\|\tilde h\|_{H^{1,0}_\mu}\|\partial_y \tilde h\|_{H^{1,0}_\mu}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\partial_t \tilde h\|_{H^{2,0}_\mu} \le \|u\|_{H^{2,0}_\mu} + \|u\|_{H^{1,1}_\mu}\|\tilde h\|_{H^{2,0}_\mu} + \langle t\rangle^{\frac14}\|u\|_{H^{1,0}_\mu}\|\partial_y \tilde h\|_{H^{1,0}_\mu}\\ + \|\tilde h\|_{H^{1,1}_\mu}\|u\|_{H^{2,0}_\mu} + \langle t\rangle^{\frac14}\|\tilde h\|_{H^{1,0}_\mu}\|\partial_y \tilde h\|_{H^{1,0}_\mu}.\end{aligned}$$

Then, we have

$$\begin{aligned}\|\partial_t u(0,\cdot)\|_{H^{1,0}_\mu} \le \|\partial_y^2 u\|_{H^{1,0}_\mu} + \|\tilde h_0\|_{H^{2,0}_\mu} + \|u_0\|_{H^{1,1}_\mu}\|u_0\|_{H^{2,0}_\mu} + \|u_0\|_{H^{1,0}_\mu}\|\partial_y u_0\|_{H^{1,0}_\mu}\\ + \|\tilde h\|_{H^{1,1}_\mu}\|\tilde h\|_{H^{2,0}_\mu} + \|\tilde h_0\|_{H^{1,0}_\mu}\|\partial_y \tilde h_0\|_{H^{1,0}_\mu}.\end{aligned}$$

and

$$\begin{aligned}\|\partial_t \tilde h(0,\cdot)\|_{H^{1,0}_\mu} \le \|u_0\|_{H^{2,0}_\mu} + C\|u_0\|_{H^{1,1}_\mu}\|\tilde h_0\|_{H^{3,0}_\mu} + C\|u_0\|_{H^{1,0}_\mu}\|\partial_y \tilde h_0\|_{H^{2,0}_\mu}\\ + \|\tilde h_0\|_{H^{1,1}_\mu}\|u_0\|_{H^{2,0}_\mu} + C\|\tilde h_0\|_{H^{1,0}_\mu}\|\partial_y u_0\|_{H^{2,0}_\mu}.\end{aligned}$$

□

Lemma 4.5. *It holds that*

$$\begin{aligned}\|\partial_y\partial_t u\|_{H^{1,0}_\mu} \le \|\partial_y^3 u\|_{H^{1,0}_\mu} + \|\partial_y \tilde h\|_{H^{2,0}_\mu} + \|u\|_{H^{1,1}_\mu}\|\partial_y u\|_{H^{2,0}_\mu} + C(T_\varepsilon)\|u\|_{H^{1,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu}\\ + \|\tilde h\|_{H^{1,1}_\mu}\|\partial_y \tilde h\|_{H^{2,0}_\mu} + C(T_\varepsilon)\|\tilde h\|_{H^{1,0}_\mu}\|\tilde h\|_{H^{1,2}_\mu}.\end{aligned}$$

for any $t \in [0, T_\varepsilon]$.

Proof. Thanks to the Hölder inequality, one has

$$\begin{aligned}\|\partial_y\partial_t u\|_{H^{1,0}_\mu} &\le \|\partial_y^3 u\|_{H^{1,0}_\mu} + \|\partial_y \tilde h\|_{H^{2,0}_\mu} + \|\partial_y(u\partial_x u + v\partial_y u - \tilde h\partial_x \tilde h + g\partial_y \tilde h)\|_{H^{1,0}_\mu}\\ &\le \|\partial_y^3 u\|_{H^{1,0}_\mu} + \|\partial_y \tilde h\|_{H^{2,0}_\mu} + \|u\|_{H^{1,1}_\mu}\|\partial_y u\|_{H^{2,0}_\mu} + \langle t\rangle^{\frac14}\|u\|_{H^{1,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu}\\ &+ \|\tilde h\|_{H^{1,1}_\mu}\|\partial_y \tilde h\|_{H^{2,0}_\mu} + \langle t\rangle^{\frac14}\|\tilde h\|_{H^{1,0}_\mu}\|\tilde h\|_{H^{1,2}_\mu}\end{aligned}$$

which implies

$$\begin{aligned}\|\partial_y\partial_t u\|_{H^{1,0}_\mu} \le \|\partial_y^3 u\|_{H^{1,0}_\mu} + \|\partial_y \tilde h\|_{H^{2,0}_\mu} + \|u\|_{H^{1,1}_\mu}\|\partial_y u\|_{H^{2,0}_\mu} + C(T_\varepsilon)\|u\|_{H^{1,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu}\\ + \|\tilde h\|_{H^{1,1}_\mu}\|\partial_y \tilde h\|_{H^{2,0}_\mu} + C(T_\varepsilon)\|\tilde h\|_{H^{1,0}_\mu}\|\tilde h\|_{H^{1,2}_\mu}.\end{aligned}$$

□

4.1. **Nonlinear term estimates.**

Lemma 4.6. *It holds that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u_\alpha\|^2_{H^{3,0}_\mu}+\|\tilde h_\alpha\|^2_{H^{3,0}_\mu})+\frac{1}{2}\|\partial_y u_\alpha\|^2_{H^{3,0}_\mu}\\&\le CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac34}E(t)^{\frac34}+C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)^{\frac12}E(t)^{\frac32}+CD(t)E(t).\end{aligned}\tag{4.2}$$

Proof. Making $H^{3,0}_\mu$-inner product between the equations (3.3) and $(u_\alpha,\tilde h_\alpha)$, we obtain

$$\begin{aligned}(\partial_t u_\alpha,u_\alpha)_{H^{3,0}_\mu}&-(\partial_y^3 u_\alpha,u_\alpha)_{H^{3,0}_\mu}-(\partial_x\tilde h_\alpha,u_\alpha)_{H^{3,0}_\mu}\\&+(\partial_t\tilde h_\alpha,\tilde h_\alpha)_{H^{3,0}_\mu}-(\partial_x u_\alpha,\tilde h_\alpha)_{H^{3,0}_\mu}+(T_u\partial_x u_\alpha,u_\alpha)_{H^{3,0}_\mu}\\&-(T_{\tilde h}\partial_x\tilde h_\alpha,u_\alpha)_{H^{3,0}_\mu}-(T_{\tilde h}\partial_x u_\alpha,\tilde h_\alpha)_{H^{3,0}_\mu}+(T_u\partial_x\tilde h_\alpha,\tilde h_\alpha)_{H^{3,0}_\mu}\\&=(G_1,u_\alpha)_{H^{3,0}_\mu}+(G_2,\tilde h_\alpha)_{H^{3,0}_\mu}.\end{aligned}$$

where $\mu=e^\theta$ and $\theta=\frac{y^2}{8\langle t\rangle}$. Thanks to integration by parts and using the Young inequality, we have

$$\begin{aligned}(\partial_t u_\alpha,u_\alpha)_{H^{3,0}_\mu}-(\partial_y^3 u_\alpha,u_\alpha)_{H^{3,0}_\mu}&\ge\frac{1}{2}\frac{d}{dt}\|u_\beta\|^2_{H^{3,0}_\mu}-\int_{R_+}\partial_t\theta\|e^\theta u_\alpha\|^2_{H^3_x}dy\\&+\|\partial_y u_\alpha\|^2_{H^{3,0}_\mu}+2\int_{R_+}2\partial_y\theta(e^\theta\partial_y u_\alpha,e^\theta u_\alpha)^2_{H^3_x}dy\\&\ge\frac{1}{2}\frac{d}{dt}\|u_\alpha\|^2_{H^{3,0}_\mu}+\frac{1}{2}\|\partial_y u_\alpha\|^2_{H^{3,0}_\mu}-\int_{R_+}(\partial_t\theta+2(\partial_y\theta)^2)\|e^\theta u_\alpha\|^2_{H^3_x}dy\\&\ge\frac{1}{2}\frac{d}{dt}\|u_\alpha\|^2_{H^{3,0}_\mu}+\frac{1}{2}\|\partial_y u_\alpha\|^2_{H^{3,0}_\mu}.\end{aligned}$$

where we have used the fact $\partial_t\theta+2(\partial_y\theta)^2\le0$. Similarly, one gets

$$(\partial_t\tilde h_\alpha,\tilde h_\alpha)_{H^{3,0}_\mu}=\frac{1}{2}\frac{d}{dt}\|\tilde h_\alpha\|^2_{H^{3,0}_\mu}+\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde h_\beta\|^2_{H^3_x}dy.$$

