

SOME NEW SIMPSON'S TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS IN QUANTUM CALCULUS

MUHAMMAD AAMIR ALI, HÜSEYİN BUDAK, ZHIYUE ZHANG, AND HÜSEYİN YILDIRIM

ABSTRACT. In this article, by using the notion of newly defined q_1q_2 -derivatives and integrals some new Simpson's type inequalities for co-ordinated convex functions are shown. The outcomes raised in this paper are extensions and generalizations of the comparable results in the literature on Simpson's inequalities for co-ordinated convex functions.

1. INTRODUCTION

Quantum calculus or q -calculus is sometimes referred to as calculus without limits. In this, we gain q -analogues of mathematical items that may be got back as $q \rightarrow 1$. There are two kinds of q -addition, the Nalli-Ward-Al-Salam q -addition (NWA) and the Jackson-Hahn-Cigler q -addition (JHC). The first one is commutative and associative, at the same time as the second one is neither. That is why from time to time multiple q -analogs exist. These operators form the basis of the technique which unites hypergeometric collection and q -hypergeometric collection and which gives many formulations of q -calculus a natural shape. The past of quantum calculus may be traced back to Euler (1707–1783), who first added the q in the tracks of Newton's infinite series. In recent years many researchers have shown an eager hobby in studying and investigating quantum calculus accordingly it emerges as an interdisciplinary subject. This is of course because of the fact that quantum analysis is very helpful in several fields and has huge applications in various areas of natural and carried out sciences such as computer science and particle physics, and additionally acts as an critical tool for researchers operating with analytic number theory or in theoretical physics. Quantum calculus can be considered as a bridge among Mathematics and Physics. Many scientists who use quantum calculus are physicists, as quantum calculus has many applications in quantum group theory. For some recent trends in quantum calculus involved readers are referred to [8–12, 21].

In recent decades the idea of convex functions has been drastically studied because of its fantastic significance in numerous fields of pure and applied sciences. Theory of inequalities and concept of convex functions are closely related to each other, thus diverse inequalities can be found within the literature which are proved for convex and differentiable convex functions of single and double variables, see [3–5, 13–17, 20, 23].

Simpson's rules are well-known technique for the numerical integration and the numerical estimations of definite integrals. This method is known to be developed by Thomas Simpson (1710–1761). However, Johannes Kepler used a similar approximation about 100 years ago, so this method is also known as Kepler's rule.

Simpson's quadrature formula (Simpson's 1/3 rule) stated as:

$$\int_a^b F(x)dx \approx \frac{1}{6} \left[F(a) + 4F\left(\frac{a+b}{2}\right) + F(b) \right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimations known as Simpson's inequality:

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Theorem 1. Suppose that $F : [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) , and let $\|F^{(4)}\|_\infty = \sup_{x \in (a, b)} |F^{(4)}(x)| < \infty$. Then, one has the inequality

$$\left| \frac{1}{3} \left[\frac{F(a) + F(b)}{2} + 2F\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b F(x) dx \right| \leq \frac{1}{2880} \|F^{(4)}\|_\infty (b-a)^4.$$

In recent years, many authors have focused on Simpson type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson and Newton type results for convex mappings, because convexity theory is effective and strong method for solving a great number of problems which arise different branches in pure and applied mathematics. For example, Dragomir et al. presented new Simpson type results and their applications to quadrature formula in numerical integration in [7]. What is more, some inequalities of Simpson type for s -convex functions are deduced by Alomari et al. in [2]. Afterwards, Sarikaya et al. observed the variants of Simpson type inequalities based on s -convexity in [19]. Kunt et al. also gave the simpson's type inequalities for the functions of two variables in [18].

Inspired by this ongoing studies, we establish some new quantum analogues of Simpson's inequalities for q -differentiable co-ordinated convex functions. This is the primary motivation of this paper. The ideas and strategies of the paper may open new venues of further research in this field.

2. PRELIMINARIES OF q -CALCULUS AND SOME INEQUALITIES

In this section we present some required definitions and related inequalities about q -calculus. For more information about q -calculus, one can refer to ([9–11])

Definition 1. [21] For a continuous function $F : [a, b] \rightarrow \mathbb{R}$, then q - derivative of F at $x \in [a, b]$ is characterized by the expression

$$(2.1) \quad {}_a d_q F(x) = \frac{F(x) - F(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a.$$

Since $F : [a, b] \rightarrow \mathbb{R}$ is a continuous function, thus we have ${}_a d_q F(a) = \lim_{x \rightarrow a} {}_a d_q F(x)$. The function F is said to be q - differentiable on $[a, b]$ if ${}_a d_q F(t)$ exists for all $x \in [a, b]$. If $a = 0$ in (2.1), then ${}_0 d_q F(x) = d_q F(x)$, where $d_q F(x)$ is familiar q -derivative of F at $x \in [a, b]$ defined by the expression (see [12])

$$(2.2) \quad d_q F(x) = \frac{F(x) - F(qx)}{(1-q)x}, \quad x \neq 0.$$

Definition 2. [21] Let $F : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the q_a -definite integral on $[a, b]$ is defined as

$$(2.3) \quad \int_a^x F(t) {}_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n F(q^n x + (1-q^n)a)$$

for $x \in [a, b]$.

We have to give following notation which will be used many times in the next sections (see [12]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Moreover, we will need to the following Lemma in our main results:

Lemma 1. [22] We have the equality

$$\int_a^b (x-a)^\alpha {}_a d_q x = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q}$$

for $\alpha \in \mathbb{R} \setminus \{-1\}$.

On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 3. [5] Let $F : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the q^b -definite integral on $[a, b]$ is defined as

$$\int_a^b F(t) {}^b d_{q^b} t = (1 - q) (b - a) \sum_{n=0}^{\infty} q^n F(q^n a + (1 - q^n) b)$$

for $x \in [a, b]$.

Theorem 2. [5] If $F : [a, b] \rightarrow \mathbb{R}$ be a convex differentiable function on $[a, b]$ and $0 < q < 1$. Then, q -Hermite-Hadamard inequalities

$$(2.4) \quad F\left(\frac{a + qb}{[2]_q}\right) \leq \frac{1}{b - a} \int_a^b F(x) {}^b d_{q^b} x \leq \frac{F(a) + qF(b)}{[2]_q}.$$

In [15], Latif defined q_{ac} -integral and partial q -derivatives for two variables functions as follows:

Definition 4. Suppose that $F : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous function. Then, the definite q_{ac} -integral on $[a, b] \times [c, d]$ is defined by

$$\begin{aligned} \int_a^x \int_c^y F(t, s) {}^c d_{q_2} s {}^a d_{q_1} t &= (1 - q_1) (1 - q_2) (x - a) (y - c) \\ &\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1 - q_1^n) x, q_2^m c + (1 - q_2^m) y) \end{aligned}$$

for $(x, y) \in [a, b] \times [c, d]$.