From integration by parts, one can deduce

$$-(\partial_x\tilde h_\alpha,u_\alpha)_{H^{3,0}_\mu}-(\partial_x u_\alpha,\tilde h_\alpha)_{H^{3,0}_\mu}=0.$$

The other terms on the left hand side can be handled as those in [6]. But the nonlinear term on the right hand side is somewhat different. Let us estimate the nonlinear term G_1 and G_2.

$$\begin{aligned}(G_1,u_\alpha)_{H^{3,0}_\mu}&=(\sum_{i=1}^8 G_{1i},u_\alpha)_{H^{3,0}_\mu}+(G_{19},u_\alpha)_{H^{3,0}_\mu}\\&\le\sum_{i=1}^8\|G_{1i}\|_{H^{3,0}_\mu}\|u_\beta\|_{H^{3,0}_\mu}+|(G_9,u_\alpha)_{H^{3,0}_\mu}|.\end{aligned}$$

From Lemma 4.4, we deduce

$$\sum_{i=1}^8\|G_{1i}\|_{H^{3,0}_\mu}\|u_\beta\|_{H^{3,0}_\mu}\le C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)^{\frac12}E(t)^{\frac32}.\tag{4.3}$$

Since $\partial_y \tilde{h}$ loss one derivative in the direction y, we are able to use integration by parts to transform the derivative ∂_y to u_α. More precisely,

$$
\begin{aligned}
(G_{19}, u_\alpha)_{H^{3,0}_\mu} &= (T_{\frac{\partial_y u}{h}} \partial_y \tilde{h}, u_\alpha)_{H^{3,0}_\mu} \\
&= (\partial_y (T_{\frac{\partial_y u}{h}} \partial_y \tilde{h}), u_\alpha)_{H^{3,0}_\mu} - (T_{\partial_y \frac{\partial_y u}{h}} \tilde{h}, u_\alpha)_{H^{3,0}_\mu} \\
&= -(T_{\frac{\partial_y u}{h}} \tilde{h}, e^{2\theta} \partial_y u_\alpha)_{H^{3,0}} - (T_{\frac{\partial_y u}{h}} \tilde{h}, 2\partial_y \theta e^{2\theta} u_\alpha)_{H^{3,0}} - (T_{\partial_y \frac{\partial_y u}{h}} \tilde{h}, u_\alpha)_{H^{3,0}_\mu} \\
&= H_1 + H_2 + H_3.
\end{aligned}
$$

We first bound H_1, H_2 and H_3.

$$
\begin{aligned}
-(T_{\partial_y \frac{\partial_y u}{h}} h, e^{2\theta} \partial_y u_\alpha)_{H^{3,0}} &\le \|(\partial_y \tilde{h} \partial_y u + \partial_y^2 u)\|_{L^\infty} \|\tilde{h}\|_{H^{3,0}_\mu} \|\partial_y u_\alpha\|_{H^{3,0}_\mu} \\
&\le \|\partial_y \tilde{h}\|_{L^\infty} \|\partial_y u\|_{L^\infty} \|\tilde{h}\|_{H^{3,0}_\mu} \|\partial_y u_\alpha\|_{H^{3,0}_\mu} \\
&+ \|\partial_y^2 u\|_{L^\infty} \|\tilde{h}\|_{H^{3,0}_\mu} \|\partial_y u_\alpha\|_{H^{3,0}_\mu} \\
&\le \|\partial_y \tilde{h}\|^{\frac{1}{2}}_{H^{1,0}_\mu} \|\partial_y^2 \tilde{h}\|^{\frac{1}{2}}_{H^{1,0}_\mu} \|\partial_y u\|^{\frac{1}{2}}_{H^{1,0}_\mu} \|\partial_y^2 u\|^{\frac{1}{2}}_{H^{1,0}_\mu} \|\tilde{h}\|_{H^{3,0}_\mu} \|\partial_y u_\alpha\|_{H^{3,0}_\mu} \\
&+ \|\partial_y^2 u\|^{\frac{1}{2}}_{H^{1,0}_\mu} \|\partial_y^3 u\|^{\frac{1}{2}}_{H^{1,0}_\mu} \|\tilde{h}\|_{H^{3,0}_\mu} \|\partial_y u_\alpha\|_{H^{3,0}_\mu} \\
&\le CD(t)^{\frac{1}{2}} E(t) + CD(t)^{\frac{3}{4}} E(t)^{\frac{3}{4}}.
\end{aligned}
$$

In view of the Hölder inequality, Lemma 2.1 and the Young inequality, one has

$$
\begin{aligned}
(T_{\frac{\partial_y u}{h}} \tilde{h}, 2\partial_y \theta e^{2\theta} u_\alpha)_{H^{3,0}} &\le C\|\partial_y u\|^2_{L^\infty} \|\partial_y u_\alpha\|^2_{H^{3,0}_\mu} + \frac{1}{4} \int_{R_+} \frac{y^2}{4\langle t\rangle^2} \|e^\theta \tilde{h}\|^2_{H^3_x} dy \\
&\le CD(t)E(t) + \frac{1}{4} \int_{R_+} \frac{y^2}{4\langle t\rangle^2} \|e^\theta \tilde{h}\|^2_{H^3_x} dy.
\end{aligned}
$$

Using Lemma 2.1 again, we obtain

$$
\begin{aligned}
H_3 &\le C\|T_{\partial_y \frac{\partial_y u}{h}} \tilde{h}\|_{H^{3,0}_\mu} \|u_\alpha\|_{H^{3,0}_\mu} \\
&\le C\|\partial_y \tilde{h}\|_{L^\infty} \|\partial_y u\|_{L^\infty} \|\tilde{h}\|_{H^{3,0}_\mu} \|u_\alpha\|_{H^{3,0}_\mu} + \|\partial_y^2 u\|_{L^\infty} \|\tilde{h}\|_{H^{3,0}_\mu} \|u_\alpha\|_{H^{3,0}_\mu} \\
&\le CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}.
\end{aligned}
$$

Thus, one has

$$
(G_{19}, u_\alpha)_{H^{3,0}_\mu} \le CD(t)^{\frac{3}{4}} E(t)^{\frac{3}{4}} + \frac{1}{4} \int_{R_+} \frac{y^2}{4\langle t\rangle^2} \|e^\theta \tilde{h}\|^2_{H^3_x} dy + CD(t)^{\frac{1}{2}} E(t) + CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}},
$$

which along with (4.3) gives rise to

$$
\begin{aligned}
(G_1, u_\alpha)_{H^{3,0}_\mu} \le& \frac{1}{4} \int_{R_+} \frac{y^2}{4\langle t\rangle^2} \|e^\theta \tilde{h}\|^2_{H^3_x} dy + C\langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) \\
&+ C\langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + CD(t)^{\frac{3}{4}} E(t)^{\frac{3}{4}}.
\end{aligned}
$$

Now we are reminded to estimate $(G_2, \tilde{h}_\alpha)_{H^{3,0}_\mu}$.

$$
(G_2, \tilde{h}_\alpha)_{H^{3,0}_\mu} \le \sum_{i=1}^{8} \|G_{2i}\|_{H^{3,0}_\mu} \|\tilde{h}_\alpha\|_{H^{3,0}_\mu}.
$$

Now we have to pay our attention to estimate $\|G_{24}\|_{H^{3,0}_\mu}$, which cannot be found in [6].