Lemma 2. [13] If the assumptions of Definition 4 holds, then

$$\begin{aligned} \int_{y_1}^y \int_{x_1}^x F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s &= \int_{y_1}^y \int_a^x F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s - \int_{y_1}^y \int_a^{x_1} F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s \\ &= \int_c^y \int_a^x F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s - \int_c^{y_1} \int_a^x F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s \\ &\quad - \int_c^y \int_a^{x_1} F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s + \int_c^{y_1} \int_a^{x_1} F(t, s) {}^a d_{q_1} t {}^c d_{q_2} s. \end{aligned}$$

Definition 5. [15] Let $F : [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function of two variables. Then the partial q_1 -derivatives, q_2 -derivatives and $q_1 q_2$ -derivatives at $(x, y) \in [a, b] \times [c, d]$ can be given as follows:

$$\begin{aligned} \frac{{}^b \partial_{q_1} F(x, y)}{{}^b \partial_{q_1} x} &= \frac{F(q_1 x + (1 - q_1) a, y) - F(x, y)}{(1 - q_1) (x - a)}, \quad x \neq a \\ \frac{{}^d \partial_{q_2} F(x, y)}{{}^d \partial_{q_2} y} &= \frac{F(x, q_2 y + (1 - q_2) c) - F(x, y)}{(1 - q_2) (y - c)}, \quad y \neq c \\ \frac{{}_a, {}_c \partial_{q_1, q_2}^2 F(x, y)}{{}_a \partial_{q_1} x {}_c \partial_{q_2} y} &= \frac{1}{(x - a) (y - c) (1 - q_1) (1 - q_2)} [F(q_1 x + (1 - q_1) a, q_2 y + (1 - q_2) c) \\ &\quad - F(q_1 x + (1 - q_1) a, y) - F(x, q_2 y + (1 - q_2) c) + F(x, y)], \quad x \neq a, y \neq c. \end{aligned}$$

For more details related to q -integrals and derivatives for the functions of two variables one can see ([15]).

On the other hand, Budak et al. gave the following definitions of q_a^d , q_b^c and q^{bd} integrals and related inequalities of Hermite-Hadamard type:

Definition 6. [6] Suppose that $F : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous function. Then the following q_a^d , q_c^b and q^{bd} integrals on $[a, b] \times [c, d]$ are defined by

$$(2.5) \quad \int_a^x \int_y^d F(t, s) {}^d d_{q_2} s {}_a d_{q_1} t = (1 - q_1)(1 - q_2)(x - a)(d - y) \\ \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_2 x + (1 - q_2)a, q_1 y + (1 - q_1)d)$$

$$(2.6) \quad \int_x^b \int_c^y F(t, s) {}_c d_{q_2} s {}^b d_{q_1} t = (1 - q_1)(1 - q_2)(b - x)(y - c) \\ \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_2 x + (1 - q_2)b, q_1 y + (1 - q_1)c)$$

and

$$(2.7) \quad \int_x^b \int_y^d F(t, s) {}^d d_{q_2} s {}^b d_{q_1} t = (1 - q_1)(1 - q_2)(b - x)(d - y) \\ \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_2 x + (1 - q_2)b, q_1 y + (1 - q_1)d)$$

respectively, for $(x, y) \in [a, b] \times [c, d]$.

Theorem 3. [6] Let $F : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be coordinated on $[a, b] \times [c, d]$. Then we have the inequalities

$$(2.8) \quad F\left(\frac{q_1 a + b}{[2]_{q_1}}, \frac{c + q_2 d}{[2]_{q_2}}\right) \\ \leq \frac{1}{2} \left[\frac{1}{b - a} \int_a^b F\left(x, \frac{c + q_2 d}{[2]_{q_2}}\right) {}_a d_{q_1} x + \frac{1}{d - c} \int_c^d F\left(\frac{q_1 a + b}{[2]_{q_1}}, y\right) {}^d d_{q_2} y \right] \\ \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d F(x, y) {}^d d_{q_2} y {}_a d_{q_1} x \\ \leq \frac{q_1}{2 [2]_{q_1} (d - c)} \int_c^d F(a, y) {}^d d_{q_2} y + \frac{1}{2 [2]_{q_1} (d - c)} \int_c^d F(b, y) {}^d d_{q_2} y \\ + \frac{1}{2 [2]_{q_2} (b - a)} \int_a^b F(x, c) {}_a d_{q_1} x + \frac{q_2}{2 [2]_{q_2} (b - a)} \int_a^b F(x, d) {}_a d_{q_1} x \\ \leq \frac{q_1 F(a, c) + q_1 q_2 F(a, d) + F(b, c) + q_2 F(b, d)}{[2]_{q_1} [2]_{q_2}}$$

for all $q_1, q_2 \in (0, 1)$.

Theorem 4. [6] Let $F : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be coordinated on $[a, b] \times [c, d]$. Then we have the inequalities

$$\begin{aligned}
 (2.9) \quad & F\left(\frac{a+q_1b}{[2]_{q_1}}, \frac{q_2c+d}{[2]_{q_2}}\right) \\
 & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b F\left(x, \frac{q_2c+d}{[2]_{q_2}}\right) {}^b d_{q_1}x + \frac{1}{d-c} \int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}}, y\right) {}^c d_{q_2}y \right] \\
 & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) {}^c d_{q_2}y {}^b d_{q_1}x \\
 & \leq \frac{1}{2[2]_{q_1}(d-c)} \int_c^d F(a, y) {}^c d_{q_2}y + \frac{q_1}{2[2]_{q_1}(d-c)} \int_c^d F(b, y) {}^c d_{q_2}y \\
 & + \frac{q_2}{2[2]_{q_2}(b-a)} \int_a^b F(x, c) {}^b d_{q_1}x + \frac{1}{2[2]_{q_2}(b-a)} \int_a^b F(x, d) {}^b d_{q_1}x \\
 & \leq \frac{q_2F(a, c) + F(a, d) + q_1q_2F(b, c) + q_1F(b, d)}{[2]_{q_1}[2]_{q_2}}
 \end{aligned}$$

for all $q_1, q_2 \in (0, 1)$.