A direct calculation gives

$$\begin{aligned}
\partial_t \frac{\partial_y \tilde{h}}{h} &= \frac{-\partial_t \tilde{h}\partial_y \tilde{h}}{h^2} + \frac{\partial_y \partial_t \tilde{h}}{h} \\
&= \frac{-u\partial_x \tilde{h}\partial_y \tilde{h} - v\partial_y u \partial_y \tilde{h} + h\partial_x \tilde{u}\partial_y \tilde{h} + g\partial_y u \partial_y \tilde{h}}{h^2} \\
&- \frac{\partial_y u\partial_x \tilde{h} + u\partial_y\partial_x \tilde{h} - \partial_x u \partial_y \tilde{h} + v\partial_y^2 \tilde{h} - \partial_y h \partial_x u - h\partial_x\partial_y u + \partial_x \tilde{h}\partial_y u - g\partial_y^2 u}{h} \\
&= J_1 + J_2.
\end{aligned}$$

Then

$$\|T_{\partial_t \frac{\partial_y \tilde{h}}{h}} h_1\|_{H^{3,0}_\mu} = \|T_{J_1} h_1\|_{H^{3,0}_\mu} + \|T_{J_2} h_1\|_{H^{3,0}_\mu}$$

Now we deal with $\|T_{J_1} h_1\|_{H^{3,0}_\mu}$ and $\|T_{J_2} h_1\|_{H^{3,0}_\mu}$, respectively. Thanks to Lemma 4.2, we have

$$\begin{aligned}
\|T_{J_1} h_1\|_{H^{3,0}_\mu} &\leq \|T_{-u\partial_x \tilde{h}\partial_y \tilde{h} - v\partial_y u \partial_y \tilde{h} + g\partial_y u \partial_y \tilde{h}} h_1\|_{H^{3,0}_\mu} + \|T_{\partial_x \tilde{u}\partial_y \tilde{h}} h_1\|_{H^{3,0}_\mu} \\
&\leq \| - u\partial_x \tilde{h}\partial_y \tilde{h} - v\partial_y u \partial_y \tilde{h} + g\partial_y u \partial_y \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|h_1\|_{L^\infty_y H^3_x} + \|\partial_x \tilde{u}\partial_y \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|h_1\|_{L^\infty_y H^3_x} \\
&\leq \langle t\rangle^{\frac{1}{4}} \|u\|_{L^\infty} \|\partial_x \tilde{h}\|_{L^\infty} \|\partial_y \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|\tilde{h}\|_{H^{3,0}_\mu} \\
&+ \langle t\rangle^{\frac{1}{4}} \|v\|_{L^\infty} \|\partial_y u\|_{L^\infty} \|\partial_y \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|\tilde{h}\|_{H^{3,0}_\mu} \\
&+ \langle t\rangle^{\frac{1}{4}} \|\partial_x u\|_{L^\infty} \|\partial_y \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|\tilde{h}\|_{H^{3,0}_\mu} \\
&+ \langle t\rangle^{\frac{1}{4}} \|g\|_{L^\infty} \|\partial_y u\|_{L^\infty} \|\partial_y \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|\tilde{h}\|_{H^{3,0}_\mu} \\
&\leq \langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}} + \langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + \langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
&\|T_{J_2} h_1\|_{H^{3,0}_\mu} \leq \|T_{\partial_y u\partial_x \tilde{h} + u\partial_y\partial_x \tilde{h} - \partial_x u \partial_y \tilde{h} + v\partial_y^2 \tilde{h} - \partial_y h \partial_x u + \partial_x \tilde{h}\partial_y u - g\partial_y^2 u} h_1\|_{H^{3,0}_\mu} + \|T_{\partial_x \partial_y u} h_1\|_{H^{3,0}_\mu} \\
&\leq \|\partial_y u\partial_x \tilde{h} + u\partial_y\partial_x \tilde{h} - \partial_x u \partial_y \tilde{h} + v\partial_y^2 \tilde{h}\|_{L^\infty_x L^2_{y,\mu}} \|h_1\|_{L^\infty_y H^3_x} \\
&+ \| - \partial_y h \partial_x u - h\partial_x\partial_y u + \partial_x \tilde{h}\partial_y u - g\partial_y^2 u\|_{L^\infty_x L^2_{y.\mu}} \|h_1\|_{L^\infty_y H^3_x} + \|\partial_x \partial_y u\|_{L^\infty_x L^2_{y.\mu}} \|h_1\|_{L^\infty_y H^3_x} \\
&\leq C\langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + C\langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t).
\end{aligned}$$

Then, we can deduce

$$\|G_{24}\|_{H^{3,0}_\mu} \leq C\langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + C\langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t) + C\langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}}. \tag{4.4}$$

Thanks to Lemma 4.4 in [6] again, we have

$$\sum_{i=1, i\neq 4}^{8} \|G_{2i}\|_{H^{3,0}_\mu} \leq C\langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + C\langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t).$$

which together with (4.4) yields

$$(G_2, \tilde{h}_\alpha)_{H^{3,0}_\mu} \leq \|G_2\|_{H^{3,0}_\mu} \|\tilde{h}_\alpha\|_{H^{3,0}_\mu} \leq C\langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + C\langle t\rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}}.$$

Collecting all the above estimates, we can build

$$\frac{1}{2}\frac{d}{dt}(\|u_\alpha\|^2_{H^{3,0}_\mu} + \|\tilde{h}_\alpha\|^2_{H^{3,0}_\mu}) + \frac{1}{2}\|\partial_y u_\alpha\|^2_{H^{3,0}_\mu} + \frac{1}{2}\int_{R_+} \frac{y^2}{4\langle t\rangle^2} \|e^\theta \tilde{h}_\beta\|^2_{H^3_x} dy$$

$$\le CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac34}E(t)^{\frac34}+C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)^{\frac12}E(t)^{\frac32}+CD(t)E(t).$$

□

5. Vertical estimate

In this section, we will derive the high derivative norm in the vertical variable y. We also assume that $(u,\tilde h)$ is a solution of (1.2) on $[0,T]$ satisfying (4.1).

5.1. Vertical estimates of the velocity field.

Proposition 5.1. *It holds that*

$$\begin{aligned}\frac12\frac{d}{dt}(\|u\|^2_{H^{1,0}_\mu}+\|\tilde h\|^2_{H^{1,0}_\mu})+\frac12\|\partial_y u\|^2_{H^{1,0}_\mu}+\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde h\|^2_{H^1_x}dy\\ \le CD(t)^{\frac14}E(t)^{\frac54}+\langle t\rangle^{\frac14}D(t)^{\frac12}E(t).\end{aligned}\tag{5.1}$$

Proof. Performing $H^{1,0}_\mu$ inner product with $(u,\tilde h)$ to the equation (1.2), we obtain

$$\begin{aligned}&(\partial_t u,u)_{H^{1,0}_\mu}+(\partial_t\tilde h,\tilde h)_{H^{1,0}_\mu}-(\partial_y^2u,u)_{H^{1,0}_\mu}\\ &=(\partial_x\tilde h,u)_{H^{1,0}_\mu}+(\partial_x u,\tilde h)_{H^{1,0}_\mu}+\Big(-u\partial_x u-v\partial_y u+\tilde h\partial_x\tilde h+g\partial_y\tilde h,u\Big)_{H^{1,0}_\mu}\\ &+\Big(-u\partial_x\tilde h-v\partial_y\tilde h+\tilde h\partial_x u+g\partial_y u,\tilde h\Big)_{H^{1,0}_\mu}\\ &=I_1+I_2+I_3+I_4.\end{aligned}\tag{5.2}$$

It is easy to check that by integration by parts

$$I_1+I_2=0,$$

and

$$\begin{aligned}(\partial_t u,u)_{H^{1,0}_\mu}+(\partial_t\tilde h,\tilde h)_{H^{1,0}_\mu}-(\partial_y^2u,u)_{H^{1,0}_\mu}&\ge\frac12\frac{d}{dt}(\|u\|^2_{H^{1,0}_\mu}+\|h\|^2_{H^{1,0}_\mu})+\frac12\|\partial_y u\|^2_{H^{1,0}_\mu}\\ &-\int_{R_+}(\partial_t\theta+2(\partial_y\theta)^2)\|e^\theta u\|^2_{H^1_x}dy+\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde h\|^2_{H^1_x}dy\\ &\ge\frac12\frac{d}{dt}(\|u\|^2_{H^{1,0}_\mu}+\|\tilde h\|^2_{H^{1,0}_\mu})+\frac12\|\partial_y u\|^2_{H^{1,0}_\mu}+\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde h\|^2_{H^1_x}dy.\end{aligned}\tag{5.3}$$

The estimate of I_3+I_4 can be found in [6], that is,

$$I_3+I_4\le CD(t)^{\frac14}E(t)^{\frac54}+\langle t\rangle^{\frac14}D(t)^{\frac12}E(t).$$

Thus we deduce that

$$\begin{aligned}\frac12\frac{d}{dt}(\|u\|^2_{H^{1,0}_\mu}+\|h\|^2_{H^{1,0}_\mu})+\frac12\|\partial_y u\|^2_{H^{1,0}_\mu}+\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde h\|^2_{H^1_x}dy\\ \le CD(t)^{\frac14}E(t)^{\frac54}+\langle t\rangle^{\frac14}D(t)^{\frac12}E(t).\end{aligned}$$

□

The following Lemma gives the $H^{1,1}_\mu$ estimate.

Proposition 5.2. *It holds that*

$$\frac{1}{2}\frac{d}{dt}(\|\partial_y u\|^2_{H^{1,0}_\mu} + \|\partial_y \tilde{h}\|^2_{H^{1,0}_\mu}) + \frac{1}{2}\|\partial_y^2 u\|^2_{H^{1,0}_\mu} + \int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y \tilde{h}\|^2_{H^1_x} dy \leq D(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + \langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \tag{5.4}$$

Proof. We apply ∂_y to the result equations (1.2) and then take $H^{1,0}_\mu$-inner product with $(\partial_y u, \partial_y \tilde{h})$ to get

$$\begin{aligned}
&(\partial_t\partial_y u, \partial_y u)_{H^{1,0}_\mu} + (\partial_t\partial_y \tilde{h}, \partial_y \tilde{h})_{H^{1,0}_\mu} - (\partial_y^3 u, \partial_y u)_{H^{1,0}_\mu} \\
&\quad = \Big(\partial_y(-u\partial_x u - v\partial_y u + \tilde{h}\partial_x\tilde{h} + g\partial_y\tilde{h}), \partial_y u\Big)_{H^{1,0}_\mu} \\
&\qquad + \Big(\partial_y(-u\partial_x\tilde{h} - v\partial_y\tilde{h} + \tilde{h}\partial_x u + g\partial_y u), \partial_y\tilde{h}\Big)_{H^{1,0}_\mu} \\
&\quad = (\partial_x\partial_y\tilde{h}, \partial_y u)_{H^{1,0}_\mu} + (\partial_x\partial_y u, \partial_y\tilde{h})_{H^{1,0}_\mu} + (-u\partial_x\partial_y u - v\partial_y^2 u + \tilde{h}\partial_x\partial_y\tilde{h} \\
&\qquad + g\partial_y^2\tilde{h}, \partial_y u)_{H^{1,0}_\mu} + (-\partial_y u\partial_x\tilde{h} - u\partial_x\partial_y\tilde{h} + \partial_x u\partial_y\tilde{h} - v\partial_y^2\tilde{h}, \partial_y\tilde{h})_{H^{1,0}_\mu} \\
&\qquad + (\partial_y\tilde{h}\partial_x u + \tilde{h}\partial_x\partial_y u - \partial_x\tilde{h}\partial_y u + g\partial_y^2 u, \partial_y\tilde{h})_{H^{1,0}_\mu} \\
&\quad = A_1 + A_2 + A_3.
\end{aligned} \tag{5.5}$$

where we have used the fact

$$(\partial_x\partial_y\tilde{h}, \partial_y u)_{H^{1,0}_\mu} + (\partial_x\partial_y u, \partial_y\tilde{h})_{H^{1,0}_\mu} = 0.$$

Applying the same line to (5.3) yields

$$\begin{aligned}
&(\partial_t\partial_y u, \partial_y u)_{H^{1,0}_\mu} + (\partial_t\partial_y \tilde{h}, \partial_y \tilde{h})_{H^{1,0}_\mu} - (\partial_y^3 u, \partial_y u)_{H^{1,0}_\mu} \\
&\geq \frac{1}{2}\frac{d}{dt}(\|\partial_y u\|^2_{H^{1,0}_\mu} + \|\partial_y \tilde{h}\|^2_{H^{1,0}_\mu}) + \frac{1}{2}\|\partial_y^2 u\|^2_{H^{1,0}_\mu} + \int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y \tilde{h}\|^2_{H^1_x} dy.
\end{aligned} \tag{5.6}$$

A straight calculation gives

$$\begin{aligned}
A_1 \leq\ & \|u\|_{L^\infty}\|\partial_y u\|_{H^{1,0}_\mu}\|\partial_y u\|_{H^{1,0}_\mu} + \|v\|_{L^\infty}\|\partial_y^2 u\|_{H^{1,0}_\mu}\|\partial_y u\|_{H^{1,0}_\mu} \\
& + \|\tilde{h}\|_{L^\infty}\|\partial_y\tilde{h}\|_{H^{2,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} + \|g\|_{L^\infty}\|\partial_y^2\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y u\|_{H^{1,0}_\mu} \\
\leq\ & D(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + \langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t),
\end{aligned}$$

$$\begin{aligned}
A_2 \leq\ & \|\partial_y u\|_{L^\infty}\|\partial_x\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} + \|u\|_{L^\infty}\|\partial_y\tilde{h}\|_{H^{2,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} \\
& + \|\partial_x u\|_{L^\infty}\|\partial_y\tilde{h}\|^2_{H^{1,0}_\mu} + \|g\|_{L^\infty}\|\partial_y^2\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} \\
\leq\ & D(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + \langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t),
\end{aligned}$$

and

$$\begin{aligned}
A_3 \leq\ & \|\partial_x u\|_{L^\infty}\|\partial_y\tilde{h}\|^2_{H^{1,0}_\mu} + \|\tilde{h}\|_{L^\infty}\|\partial_y u\|_{H^{2,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} \\
& + \|\partial_y u\|_{L^\infty}\|\partial_x\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} + \|g\|_{L^\infty}\|\partial_y^2 u\|_{H^{1,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu} \\
\leq\ & D(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + \langle t\rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t).
\end{aligned}$$

Collecting the estimates of A_1, A_2 and A_3, we have

$$\frac{1}{2}\frac{d}{dt}(\|\partial_y u\|^2_{H^{1,0}_\mu} + \|\partial_y \tilde{h}\|^2_{H^{1,0}_\mu}) + \frac{1}{2}\|\partial_y^2 u\|^2_{H^{1,0}_\mu} + \int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y \tilde{h}\|^2_{H^1_x} dy$$
$$\leq D(t)^{\frac14}E(t)^{\frac54} + \langle t\rangle^{\frac14} D(t)^{\frac12}E(t).$$

□

Proposition 5.3. *It holds that*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\partial_y^2 u\|^2_{H^{1,0}_\mu} + \|\partial_y^2 \tilde{h}\|^2_{H^{1,0}_\mu}) + \frac{1}{2}\|\partial_y^3 u\|^2_{H^{1,0}_\mu} + \frac{1}{2}\int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y^2 \tilde{h}\|^2_{H^1_x} dy\\ \leq \frac{1}{16}\int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y \tilde{h}\|^2_{H^1_x} dy + C\langle t\rangle^{\frac12} D(t)E(t) + D(t)^{\frac14}E(t)^{\frac54} + \langle t\rangle^{\frac14} D(t)^{\frac12}E(t).\end{aligned} \tag{5.7}$$

Proof. Taking ∂_y^2 to (1.2) and making the $H^{1,0}_\mu$-inner product with $(\partial_y^2 u, \partial_y^2 \tilde{h})$, we obtain

$$\begin{aligned}(\partial_t\partial_y^2 u,&\partial_y^2 u)_{H^{1,0}_\mu} + (\partial_t \partial_y^2\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_\mu} - (\partial_y^4 u, \partial_y^2 u)_{H^{1,0}_\mu}\\ =&(\partial_y^2(-u\partial_x u - v\partial_y u), \partial_y^2 u)_{H^{1,0}_\mu} + (\partial_y^2(\tilde{h}\partial_x\tilde{h} + g\partial_y\tilde{h}), \partial_y^2 u)_{H^{1,0}_\mu}\\ &+ (\partial_y^2(-u\partial_x\tilde{h} - v\partial_y\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_\mu} + (\partial_y^2(\tilde{h}\partial_x u + g\partial_y u), \partial_y^2\tilde{h})_{H^{1,0}_\mu}\\ =&B_1 + B_2 + B_3 + B_4.\end{aligned} \tag{5.8}$$

where we have used the fact

$$-(\partial_x\partial_y^2\tilde{h}, \partial_y^2 u)_{H^{1,0}_\mu} - (\partial_x\partial_y^2 u, \partial_y^2\tilde{h})_{H^{1,0}_\mu} = 0.$$