Theorem 5. [6] Let $F : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be coordinated on $[a, b] \times [c, d]$. Then we have the inequalities

$$\begin{aligned}
 (2.10) \quad & F\left(\frac{a+q_1b}{[2]_{q_1}}, \frac{c+q_2d}{[2]_{q_2}}\right) \\
 & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b F\left(x, \frac{c+q_2d}{[2]_{q_2}}\right) {}^b d_{q_1}x + \frac{1}{d-c} \int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}}, y\right) {}^d d_{q_2}y \right] \\
 & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) {}^d d_{q_2}y {}^b d_{q_1}x \\
 & \leq \frac{1}{2[2]_{q_1}(d-c)} \int_c^d F(a, y) {}^d d_{q_2}y + \frac{q_1}{2[2]_{q_1}(d-c)} \int_c^d F(b, y) {}^d d_{q_2}y \\
 & + \frac{1}{2[2]_{q_2}(b-a)} \int_a^b F(x, c) {}^d d_{q_2}y + \frac{q_2}{2[2]_{q_2}(b-a)} \int_a^b F(x, d) {}^b d_{q_1}x \\
 & \leq \frac{F(a, c) + q_2F(a, d) + q_1F(b, c) + q_1q_2F(b, d)}{[2]_{q_1}[2]_{q_2}}
 \end{aligned}$$

for all $q_1, q_2 \in (0, 1)$.

Theorem 6. (q_1q_2 -Hölder's inequality for two variables functions, [15]) Let $x, y > 0$, $0 < q_1, q_2 < 1$, $p_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{r_1} = 1$. Then

$$\int_0^x \int_0^y |F(x, y) G(x, y)| {}^d d_{q_1}x {}^d d_{q_2}y \leq \left(\int_0^x \int_0^y |F(x, y)|^{p_1} {}^d d_{q_1}x {}^d d_{q_2}y \right)^{\frac{1}{p_1}} \left(\int_0^x \int_0^y |G(x, y)|^{r_1} {}^d d_{q_1}x {}^d d_{q_2}y \right)^{\frac{1}{r_1}}.$$

We can give the following new partial q -derivatives for functions of two variables

Definition 7. [1] Let $F : [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function of two variables. Then the partial q_1 -derivatives, q_2 -derivatives and q_1q_2 -derivatives at $(x, y) \in [a, b] \times [c, d]$ can be given as

follows:

$$\begin{aligned}
\frac{{}^b\partial_{q_1} F(x, y)}{{}^b\partial_{q_1} x} &= \frac{F(q_1 x + (1 - q_1)b, y) - F(x, y)}{(1 - q_1)(b - x)}, \quad x \neq b \\
\frac{{}^d\partial_{q_2} F(x, y)}{{}^b\partial_{q_2} y} &= \frac{F(x, q_2 y + (1 - q_2)d) - F(x, y)}{(1 - q_2)(d - y)}, \quad d \neq y \\
\frac{{}^d\partial_{q_1, q_2}^2 F(x, y)}{{}_a\partial_{q_1} x {}^d\partial_{q_2} y} &= \frac{1}{(x - a)(d - y)(1 - q_1)(1 - q_2)} [F(q_1 x + (1 - q_1)a, q_2 y + (1 - q_2)d) \\
&\quad - F(q_1 x + (1 - q_1)a, y) - F(x, q_2 y + (1 - q_2)d) + F(x, y)], \quad x \neq a, y \neq d, \\
\frac{{}^b\partial_{q_1, q_2}^2 F(x, y)}{{}^b\partial_{q_1} x {}^c\partial_{q_2} y} &= \frac{1}{(b - x)(y - c)(1 - q_1)(1 - q_2)} [F(q_1 x + (1 - q_1)b, q_2 y + (1 - q_2)c) \\
&\quad - F(q_1 x + (1 - q_1)b, y) - F(x, q_2 y + (1 - q_2)c) + F(x, y)], \quad x \neq b, y \neq c, \\
\frac{{}^b, {}^d\partial_{q_1, q_2}^2 F(x, y)}{{}^b\partial_{q_1} x {}^d\partial_{q_2} y} &= \frac{1}{(b - x)(d - y)(1 - q_1)(1 - q_2)} [F(q_1 x + (1 - q_1)b, q_2 y + (1 - q_2)d) \\
&\quad - F(q_1 x + (1 - q_1)b, y) - F(x, q_2 y + (1 - q_2)d) + F(x, y)], \quad x \neq b, y \neq d.
\end{aligned}$$

3. A CRUCIAL LEMMA

We deal with identity which is necessary to attain our main estimations in this section.

Let's start with the following useful Lemma.

Lemma 3. *Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{{}^b, {}^d\partial_{q_1, q_2}^2 F(t, s)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. Then following identity holds for $q_1 q_2$ -integrals:*

$$\begin{aligned}
(3.1) \quad & {}^b, {}^d\mathcal{I}_{q_1, q_2}(F) \\
&= (b - a)(d - c) \int_0^1 \int_0^1 \Lambda_{q_1}(t) \Lambda_{q_2}(s) \frac{{}^b, {}^d\partial_{q_1, q_2}^2 F(ta + (1 - t)b, sc + (1 - s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} {}^b d_{q_1} t {}^d d_{q_2} s,
\end{aligned}$$

where $q_1, q_2 \in (0, 1)$,

$$\begin{aligned}
{}^b, {}^d\mathcal{I}_{q_1, q_2}(F) &= \frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9} \\
&\quad + \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{36} \\
&\quad - \frac{1}{6(b - a)} \int_a^b \left[F(x, c) + 4F\left(x, \frac{c+d}{2}\right) + F(x, d) \right] {}^b d_{q_1} x \\
&\quad - \frac{1}{6(d - c)} \int_c^d \left[F(a, y) + 4F\left(\frac{a+b}{2}, y\right) + F(b, y) \right] {}^b d_{q_2} y \\
&\quad + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d F(x, y) {}^b d_{q_1} x {}^d d_{q_2} y
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_{q_1}(t) &= \begin{cases} q_1 t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ q_1 t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right), \end{cases} \\
\Lambda_{q_2}(s) &= \begin{cases} q_2 s - \frac{1}{6}, & s \in \left[0, \frac{1}{2}\right) \\ q_2 s - \frac{5}{6}, & s \in \left[\frac{1}{2}, 1\right). \end{cases}
\end{aligned}$$

Proof. Because of the Lemma 2 and the definition of $\Lambda_{q_1}(t)$ and $\Lambda_{q_2}(s)$, it is easy to see that

$$\begin{aligned}
 (3.2) \quad & \int_0^1 \int_0^1 \Lambda_{q_1}(t) \Lambda_{q_2}(s) \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
 &= \frac{4}{9} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
 &\quad + \frac{2}{3} \int_0^{\frac{1}{2}} \int_0^1 \left(q_2 s - \frac{5}{6}\right) \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
 &\quad + \frac{2}{3} \int_0^1 \int_0^{\frac{1}{2}} \left(q_1 t - \frac{5}{6}\right) \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
 &\quad + \int_0^1 \int_0^1 \left(q_1 t - \frac{5}{6}\right) \left(q_2 s - \frac{5}{6}\right) \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s. \\
 &= I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