Reasoning as (5.3) and using the boundary condition $\partial_y^2 u|_{y=0} = 0$ yield

$$\begin{aligned}(\partial_t\partial_y^2 u, \partial_y^2 u)_{H^{1,0}_\mu} + (\partial_t \partial_y^2\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_\mu} - (\partial_y^4 u, \partial_y^2 u)_{H^{1,0}_\mu}\\ \geq \frac{1}{2}\frac{d}{dt}(\|\partial_y^2 u\|^2_{H^{1,0}_\mu} + \|\partial_y^2 \tilde{h}\|^2_{H^{1,0}_\mu}) + \frac{1}{2}\|\partial_y^3 u\|^2_{H^{1,0}_\mu} + \int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y^2 \tilde{h}\|^2_{H^1_x} dy.\end{aligned} \tag{5.9}$$

Leibniz law and Lemma 4.5 tell us

$$\begin{aligned}B_1 &= (-\partial_y u\partial_x\partial_y u - u\partial_x\partial_y^2 u + \partial_x u\partial_y^2 u - v\partial_y^3 u, \partial_y^2 u)_{H^{1,0}_\mu}\\ &\leq \|\partial_y u\|_{L^\infty}\|\partial_y u\|_{H^{2,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu} + \|u\|_{L^\infty}\|\partial_y^2 u\|_{H^{2,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu}\\ &+ \|\partial_x u\|_{L^\infty}\|\partial_y^2 u\|_{H^{1,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu} + \|v\|_{L^\infty}\|\partial_y^3 u\|_{H^{1,0}_\mu}\|\partial_y^2 u\|_{H^{1,0}_\mu}\\ &\leq D(t)^{\frac14}E(t)^{\frac54} + \langle t\rangle^{\frac14} D(t)^{\frac12}E(t).\end{aligned}$$

Similarly, one has

$$\begin{aligned}B_4 &= (\partial_y^2\tilde{h}\partial_x u + 2\partial_y\tilde{h}\partial_x\partial_y u + \tilde{h}\partial_x\partial_y^2 u - \partial_x\partial_y\tilde{h}\partial_y u - 2\partial_x\tilde{h}\partial_y^2 u + g\partial_y^3 u, \partial_y^2\tilde{h})_{H^{1,0}_\mu}\\ &\leq D(t)^{\frac14}E(t)^{\frac54} + \langle t\rangle^{\frac14} D(t)^{\frac12}E(t).\end{aligned}$$

Thanks to integration by parts, we obtain

$$\begin{aligned}B_2 &= (\partial_y^2(\tilde{h}\partial_x\tilde{h}, \partial_y^2 u)_{H^{1,0}_\mu} - (\partial_y(g\partial_y\tilde{h}), e^{2\theta}\partial_y^3 u)_{H^{1,0}} - (\partial_y(g\partial_y\tilde{h}), 2e^{2\theta}\partial_y\theta\partial_y^2 u)_{H^{1,0}}\\ &= B_{21} + B_{22} + B_{23}.\end{aligned}$$

A direct calculation gives

$$
\begin{aligned}
B_{21} &= (\partial_y^2\tilde h\partial_x\tilde h + 2\partial_y\tilde h\partial_x\partial_y\tilde h, \partial_y^2 u)_{H^{1,0}_\mu} + (\tilde h\partial_x\partial_y^2\tilde h, \partial_y^2 u)_{H^{1,0}_\mu}\\
&= (\partial_y^2\tilde h\partial_x\tilde h + 2\partial_y\tilde h\partial_x\partial_y\tilde h, \partial_y^2 u)_{H^{1,0}_\mu} - (\partial_x\tilde h\partial_y^2\tilde h, \partial_y^2 u)_{H^{1,0}_\mu} - (\tilde h\partial_y^2\tilde h, \partial_x\partial_y^2 u)_{H^{1,0}_\mu}\\
&\le D(t)^{\frac14}E(t)^{\frac54} + D(t)^{\frac12}E(t).
\end{aligned}
$$

$$
B_{22} = (\partial_x\tilde h\partial_y\tilde h + g\partial_y^2\tilde h, \partial_y^3 u)_{H^{1,0}_\mu} \le \langle t\rangle^{\frac14}D(t)^{\frac12}E(t).
$$

To bound B_{23}, thanks to the Hölder's inequality and Young inequality, one gets

$$
\begin{aligned}
B_{23} =&(\partial_x\tilde h\partial_y\tilde h - g\partial_y^2\tilde h, 2e^{2\theta}\partial_y\theta\partial_y^2 u)_{H^{1,0}}\\
\le&\frac{1}{16}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde h\|^2_{H^1_x}dy + \frac{1}{16}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y^2\tilde h\|^2_{H^1_x}dy\\
&+ (\|\partial_x\tilde h\|^2_{L^\infty} + \|g\|^2_{L^\infty})\|\partial_y^2 u\|^2_{H^{1,0}_\mu}\\
\le&\frac{1}{16}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde h\|^2_{H^1_x}dy + \frac{1}{16}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y^2\tilde h\|^2_{H^1_x}dy\\
&+ C\langle t\rangle^{\frac12}D(t)E(t).
\end{aligned}
$$

In view of integration by parts and $\partial_x u + \partial_y v = 0$, we deduce

$$
\begin{aligned}
B_3 &= -(\partial_y^2 u\partial_x\tilde h + 2\partial_y u\partial_x\partial_y\tilde h - \partial_x\partial_y u\partial_y\tilde h - 2\partial_x u\partial_y^2\tilde h, \partial_y^2\tilde h)_{H^{1,0}_\mu} + (v\partial_y\theta e^\theta\partial_y^2\tilde h, e^\theta\partial_y^2\tilde h)_{H^{1,0}}\\
&\le \frac{1}{16}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y^2\tilde h\|^2_{H^1_x}dy + C\langle t\rangle^{\frac12}D(t)E(t) + CD(t)^{\frac14}E(t)^{\frac54} + CD(t)^{\frac12}E(t).
\end{aligned}
$$

Collecting all the above estimates, it leads to

$$
\begin{aligned}
&\frac12\frac{d}{dt}(\|\partial_y^2 u\|^2_{H^{1,0}_\mu} + \|\partial_y^2\tilde h\|^2_{H^{1,0}_\mu}) + \frac12\|\partial_y^3 u\|^2_{H^{1,0}_\mu} + \frac12\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y^2\tilde h\|^2_{H^1_x}dy\\
&\quad\le \frac{1}{16}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde h\|^2_{H^1_x}dy + C\langle t\rangle^{\frac12}D(t)E(t) + D(t)^{\frac14}E(t)^{\frac54} + \langle t\rangle^{\frac14}D(t)^{\frac12}E(t).
\end{aligned}
$$

□

5.2. **The dissipation estimates of the magnetic field.**

Proposition 5.4. *It holds that*

$$
\begin{aligned}
\frac12\frac{d}{dt}(\|\partial_t u\|^2_{H^{1,0}_\mu} + \|\partial_t\tilde h\|^2_{H^{1,0}_\mu}) + \frac12\|\partial_y\partial_t u\|^2_{H^{1,0}_\mu} + \int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde h\|^2_{H^1_x}dy\\
\le C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t) + C\langle t\rangle^{\frac12}D(t)E(t).
\end{aligned}
$$

Proof. Applying ∂_t to the equation (1.2) and making $H^{1,0}_\mu$ with $(\partial_t u, \partial_t\tilde h)$, we obtain

$$
\begin{aligned}
&(\partial_t^2 u, \partial_t u)_{H^{1,0}_\mu} - (\partial_y^2\partial_t u, \partial_t u)_{H^{1,0}_\mu} + (\partial_t^2\tilde h, \partial_t\tilde h)_{H^{1,0}_\mu} = (\partial_x\partial_t\tilde h, \partial_t u)_{H^{1,0}_\mu} + (\partial_x\partial_t u, \partial_t\tilde h)_{H^{1,0}_\mu}\\
&+ (\partial_t(\tilde\partial_x\tilde h + g\partial_y\tilde h - u\partial_x u - v\partial_y u), \partial_t u)_{H^{1,0}_\mu} + (\partial_t(\tilde h\partial_x u + g\partial_y u - u\partial_x\tilde h - v\partial_y\tilde h), \partial_t\tilde h)_{H^{1,0}_\mu}\\
&= (\partial_t(\tilde h\partial_x\tilde h + g\partial_y\tilde h - u\partial_x u - v\partial_y u), \partial_t u)_{H^{1,0}_\mu} + (\partial_t(\tilde h\partial_x u + g\partial_y u - u\partial_x\tilde h - v\partial_y\tilde h), \partial_t\tilde h)_{H^{1,0}_\mu}\\
&= D_1 + D_2.
\end{aligned}
$$

where we have used that

$$(\partial_x\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}+(\partial_x\partial_t u,\partial_t\tilde{h})_{H^{1,0}_\mu}=0.$$