From Definition 7, we have

$$\begin{aligned}
 & \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \\
 &= \frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)ts} [F(tq_1 a + (1-tq_1)b, sq_2 c + (1-sq_2)d) \\
 &\quad - F(tq_1 a + (1-tq_1)b, sc + (1-s)d) - F(ta + (1-t)b, sq_2 c + (1-sq_2)d) \\
 &\quad + F(ta + (1-t)b, sc + (1-s)d)].
 \end{aligned}$$

It is need to calculate the integrals in the right side of (3.2) in order to finish this proof. By using the definition of $q_1 q_2$ -integrals, we obtain that

$$\begin{aligned}
 (3.3) \quad & \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
 &= \frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{1}{ts} [F(tq_1 a + (1-tq_1)b, sq_2 c + (1-sq_2)d) \\
 &\quad - F(tq_1 a + (1-tq_1)b, sc + (1-s)d) - F(ta + (1-t)b, sq_2 c + (1-sq_2)d) \\
 &\quad + F(ta + (1-t)b, sc + (1-s)d)] d_{q_1} t d_{q_2} s \\
 &= \frac{1}{(b-a)(d-c)} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F\left(\frac{q_1^{n+1}}{2} a + \left(1 - \frac{q_1^{n+1}}{2}\right) b, \frac{q_2^{m+1}}{2} c + \left(1 - \frac{q_2^{m+1}}{2}\right) d\right) \right. \\
 &\quad - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F\left(\frac{q_1^{n+1}}{2} a + \left(1 - \frac{q_1^{n+1}}{2}\right) b, \frac{q_2^m}{2} c + \left(1 - \frac{q_2^m}{2}\right) d\right) \\
 &\quad - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F\left(\frac{q_1^n}{2} a + \left(1 - \frac{q_1^n}{2}\right) b, \frac{q_2^{m+1}}{2} c + \left(1 - \frac{q_2^{m+1}}{2}\right) d\right) \\
 &\quad \left. + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F\left(\frac{q_1^n}{2} a + \left(1 - \frac{q_1^n}{2}\right) b, \frac{q_2^m}{2} c + \left(1 - \frac{q_2^m}{2}\right) d\right) \right\} \\
 &= \frac{1}{(b-a)(d-c)} \left[F(b, d) - F\left(\frac{a+b}{2}, d\right) - F\left(b, \frac{c+d}{2}\right) + F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right].
 \end{aligned}$$

So

$$I_1 = \frac{4}{9(b-a)(d-c)} \left[F(b, d) - F\left(\frac{a+b}{2}, d\right) - F\left(b, \frac{c+d}{2}\right) + F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right]$$

Now from Definition 6, we obtain the following

$$\begin{aligned}
(3.4) \quad & \int_0^{\frac{1}{2}} \int_0^1 s \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
&= \frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)} \int_0^{\frac{1}{2}} \int_0^1 \frac{1}{t} [F(tq_1 a + (1-tq_1)b, sq_2 c + (1-sq_2)d) \\
&\quad - F(tq_1 a + (1-tq_1)b, sc + (1-s)d) - F(ta + (1-t)b, sq_2 c + (1-sq_2)d) \\
&\quad + F(ta + (1-t)b, sc + (1-s)d)] d_{q_1} t d_{q_2} s \\
&= \frac{1}{(b-a)(d-c)} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_2^m \left(F\left(\frac{q_1^{n+1}}{2} a + \left(1 - \frac{q_1^{n+1}}{2}\right) b, q_2^{m+1} c + (1 - q_2^{m+1}) d\right) \right. \right. \\
&\quad - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_2^m \left(F\left(\frac{q_1^{n+1}}{2} a + \left(1 - \frac{q_1^{n+1}}{2}\right) b, q_2^m c + (1 - q_2^m) d\right) \right) \\
&\quad - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_2^m \left(F\left(\frac{q_1^n}{2} a + \left(1 - \frac{q_1^n}{2}\right) b, q_2^{m+1} c + (1 - q_2^{m+1}) d\right) \right) \\
&\quad \left. + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_2^m \left(F\left(\frac{q_1^n}{2} a + \left(1 - \frac{q_1^n}{2}\right) b, q_2^m c + (1 - q_2^m) d\right) \right) \right\} \\
&= \frac{1}{(b-a)(d-c)} \left\{ \sum_{m=0}^{\infty} q_2^m \left[\sum_{n=0}^{\infty} F\left(\frac{q_1^{n+1}}{2} a + \left(1 - \frac{q_1^{n+1}}{2}\right) b, q_2^{m+1} c + (1 - q_2^{m+1}) d\right) \right. \right. \\
&\quad \left. \left. - \sum_{n=0}^{\infty} \left(F\left(\frac{q_1^n}{2} a + \left(1 - \frac{q_1^n}{2}\right) b, q_2^{m+1} c + (1 - q_2^{m+1}) d\right) \right) \right] \right. \\
&\quad \left. + \sum_{m=0}^{\infty} q_2^m \left[\sum_{n=0}^{\infty} F\left(\frac{q_1^n}{2} a + \left(1 - \frac{q_1^n}{2}\right) b, q_2^m c + (1 - q_2^m) d\right) \right. \right. \\
&\quad \left. \left. - \sum_{n=0}^{\infty} F\left(\frac{q_1^{n+1}}{2} a + \left(1 - \frac{q_1^{n+1}}{2}\right) b, q_2^m c + (1 - q_2^m) d\right) \right] \right\} \\
&= \frac{1}{(b-a)(d-c)} \left\{ \sum_{m=0}^{\infty} q_2^m F(b, q_2^{m+1} c + (1 - q_2^{m+1}) d) - \sum_{m=0}^{\infty} q_2^m F(b, q_2^m c + (1 - q_2^m) d) \right. \\
&\quad \left. + \sum_{m=0}^{\infty} q_2^m F\left(\frac{a+b}{2}, q_2^m c + (1 - q_2^m) d\right) - \sum_{m=0}^{\infty} q_2^m F\left(\frac{a+b}{2}, q_2^{m+1} c + (1 - q_2^{m+1}) d\right) \right\} \\
&= \frac{1}{(b-a)(d-c)} \left\{ \frac{1-q_2}{q_2} \sum_{m=0}^{\infty} q_2^m F(b, q_2^m c + (1 - q_2^m) d) - \frac{1}{q_2} F(b, c) \right. \\
&\quad \left. - \frac{1-q_2}{q_2} \sum_{m=0}^{\infty} q_2^m F\left(\frac{a+b}{2}, q_2^m c + (1 - q_2^m) d\right) + \frac{1}{q_2} F\left(\frac{a+b}{2}, c\right) \right\} \\
&= \frac{1}{(b-a)(d-c)} \left[\frac{1}{d-c} \int_c^d F(b, y) {}^d d_{q_2} y - \frac{1}{d-c} \int_c^d F\left(\frac{a+b}{2}, y\right) {}^d d_{q_2} y \right. \\
&\quad \left. - \frac{1}{q_2} F(b, c) + \frac{1}{q_2} F\left(\frac{a+b}{2}, y\right) \right].
\end{aligned}$$

By using the similar operations used in (3.3), we have

$$\begin{aligned}
(3.5) \quad & \int_0^{\frac{1}{2}} \int_0^1 s \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
&= \frac{1}{(b-a)(d-c)} \left[F(b, d) - F\left(\frac{a+b}{2}, d\right) - F(b, c) + F\left(\frac{a+b}{2}, c\right) \right].
\end{aligned}$$

From (3.4) and (3.5), we obtain that

$$\begin{aligned} I_2 &= \frac{2}{3(b-a)(d-c)} \left\{ \frac{1}{d-c} \int_c^d F(b, y) {}^d d_{q_2} y - \frac{1}{d-c} \int_c^d F\left(\frac{a+b}{2}, y\right) {}^d d_{q_2} y \right. \\ &\quad \left. - F(b, c) + F\left(\frac{a+b}{2}, c\right) \right\} - \frac{10}{18(b-a)(d-c)} \left\{ F(b, d) - F\left(\frac{a+b}{2}, d\right) \right. \\ &\quad \left. - F(b, c) + F\left(\frac{a+b}{2}, c\right) \right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_3 &= \frac{2}{3(b-a)(d-c)} \left\{ \frac{1}{b-a} \int_a^b F(x, d) {}^b d_{q_1} x - \frac{1}{b-a} \int_a^b F\left(x, \frac{c+d}{2}\right) {}^b d_{q_1} x \right. \\ &\quad \left. - F(a, d) + F\left(a, \frac{c+d}{2}\right) \right\} - \frac{10}{18(b-a)(d-c)} \left\{ F(b, d) - F\left(b, \frac{c+d}{2}\right) \right. \\ &\quad \left. - F(a, d) + F\left(a, \frac{c+d}{2}\right) \right\}. \end{aligned}$$

Also, we have

$$\begin{aligned} (3.6) \quad & \int_0^1 \int_0^1 \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\ &= \frac{1}{(b-a)(d-c)} [F(b, d) - F(a, d) - F(b, c) + F(a, c)], \end{aligned}$$

$$\begin{aligned} (3.7) \quad & \int_0^1 \int_0^1 s \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\ &= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{q_2(d-c)} \int_c^d F(b, y) {}^d d_{q_2} y - \frac{1}{q_2(d-c)} \int_c^d F(a, y) {}^d d_{q_2} y \right. \\ &\quad \left. - \frac{1}{q_2} F(b, c) + \frac{1}{q_2} F(a, c) \right\}, \end{aligned}$$

$$\begin{aligned} (3.8) \quad & \int_0^1 \int_0^1 t \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\ &= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{q_1(b-a)} \int_a^b F(x, d) {}^b d_{q_1} x - \frac{1}{q_1(b-a)} \int_a^b F(x, c) {}^b d_{q_1} x \right. \\ &\quad \left. - \frac{1}{q_1} F(a, d) + \frac{1}{q_1} F(a, c) \right\} \end{aligned}$$

and

$$\begin{aligned}
(3.9) \quad & \int_0^1 \int_0^1 ts \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s \\
&= \frac{1}{(b-a)(d-c)} \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^{n+1}a + (1-q_1^{n+1})b, q_2^{m+1}c + (1-q_2^{m+1})d) \right. \\
&\quad - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^{n+1}a + (1-q_1^{n+1})b, q_2^m c + (1-q_2^m)d) \\
&\quad - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^{m+1}c + (1-q_2^{m+1})d) \\
&\quad \left. + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d) \right\} \\
&= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{q_1 q_2} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d) \right. \right. \\
&\quad \left. - \sum_{m=0}^{\infty} q_2^m F(a, q_2^m c + (1-q_2^m)d) - \sum_{n=0}^{\infty} q_1^n F(q_1^n a + (1-q_1^n)b, c) + F(a, c) \right] \\
&\quad - \frac{1}{q_1} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d) \right. \\
&\quad \left. - \sum_{m=0}^{\infty} q_2^m F(a, q_2^m c + (1-q_2^m)d) \right] - \frac{1}{q_2} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d) \right. \\
&\quad \left. - \sum_{n=0}^{\infty} q_1^n F(q_1^n a + (1-q_1^n)b, c) \right] + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d) \Big\} \\
&= \frac{1}{(b-a)(d-c)} \left\{ \frac{(1-q_1)(1-q_2)}{q_1 q_2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d) \right. \\
&\quad \left. - \frac{1-q_2}{q_1 q_2} \sum_{m=0}^{\infty} q_2^m F(a, q_2^m c + (1-q_2^m)d) - \frac{1-q_1}{q_1 q_2} \sum_{n=0}^{\infty} q_1^n F(q_1^n a + (1-q_1^n)b, c) + \frac{1}{q_1 q_2} F(a, c) \right\} \\
&= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{q_1 q_2 (b-a)(d-c)} \int_a^b \int_c^d F(x, y) {}^b d_{q_1} x {}^d d_{q_2} y - \frac{1}{q_1 q_2 (b-a)} \int_a^b F(x, c) {}^b d_{q_1} x \right. \\
&\quad \left. - \frac{1}{q_1 q_2 (d-c)} \int_c^d F(a, x) {}^d d_{q_2} y + \frac{1}{q_1 q_2} F(a, c) \right\}.
\end{aligned}$$