In view of integration by parts and using the Young inequality, we infer

$$\begin{aligned}
&(\partial_t^2 u,\partial_t u)_{H^{1,0}_\mu}-(\partial_y^2\partial_t u,\partial_t u)_{H^{1,0}_\mu}+(\partial_t^2\tilde{h},\partial_t\tilde{h})_{H^{1,0}_\mu}\\
&\geq\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|^2_{H^{1,0}_\mu}+\|\partial_t\tilde{h}\|^2_{H^{1,0}_\mu})+\frac{1}{2}\|\partial_y\partial_t u\|^2_{H^{1,0}_\mu}\\
&+\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy-\int_{\mathbf{R}_+}(\partial_t\theta+2(\partial_y\theta)^2)\|e^\theta\partial_t u\|^2_{H^1_x}dy\\
&\geq\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|^2_{H^{1,0}_\mu}+\|\partial_t\tilde{h}\|^2_{H^{1,0}_\mu})+\frac{1}{2}\|\partial_y\partial_t u\|^2_{H^{1,0}_\mu}+\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy.
\end{aligned}$$

Now we deal with the nonlinear terms. Thanks to Leibnz law and the Hölder inequality, one has

$$\begin{aligned}
D_1&=(\partial_t(\tilde{h}\partial_x\tilde{h}-u\partial_x u-v\partial_y u),\partial_t u)_{H^{1,0}_\mu}+(\partial_t g\partial_y\tilde{h},\partial_t u)_{H^{1,0}_\mu}+(g\partial_y\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}\\
&\leq\|\partial_x\tilde{h}\|_{L^\infty}\|\partial_t\tilde{h}\|_{H^{1,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}+\|\tilde{h}\|_{H^{1,1}_\mu}\|\partial_t\tilde{h}\|_{H^{2,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}\\
&+\|\partial_x u\|_{L^\infty}\|\partial_t u\|_{H^{1,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}+\|u\|_{H^{1,1}_\mu}\|\partial_t u\|_{H^{2,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}\\
&+\langle t\rangle^{\frac14}\|\partial_t u\|_{H^{1,0}_\mu}\|\partial_y u\|_{H^{1,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}+\langle t\rangle^{\frac14}\|u\|_{H^{1,0}_\mu}\|\partial_y\partial_t u\|_{H^{1,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}\\
&+\langle t\rangle^{\frac14}\|\partial_t\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y\tilde{h}\|_{H^{1,0}_\mu}\|\partial_t u\|_{H^{1,0}_\mu}+C\langle t\rangle^{\frac14}\|\tilde{h}\|_{H^{1,0}_\mu}\|\partial_t\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y\partial_t u\|_{H^{1,0}_\mu}\\
&+C\langle t\rangle^{\frac12}\|\tilde{h}\|^2_{H^{1,0}_\mu}\|\partial_t u\|^2_{H^{1,0}_\mu}+\frac{1}{4}\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy+\|\tilde{h}\|_{H^{2,1}_\mu}\|\partial_t\tilde{h}\|_{H^{1,0}_\mu}\|\partial_t u\|^2_{H^{1,0}_\mu}\\
&\leq C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)E(t)+\frac{1}{4}\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy,
\end{aligned}$$

where we have used the following estimate:

$$\begin{aligned}
(g\partial_y\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}&=(\partial_y(e^{2\theta}g\partial_t\tilde{h}),\partial_t u)_{H^{1,0}}-(2\partial_y\theta g\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}+(\partial_x\tilde{h}\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}\\
&=-(g\partial_t\tilde{h},\partial_y\partial_t u)_{H^{1,0}_\mu}-(2\partial_y\theta g\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}+(\partial_x\tilde{h}\partial_t\tilde{h},\partial_t u)_{H^{1,0}_\mu}\\
&\leq C\langle t\rangle^{\frac14}\|\tilde{h}\|_{H^{1,0}_\mu}\|\partial_t\tilde{h}\|_{H^{1,0}_\mu}\|\partial_y\partial_t u\|_{H^{1,0}_\mu}+C\langle t\rangle^{\frac12}\|\tilde{h}\|^2_{H^{1,0}_\mu}\|\partial_t u\|^2_{H^{1,0}_\mu}\\
&+\frac{1}{4}\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy+\|\tilde{h}\|_{H^{2,1}_\mu}\|\partial_t\tilde{h}\|_{H^{1,0}_\mu}\|\partial_t u\|^2_{H^{1,0}_\mu}.
\end{aligned}$$

Similarly, we have

$$D_2\leq C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)E(t)+\frac{1}{4}\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy.$$

From all the above estimates, one deduces

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|^2_{H^{1,0}_\mu}+\|\partial_t\tilde{h}\|^2_{H^{1,0}_\mu})+\frac{1}{2}\|\partial_y\partial_t u\|^2_{H^{1,0}_\mu}+\int_{\mathbf{R}_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_t\tilde{h}\|^2_{H^1_x}dy&\\
\leq C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)E(t).&
\end{aligned}$$

□

It is time to build the $\|\partial_y \tilde{h}\|_{H^{2,0}_\mu}$.

Proposition 5.5. *It holds that*

$$\|\partial_y \tilde{h}\|^2_{H^{2,0}_\mu} \le C\|\partial_t\partial_y u\|^2_{H^{1,0}_\mu} + C\|\partial_y^3 u\|^2_{H^{1,0}_\mu} + C\langle t\rangle^{\frac12} D(t)E(t) + C\langle t\rangle^{\frac14} D(t)E(t)^{\frac12}. \tag{5.10}$$

Proof. Thanks to the equation (1.2), we deduce

$$\begin{aligned}(\partial_x\partial_y\tilde{h}, \partial_x\partial_y\tilde{h})_{H^{1,0}_\mu} &= (\partial_t\partial_y u - \partial_y^3 u + \partial_y(u\partial_x u + v\partial_y u - \tilde{h}\partial_x\tilde{h} - g\partial_y\tilde{h}), \partial_x\partial_y\tilde{h})_{H^{1,0}_\mu}\\ &\le C\|\partial_t\partial_y u\|^2_{H^{1,0}_\mu} + C\|\partial_y^3 u\|^2_{H^{1,0}_\mu} + \frac18\|\partial_y\tilde{h}\|^2_{H^{2,0}_\mu}\\ &+ \|\partial_y(u\partial_x u + v\partial_y u - \tilde{h}\partial_x\tilde{h} - g\partial_y\tilde{h})\|_{H^{1,0}_\mu}\|\partial_y\tilde{h}\|_{H^{2,0}_\mu}.\end{aligned} \tag{5.11}$$

A direct calculation gives

$$\begin{aligned}\|\partial_y(u\partial_x u + v\partial_y u - \tilde{h}\partial_x\tilde{h} - g\partial_y\tilde{h})\|_{H^{1,0}_\mu} &= \|u\partial_x\partial_y u + v\partial_y^2 u - \tilde{h}\partial_x\partial_y\tilde{h} - g\partial_y^2\tilde{h})\|_{H^{1,0}_\mu}\\ &\le \|u\|_{L^\infty}\|\partial_y u\|_{H^{2,0}_\mu} + \|v\|_{L^\infty}\|\partial_y^2 u\|_{H^{1,0}_\mu}\\ &+ \|\tilde{h}\|_{L^\infty}\|\partial_y\tilde{h}\|_{H^{2,0}_\mu} + \|g\|_{L^\infty}\|\partial_y^2\tilde{h}\|_{H^{1,0}_\mu}\\ &\le \langle t\rangle^{\frac14} E(t)^{\frac12} D(t)^{\frac12}.\end{aligned} \tag{5.12}$$

which together with (5.11) gives

$$\|\partial_y\tilde{h}\|^2_{H^{2,0}_\mu} \le C\|\partial_t\partial_y u\|^2_{H^{1,0}_\mu} + C\|\partial_y^3 u\|^2_{H^{1,0}_\mu} + C\langle t\rangle^{\frac14} D(t)E(t)^{\frac12}.$$