From (3.6)-(3.9), we obtain that

$$\begin{aligned}
I_4 &= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) {}^b d_{q_1} x {}^d d_{q_2} y \right. \\
&\quad \left. - \frac{1}{b-a} \int_a^b F(x, c) {}^b d_{q_1} x - \frac{1}{d-c} \int_c^d F(a, y) {}^d d_{q_2} y + F(a, c) \right\} \\
&\quad - \frac{5}{6(b-a)(d-c)} \left\{ \frac{1}{b-a} \int_a^b F(x, d) {}^b d_{q_1} x - \frac{1}{b-a} \int_a^b F(x, c) {}^b d_{q_1} x \right. \\
&\quad \left. + F(a, c) - F(a, d) \right\} - \frac{5}{6(b-a)(d-c)} \left\{ \frac{1}{d-c} \int_c^d F(b, y) {}^d d_{q_2} y \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{d-c} \int_c^d F(a, y) {}^d d_{q_2} y + F(a, c) - F(b, c) \Big\} \\
& + \frac{25}{36(b-a)(d-c)} [F(b, d) - F(a, d) - F(b, c) + F(a, c)].
\end{aligned}$$

Now using the calculated integrals $(I_1)-(I_4)$ in (3.2) and multiplying the resulting one with $(b-a)(d-c)$, then we have desired equality (3.1) which accomplishes the proof. \square

Remark 1. Under the given conditions of Lemma 3 with $q_1, q_2 \rightarrow 1^-$, then we have following identity

$$\begin{aligned}
(3.10) \quad & \frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9} \\
& + \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{36} \\
& - \frac{1}{6(b-a)} \int_a^b \left[F(x, c) + 4F\left(x, \frac{c+d}{2}\right) + F(x, d) \right] dx \\
& - \frac{1}{6(d-c)} \int_c^d \left[F(a, y) + 4F\left(\frac{a+b}{2}, y\right) + F(b, y) \right] dy \\
& + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) dx dy \\
& = (b-a)(d-c) \int_0^1 \int_0^1 \Lambda(t) \Lambda(s) \frac{\partial^2 F(ta + (1-t)b, sc + (1-s)d)}{\partial t \partial s} dt ds,
\end{aligned}$$

where

$$\Lambda(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}) \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1) \end{cases}$$

and

$$\Lambda(s) = \begin{cases} s - \frac{1}{6}, & s \in [0, \frac{1}{2}) \\ s - \frac{5}{6}, & s \in [\frac{1}{2}, 1) \end{cases}$$

which is turned out by Özdemir et al. in [18, Lemma 1].

4. SOME NEW $q_1 q_2$ -SIMPSON'S TYPE INEQUALITIES

For brevity, we give some calculated integrals before giving new estimates.

$$(4.1) \quad A_1(q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| t d_q t = \begin{cases} \frac{1-2q-2q^2}{24[2]_q[3]_q} & 0 < q < \frac{1}{3} \\ \frac{18q^2+18q-7}{216[2]_q[3]_q} & \frac{1}{3} \leq q < 1, \end{cases}$$

$$(4.2) \quad A_2(q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| (1-t) d_q t = \begin{cases} \frac{1-4q^3}{24[2]_q[3]_q} & 0 < q < \frac{1}{3} \\ \frac{36q^3+12q^2+12q+1}{216[2]_q[3]_q} & \frac{1}{3} \leq q < 1, \end{cases}$$

$$(4.3) \quad A_3(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| t d_q t = \begin{cases} \frac{15-6q-6q^2}{24[2]_q[3]_q} & 0 < q < \frac{5}{6} \\ \frac{18q^2+18q+25}{216[2]_q[3]_q} & \frac{5}{6} \leq q < 1 \end{cases}$$

$$(4.4) \quad A_4(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| (1-t) d_q t = \begin{cases} \frac{-5+8q+8q^2-8q^3}{24[2]_q[3]_q} & 0 < q < \frac{5}{6} \\ \frac{12q^2+12q+5}{216[2]_q[3]_q} & \frac{5}{6} \leq q < 1, \end{cases}$$

$$(4.5) \quad A_5(q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_q t = \frac{6q-1}{36[2]_q},$$

$$(4.6) \quad A_6(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| d_q t = \frac{5}{36[2]_q}.$$

Now we give some new quantum estimates by using the identity given in last section.

Theorem 7. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^{b, d}\partial_{q_1, q_2}^2 F(t, s)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. Then we have following inequality provided that $\left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(t, s)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right|$ is convex on $[a, b] \times [c, d]$

$$(4.7) \quad \begin{aligned} |{}^{b, d}\mathcal{I}_{q_1, q_2}(F)| &\leq (b-a)(d-c) \left[(A_1(q_1) + A_3(q_1))(A_1(q_2) + A_3(q_2)) \left| \frac{{}^{b, d}\partial_{q_1, q_2} F(a, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right. \\ &\quad + (A_1(q_1) + A_3(q_1))(A_2(q_2) + A_4(q_2)) \left| \frac{{}^{b, d}\partial_{q_1, q_2} F(a, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \\ &\quad + (A_2(q_1) + A_4(q_1))(A_1(q_2) + A_3(q_2)) \left| \frac{{}^{b, d}\partial_{q_1, q_2} F(b, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \\ &\quad \left. + (A_2(q_1) + A_4(q_1))(A_2(q_2) + A_4(q_2)) \left| \frac{{}^{b, d}\partial_{q_1, q_2} F(b, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right], \end{aligned}$$

where $q_1, q_2 \in (0, 1)$.

Proof. On taking modulus of the identity of Lemma 3.1, because of the properties of modulus, we find that

$$(4.8) \quad \begin{aligned} &|{}^{b, d}\mathcal{I}_{q_1, q_2}(F)| \\ &\leq (b-a)(d-c) \int_0^1 \int_0^1 |\Lambda_{q_1}(t) \Lambda_{q_2}(s)| \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| d_{q_1} t d_{q_2} s \end{aligned}$$

Now using the convexity of $\left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(t, s)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right|$, then (4.8) becomes

$$(4.9) \quad \begin{aligned} &|{}^{b, d}\mathcal{I}_{q_1, q_2}(F)| \\ &\leq (b-a)(d-c) \int_0^1 \Lambda_{q_2}(s) \left[\int_0^1 \Lambda_{q_1}(t) \left\{ t \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right. \right. \\ &\quad \left. \left. + (1-t) \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right\} d_{q_1} t \right] d_{q_2} s. \end{aligned}$$

Now we compute the integrals appear in right side of inequality (4.9)

$$\begin{aligned} &\int_0^1 \Lambda_{q_1}(t) \left\{ t \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| + (1-t) \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right\} d_{q_1} t \\ &= \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| d_{q_1} t \\ &\quad + \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| d_{q_1} t \\ &\quad + \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| d_{q_1} t \\ &\quad + \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| \left| \frac{{}^{b, d}\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| d_{q_1} t. \end{aligned}$$

From (4.1)-(4.4), we obtain that

$$\begin{aligned} & \int_0^1 \Lambda_{q_1}(t) \left\{ t \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| + (1-t) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right\} d_{q_1} t \\ &= \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| (A_1(q_1) + A_3(q_1)) \\ &+ \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| (A_2(q_1) + A_4(q_1)). \end{aligned}$$

Thus, we have

$$\begin{aligned} & |{}^b, d\mathcal{I}_{q_1, q_2}(F)| \\ &\leq (b-a)(d-c) \int_0^1 \Lambda_{q_2}(s) \left[\left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| (A_1(q_1) + A_3(q_1)) \right. \\ &\quad \left. + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| (A_2(q_1) + A_4(q_1)) \right] d_{q_2} s \\ &\leq (b-a)(d-c) \int_0^1 \Lambda_{q_2}(s) \left[\left\{ s \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| + (1-s) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right. \right. \\ &\quad \times (A_1(q_1) + A_3(q_1)) \left. \left. + \left\{ s \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| + (1-s) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right. \right. \right. \\ &\quad \times (A_2(q_1) + A_4(q_1)) \left. \left. \right\} \right] d_{q_2} s \\ &= (b-a)(d-c) (A_1(q_1) + A_3(q_1)) \left[\left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_0^{\frac{1}{2}} s \left| q_2 s - \frac{1}{6} \right| d_{q_2} s \right. \\ &\quad + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_0^{\frac{1}{2}} (1-s) \left| q_2 s - \frac{1}{6} \right| d_{q_2} s + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_{\frac{1}{2}}^1 s \left| q_2 s - \frac{5}{6} \right| d_{q_2} s \\ &\quad + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_{\frac{1}{2}}^1 (1-s) \left| q_2 s - \frac{5}{6} \right| d_{q_2} s \left. \right] \\ &\quad + (b-a)(d-c) (A_2(q_1) + A_4(q_1)) \left[\left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_0^{\frac{1}{2}} s \left| q_2 s - \frac{1}{6} \right| d_{q_2} s \right. \\ &\quad + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_0^{\frac{1}{2}} (1-s) \left| q_2 s - \frac{1}{6} \right| d_{q_2} s + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_{\frac{1}{2}}^1 s \left| q_2 s - \frac{5}{6} \right| d_{q_2} s \\ &\quad + \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \int_{\frac{1}{2}}^1 (1-s) \left| q_2 s - \frac{5}{6} \right| d_{q_2} s \left. \right]. \end{aligned}$$

From (4.1)-(4.4), we have

$$\begin{aligned} |{}^b, d\mathcal{I}_{q_1, q_2}(F)| &\leq (b-a)(d-c) \left[(A_1(q_1) + A_3(q_1)) (A_1(q_2) + A_3(q_2)) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right. \\ &\quad + (A_1(q_1) + A_3(q_1)) (A_2(q_2) + A_4(q_2)) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(a, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \\ &\quad + (A_2(q_1) + A_4(q_1)) (A_1(q_2) + A_3(q_2)) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, c)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \\ &\quad \left. + (A_2(q_1) + A_4(q_1)) (A_2(q_2) + A_4(q_2)) \left| \frac{{}^b, d\partial_{q_1, q_2}^2 F(b, d)}{{}^b\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right]. \end{aligned}$$

Hence proof is completed. \square

Remark 2. Under the given conditions of Theorem 7 with $q_1, q_2 \rightarrow 1^-$, then we attain the following inequality

$$\begin{aligned}
 (4.10) \quad & \left| \frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9} \right. \\
 & + \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{36} \\
 & - \frac{1}{6(b-a)} \int_a^b \left[F(x, c) + 4F\left(x, \frac{c+d}{2}\right) + F(x, d) \right] dx \\
 & - \frac{1}{6(d-c)} \int_c^d \left[F(a, y) + 4F\left(\frac{a+b}{2}, y\right) + F(b, y) \right] dy \\
 & \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) dx dy \right| \\
 & \leq \frac{25(b-a)(d-c)}{72} \left[\frac{\left| \frac{\partial^2 F(a, c)}{\partial t \partial s} \right| + \left| \frac{\partial^2 F(a, d)}{\partial t \partial s} \right| + \left| \frac{\partial^2 F(b, c)}{\partial t \partial s} \right| + \left| \frac{\partial^2 F(b, d)}{\partial t \partial s} \right|}{72} \right]
 \end{aligned}$$

which is shown by Özdemir et al. in [18, Theorem 3].

Theorem 8. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^{b, d} \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. If $\left| \frac{{}^{b, d} \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p$ is convex on $[a, b] \times [c, d]$ for some $p > 1$ and $\frac{1}{r} + \frac{1}{p} = 1$. Then we have following inequality

$$\begin{aligned}
 (4.11) \quad & \left| {}^{b, d} \mathcal{I}_{q_1, q_2}(F) \right| \\
 & \leq (b-a)(d-c) \left(\int_0^1 \int_0^1 |\Lambda_{q_1}(t) \Lambda_{q_2}(s)|^r {}^d q_1 t {}^d q_2 s \right)^{\frac{1}{r}} \\
 & \quad \left[\frac{1}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^{b, d} \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + \frac{q_2}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^{b, d} \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right. \\
 & \quad \left. + \frac{q_1}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^{b, d} \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + \frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^{b, d} \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right]^{\frac{1}{p}},
 \end{aligned}$$

where $0 < q_1, q_2 < 1$.

Proof. Applying well-known Hölder's inequality for $q_1 q_2$ -integrals to the integrals in right side of (4.8), it is found that

$$\begin{aligned}
 (4.12) \quad & \left| {}^{b, d} \mathcal{I}_{q_1, q_2}(F) \right| \\
 & \leq (b-a)(d-c) \left[\left(\int_0^1 \int_0^1 |\Lambda_{q_1}(t) \Lambda_{q_2}(s)|^r {}^d q_1 t {}^d q_2 s \right)^{\frac{1}{r}} \right. \\
 & \quad \left. \times \left(\int_0^1 \int_0^1 \left| \frac{{}^{b, d} \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p {}^d q_1 t {}^d q_2 s \right)^{\frac{1}{p}} \right].
 \end{aligned}$$

By applying convexity of $\left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^{P_1}$, then (4.12) becomes

$$\begin{aligned}
 (4.13) \quad & \left| {}^b, {}^d \mathcal{I}_{q_1, q_2}(F) \right| \\
 & \leq (b-a)(d-c) \left[\left(\int_0^1 \int_0^1 |\Lambda_{q_1}(t) \Lambda_{q_2}(s)|^r d_{q_1} t d_{q_2} s \right)^{\frac{1}{r}} \right. \\
 & \quad \times \left(\int_0^1 \int_0^1 \left[t s \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + t(1-s) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right. \right. \\
 & \quad \left. \left. + (1-t)s \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + (1-t)(1-s) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right] d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \Bigg].
 \end{aligned}$$