□

Proposition 5.6. *It holds that*

$$\begin{aligned}&\frac12\frac{d}{dt}\|(\partial_y u, \partial_y\tilde{h})\|^2_{H^{2,0}_\mu} + \frac12\|\partial_y^2 u\|^2_{H^{2,0}_\mu} + \int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde{h}\|^2_{H^2_x}dy\\ &\le C\langle t\rangle^{\frac14} D(t)^{\frac12}E(t) + C\langle t\rangle^{\frac12} D(t)E(t) + \frac14\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde{h}\|^2_{H^2_x}dy.\end{aligned}$$

Proof. Applying ∂_y to the equation (1.2) and Making $H^{2,0}_\mu$-inner product with $(\partial_y u, \partial_y\tilde{h})$ to obtain

$$\begin{aligned}(\partial_t\partial_y u, \partial_y u)_{H^{2,0}_\mu} - (\partial_y^3 u, \partial_y u)_{H^{2,0}_\mu} + (\partial_t\partial_y\tilde{h}, \partial_y\tilde{h})_{H^{2,0}_\mu} &= [(\partial_x\partial_y\tilde{h}, \partial_y u)_{H^{2,0}_\mu}\\ &+ (\partial_x\partial_y u, \partial_y\tilde{h})_{H^{2,0}_\mu}] - (\partial_y(u\partial_x u + v\partial_y u - \tilde{h}\partial_x\tilde{h} - g\partial_y\tilde{h}), \partial_y u)_{H^{2,0}_\mu}\\ &- (\partial_y(u\partial_x\tilde{h} + v\partial_y\tilde{h} - \tilde{h}\partial_x u - g\partial_y u), \partial_y\tilde{h})_{H^{2,0}_\mu}\\ &= -(\partial_y(u\partial_x u + v\partial_y u - \tilde{h}\partial_x\tilde{h} - g\partial_y\tilde{h}), \partial_y u)_{H^{2,0}_\mu}\\ &- (\partial_y(u\partial_x\tilde{h} + v\partial_y\tilde{h} - \tilde{h}\partial_x u - g\partial_y u), \partial_y\tilde{h})_{H^{2,0}_\mu} = E_7 + E_8.\end{aligned}$$

where we used the fact

$$(\partial_x\partial_y\tilde{h}, \partial_y u)_{H^{2,0}_\mu} + (\partial_x\partial_y u, \partial_y\tilde{h})_{H^{2,0}_\mu} = 0.$$

Thanks to integration by parts and the Young inequality, we obtain

$$
\begin{aligned}
&(\partial_t\partial_y u, \partial_y u)_{H^{2,0}_\mu} - (\partial_y^3 u, \partial_y u)_{H^{2,0}_\mu} + (\partial_t\partial_y \tilde{h}, \partial_y \tilde{h})_{H^{2,0}_\mu}\\
&\geq \frac{1}{2}\frac{d}{dt}(\|\partial_y u\|^2_{H^{2,0}_\mu} + \|\partial_y \tilde{h}\|^2_{H^{2,0}_\mu}) + \frac{1}{2}\|\partial_y^2 u\|^2_{H^{2,0}_\mu}\\
&+ \int_{R_+} \frac{y^2}{4\langle t\rangle^2}\|e^\theta \partial_y \tilde{h}\|^2_{H^2_x}dy - \int_{R_+}(\partial_t\theta + 2(\partial_y\theta)^2)\|e^\theta\partial_y u\|^2_{H^2_x}dy\\
&\geq \frac{1}{2}\frac{d}{dt}\|(\partial_y u, \partial_y \tilde{h})\|^2_{H^{2,0}_\mu} + \frac{1}{2}\|\partial_y^2 u\|^2_{H^{2,0}_\mu} + \int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde{h}\|^2_{H^2_x}dy.
\end{aligned}
\tag{5.13}
$$

Now we are reminded to bound the nonlinear terms. According to the Hölder inequality, we deduce

$$
\begin{aligned}
E_7 &= (u\partial_x\partial_y u + v\partial_y^2 u, \partial_y u)_{H^{2,0}_\mu} + (\tilde{h}\partial_x\tilde{h} + g\partial_y\tilde{h}, \partial_y^2 u)_{H^{2,0}_\mu} + (\tilde{h}\partial_x\tilde{h} + g\partial_y\tilde{h}, 2\partial_y\theta e^{2\theta}\partial_y u)_{H^{2,0}}\\
&\leq \|u\|_{L^\infty}\|\partial_y u\|_{H^{3,0}_\mu}\|\partial_y u\|_{H^{2,0}_\mu} + \|v\|_{L^\infty}\|\partial_y^2 u\|_{H^{2,0}_\mu}\|\partial_y u\|_{H^{2,0}_\mu}\\
&+ \|\tilde{h}\|_{L^\infty}\|\tilde{h}\|_{H^{3,0}_\mu}\|\partial_y^2 u\|_{H^{2,0}_\mu} + \|g\|_{L^\infty}\|\partial_y\tilde{h}\|_{H^{2,0}_\mu}\|\partial_y^2 u\|_{H^{2,0}_\mu} + \|\partial_x\tilde{h}\|^2_{L^\infty}\|\partial_y u\|^2_{H^{2,0}_\mu}\\
&+ \frac{1}{4}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde{h}\|^2_{H^2_x}dy + \frac{1}{4}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde{h}\|^2_{H^2_x}dy + \|g\|^2_{L^\infty}\|\partial_y u\|^2_{H^{2,0}_\mu}\\
&\leq C\langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t) + C\langle t\rangle^{\frac{1}{2}}D(t)E(t) + \frac{1}{4}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde{h}\|^2_{H^2_x}dy + \frac{1}{4}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde{h}\|^2_{H^2_x}dy.
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
E_8 &= (-\partial_y u\partial_x\tilde{h} + \partial_x u\partial_y\tilde{h}, \partial_y\tilde{h})_{H^{2,0}_\mu} - (u\partial_x(e^\theta\partial_y\tilde{h}) + v\partial_y(e^\theta\partial_y\tilde{h}), \partial_y\tilde{h})_{H^{2,0}}\\
&- (\partial_y\theta v e^\theta\partial_y\tilde{h}, e^\theta\partial_y\tilde{h})_{H^{2,0}} + (\partial_y\tilde{h}\partial_x u + \tilde{h}\partial_x\partial_y u - \partial_x\tilde{h}\partial_y u + g\partial_y^2 u, \partial_y\tilde{h})_{H^{2,0}}\\
&= (-\partial_y u\partial_x\tilde{h} + \partial_x u\partial_y\tilde{h}, \partial_y\tilde{h})_{H^{2,0}_\mu} - (\partial_y\theta v e^\theta\partial_y\tilde{h}, e^\theta\partial_y\tilde{h})_{H^{2,0}}\\
&+ (\partial_y\tilde{h}\partial_x u + \tilde{h}\partial_x\partial_y u - \partial_x\tilde{h}\partial_y u + g\partial_y^2 u, \partial_y\tilde{h})_{H^{2,0}}\\
&\leq \langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t) + \langle t\rangle^{\frac{1}{4}}D(t)E(t) + \frac{1}{4}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde{h}\|^2_{H^2_x}dy..
\end{aligned}
$$

Collecting all the estimates mentioned above yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_y u, \partial_y\tilde{h})\|^2_{H^{2,0}_\mu} + \frac{1}{2}\|\partial_y^2 u\|^2_{H^{2,0}_\mu} + \int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\partial_y\tilde{h}\|^2_{H^2_x}dy\\
&\leq C\langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t) + C\langle t\rangle^{\frac{1}{2}}D(t)E(t) + \frac{1}{4}\int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde{h}\|^2_{H^2_x}dy.
\end{aligned}
$$

□

A similar argument can build the following proposition.

Proposition 5.7. *It holds that*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u, \tilde{h})\|^2_{H^{2,0}_\mu} + \frac{1}{2}\|\partial_y u\|^2_{H^{2,0}_\mu} + \int_{R_+}\frac{y^2}{4\langle t\rangle^2}\|e^\theta\tilde{h}\|^2_{H^2_x}dy\\
&\qquad \leq \langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t) + \langle t\rangle^{\frac{1}{4}}D(t)E(t).
\end{aligned}
\tag{5.14}
$$

Now we collect all the above propositions to build the following vertical estimates.

Proposition 5.8. *It holds that*

$$\begin{aligned}&\frac{d}{dt}(\|(u,\tilde{h})\|^2_{H^{1,2}_\mu}+\|(u,\tilde{h})\|^2_{H^{2,1}_\mu}+\|(\partial_t u,\partial_t\tilde{h})\|^2_{L^2_\mu})+(\|\partial_y u\|_{H^{1,2}_\mu}\\ &+\|\partial_t\partial_y u\|^2_{H^{1,0}_\mu}+\|\partial_y\tilde{h}\|^2_{H^{2,0}_\mu}+\|\partial_y u\|^2_{H^{2,1}_\mu})\\ &\le CD(t)^{\frac14}E(t)^{\frac54}+\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+\langle t\rangle^{\frac14}D(t)E(t)^{\frac12}+C\langle t\rangle^{\frac12}E(t)^2+C\langle t\rangle^{\frac12}D(t)E(t).\end{aligned}\tag{5.15}$$

Proof. According to the estimates from Proposition 5.1 to Proposition 5.7, it is easy to conclude this proposition. So we omit it. □

6. Proof of Theorem 1.1

Motivated by [13], the approximate solution can be constructed by adding the viscosity term $\partial_x^2 u, \Delta\tilde{h}$ to the system (1.2). Thus, we only present the uniform estimates of smooth solution. With the uniform estimates, the existence and uniqueness of the solution can be obtained by showing that the approximate sequence is a Cauchy sequence in lower order Sobolev spaces.

The uniform estimate is based on a bootstrap argument. Let us first assume that $[0,T^*)$ is the maximal time interval so that

$$E(t)\le C_1\varepsilon^2.\tag{6.1}$$

where $C_1>0$ is a fixed constant. Let us also assume $T^*<\varepsilon^{-2}$. Thanks to Lemma 4.6 and Proposition 5.8, we have

$$\begin{aligned}&\frac{d}{dt}E(t)+D(t)\le C\langle t\rangle^{\frac12}D(t)E(t)+C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}D(t)^{\frac12}E(t)^{\frac32}\\ &+CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac34}E(t)^{\frac34}+C\langle t\rangle^{\frac12}E(t)^2+C\langle t\rangle^{\frac14}D(t)E(t)^{\frac12}.\end{aligned}$$

According to (6.1), it leads to

$$\begin{aligned}\frac{d}{dt}E(t)+D(t)\le{}& C\langle t\rangle^{\frac12}D(t)^{\frac12}E(t)^{\frac32}+CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac34}E(t)^{\frac34}\\ &+C\langle t\rangle^{\frac14}D(t)^{\frac12}E(t)+C\langle t\rangle^{\frac12}E(t)^2.\end{aligned}$$

Thanks to the Young inequality, we get

$$\frac{d}{dt}E(t)+D(s)\le C_2E(t)^{\frac53}+C_2\langle t\rangle^{\frac12}E(t)^2+C_2\langle t\rangle E(t)^3.$$

which along with (6.1) implies

$$\frac{d}{dt}E(t)\le(C_2\varepsilon^{\frac43}+C_2\langle t\rangle^{\frac12}\varepsilon^2+C_2\langle t\rangle\varepsilon^4)E(t).$$

Then, for any $t<\varepsilon^{-\frac43}$, one has

$$E(t)\le C_3\varepsilon^2,$$

Taking $C_3=\frac{C_1}{2}$, the theorem follows by a bootstrap argument.

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References

[1] R. Alexandre, Y. Wang, C.- J. Xu and T. Yang, Well-posedness of the Prandtl equation in Sobolev spaces, J. Amer. Math. Soc., 28(2015), 745-784.

[2] H. Bahouri, J. Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften 343, Springer-Verlag Berlin Heidelberg, 2011.

[3] A. B. Ferrari. On the blow-up of solutions of the 3-D Euler equations in a bounded domain, *Math.Phys.*, **5**(1989), 208-218.

[4] D. Chen, Y. Wang and Z. Zhang, Well-posedness of the Prandtl equation with monotonicity in Sobolev spaces, J. Differential Equations, 28 (2018), 745-784.

[5] D. Chen, Y. Wang and Z. Zhang, Well-posedness of the linearized Prandtl equation around a non-monotonic shear flow, Ann. Inst. H. Poincar Anal. Non Linaire, 35 (2018), 1119-1142.

[6] D. Chen, S. Ren, Y. Wang and Z. Zhang, Long time well-posedness of f the MHD boundary layer equation in Sobolev apace, Anal. Theory Appl., 36 (2020), 1-18.

[7] W. E and B. Engquist, Blow up of solutions of the unsteady Prandtls equation, Commun. Pure Appl. Math., 50 (1997), 1287-1293.

[8] D. Gérard-Varet and M. Prestipino, Formal derivation and stability analysis of boundary layer models in MHD, Z. Angew. Math. Phys., 68 (2017), PP76.

[9] D. Gérard-Varet and E. Dormy, On the ill-posedness of the Prandtl equation, J. Amer. Math. Soc., 23 (2010), 591-609.

[10] D. Gérard-Varet and N. Masmoudi, Well-posedness for the Prandtl system without analyt- icity or monotonicity, Ann. Sci. Ec. Norm. Supér, 48(2015), 1273-1325.

[11] I. Kukavica, V. Vicol and F. Wang, The van Dommelen and shen singularity in the Prandtl equtions, Adv. Math., 307(2017), 288-311.

[12] M. Ignatova and V. Vicol, Almost global existence for the Prandtlm boundary layer equa- tions, Arch. Rational Mech. Anal., 220(2016), 809-848.

[13] C. Liu, F. Xie, T. Yang, MHD boundary layer theory in Sobolev spaces without monotonicity. I. well-posedness theory, Commun. Pure Appl. Math., 72(2019), 63-121.

[14] C. Liu, D. Wang, F. Xie, T. Yang, Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces, J. Funct. Anal., 279(2020), 108367.

[15] W. Li and T. Yang, Well-posedness in Gevrey space for the Prandtl equations with non- degenerate critical points, J. Eur. Math. Soc., 22(2020),717-775.

[16] N. Masmoudi and T. K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, Commun. Pure Appl. Math., 68(2015), 1683-1741.

[17] O. A. Oleinik, The Prandtl system of equations in boundary layer theory, Sov. Math. Dokl. 4(1963), 583-586.

[18] O. A. Oleinik and V. N. Samokhin, Mathematical Models in Boundary Layer Theory, Ap- plied Mathematics and Mathematical Computation 15 Chapman Hall/CRC, Boca Raton, Fla., 1999.

[19] L. Prandtl, Uber flssigkeits-bewegung bei sehr kleiner reibung, in: Verhandlungen des III. Interna- tionlen Mathematiker Kongresses, Heidelberg, Teubner, Leipzig, 1904, pp. 484-491.

[20] M. Sammartino and R. E. Caflisch, Zero viscosity limit for analytic solutions of the Navier- Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, Commun. Math. Phys., 192(1998), 433-461.

[21] F. Xie, T. Yang, Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime, SIAM J. Math. Anal. 50(2018), 5749-5760.

[22] F. Xie, T. Yang, Lifespan of solutions to MHD boundary layer equations with analytic perturbation of general shear flow, Acta Math. Appl. Sin. Engl. Ser. 35(2019), 209-229.

[23] Z. Xin and L. Zhang, OntheglobalexistenceofsolutionstothePrandtlssystem, Adv.Math., 181 (2004), 88-133.

[24] C.-J. Xu and X. Zhang, Long time well-posedness of Prandtl equations in Sobolev space, J. Differen. Equa., 263 (2017), 8749-8803.

[25] P. Zhang and Z. Zhang, Long time well-posedness of Prandtl system with small and analytic initial data, J. Funct. Anal., 270 (2016), 2591-2615.