Now, if we apply the concept of Lemma 1 for $a = 0$ to the above quantum integrals, we attain

$$\begin{aligned}
 (4.14) \quad & \int_0^1 \int_0^1 t s d_{q_1} t d_{q_2} s = \left(\int_0^1 t d_{q_1} t \right) \left(\int_0^1 s d_{q_2} s \right) \\
 & = \frac{1}{[2]_{q_1} [2]_{q_2}},
 \end{aligned}$$

$$(4.15) \quad \int_0^1 \int_0^1 t(1-s) d_{q_1} t d_{q_2} s = \frac{q_2}{[2]_{q_1} [2]_{q_2}},$$

$$(4.16) \quad \int_0^1 \int_0^1 (1-t) s d_{q_1} t d_{q_2} s = \frac{q_1}{[2]_{q_1} [2]_{q_2}},$$

$$(4.17) \quad \int_0^1 \int_0^1 (1-t)(1-s) d_{q_1} t d_{q_2} s = \frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}}.$$

By substituting the calculated integrals (4.14)-(4.17) in (4.13), then we obtain the desired inequality (4.11) which finishes the proof. \square

Theorem 9. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. If $\left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p$

is convex on $[a, b] \times [c, d]$ for some $p \geq 1$. Then we have following inequality

$$\begin{aligned}
(4.18) \quad & \left| {}^b, {}^d \mathcal{I}_{q_1, q_2}(F) \right| \\
\leq & (b-a)(d-c) \left[A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \right. \\
& \times \left\{ A_1(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& + A_2(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \left. \right\}^{\frac{1}{p}} \\
& + A_5^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_1(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& + A_2(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \left. \right\}^{\frac{1}{p}} \\
& + A_6^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_3(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& + A_4(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \left. \right\}^{\frac{1}{p}} \\
& + A_6^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_3(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& + A_4(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \left. \right\}^{\frac{1}{p}} \left. \right]
\end{aligned}$$

where $0 < q_1, q_2 < 1$.

Proof. Applying well-known power mean inequality for $q_1 q_2$ -integrals to the integrals in right side of (4.8), it is found that

$$\begin{aligned}
(4.19) \quad & \left| {}^b, {}^d \mathcal{I}_{q_1, q_2}(F) \right| \\
\leq & (b-a)(d-c) \left[\left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{1}{6} \right| d_{q_1} t d_{q_2} s \right)^{1-\frac{1}{p}} \right. \\
& \times \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{1}{6} \right| \right. \\
& \times \left. \left. \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{5}{6} \right| \right)^{1-\frac{1}{p}} \\
& \times \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{5}{6} \right| \right. \\
& \times \left. \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \\
& + \left(\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{1}{6} \right| d_{q_1} t d_{q_2} s \right)^{1-\frac{1}{p}} \\
& \times \left(\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{1}{6} \right| \right. \\
& \times \left. \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \\
& + \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{5}{6} \right| d_{q_1} t d_{q_2} s \right) \\
& \times \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{5}{6} \right| d_{q_1} t d_{q_2} s \right. \\
& \times \left. \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \Bigg].
\end{aligned}$$

By applying convexity of $\left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p$, then we have

$$\begin{aligned}
(4.20) \quad & \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{1}{6} \right| d_{q_1} t d_{q_2} s \right)^{1-\frac{1}{p}} \\
& \times \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{1}{6} \right| \right. \\
& \times \left. \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \\
& \leq \left(\left(\int_0^{\frac{1}{2}} \left| q_1 t - \frac{1}{6} \right| d_{q_1} t \right) \left(\int_0^{\frac{1}{2}} \left| q_2 s - \frac{1}{6} \right| d_{q_2} s \right) \right)^{1-\frac{1}{p}} \\
& \left[\int_0^{\frac{1}{2}} \left| q_2 s - \frac{1}{6} \right| \left\{ \int_0^{\frac{1}{2}} \left| q_1 t - \frac{1}{6} \right| \left(t \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right. \right. \right. \\
& \left. \left. \left. + (1-t) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) d_{q_1} t \right\} d_{q_2} s \right]^{\frac{1}{p}} \\
& = A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \left[A_1(q_1) \int_0^{\frac{1}{2}} \left| q_2 s - \frac{1}{6} \right| \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_2} s \right.
\end{aligned}$$

$$\begin{aligned}
& + A_2(q_1) \int_0^{\frac{1}{2}} \left| q_2 s - \frac{1}{6} \right| \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_2} s \Bigg]^{\frac{1}{p}} \\
& \leq A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \left[A_1(q_1) \int_0^{\frac{1}{2}} \left| q_2 s - \frac{1}{6} \right| \left(s \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right. \right. \\
& \quad \left. \left. + (1-s) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) d_{q_2} s + A_2(q_1) \int_0^{\frac{1}{2}} \left| q_2 s - \frac{1}{6} \right| \right. \\
& \quad \left. \times \left(s \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + (1-s) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) d_{q_2} s \right]^{\frac{1}{p}} \\
& = A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \left[A_1(q_1) \left\{ A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right. \\
& \quad \left. + A_2(q_1) \left\{ A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right]^{\frac{1}{p}}.
\end{aligned}$$

By applying the similar operations, we obtain that

$$\begin{aligned}
(4.21) \quad & \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{5}{6} \right| \right)^{1-\frac{1}{p}} \\
& \times \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{1}{6} \right| \left| q_2 s - \frac{5}{6} \right| \right. \\
& \quad \left. \times \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \\
& \leq A_5^{1-\frac{1}{p}} A_6^{1-\frac{1}{p}} \left[A_1(q_1) \left\{ A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right. \\
& \quad \left. + A_2(q_1) \left\{ A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right]^{\frac{1}{p}},
\end{aligned}$$

$$\begin{aligned}
(4.22) \quad & \left(\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{1}{6} \right| d_{q_1} t d_{q_2} s \right)^{1-\frac{1}{p}} \\
& \times \left(\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{1}{6} \right| \right. \\
& \quad \left. \times \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right)^{\frac{1}{p}} \\
& \leq A_6^{1-\frac{1}{p}} A_5^{1-\frac{1}{p}} \left[A_3(q_1) \left\{ A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right. \\
& \quad \left. + A_4(q_1) \left\{ A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right]^{\frac{1}{p}},
\end{aligned}$$

$$\begin{aligned}
(4.23) \quad & \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{5}{6} \right| d_{q_1} t d_{q_2} s \right) \\
& \times \left[\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left| q_1 t - \frac{5}{6} \right| \left| q_2 s - \frac{5}{6} \right| d_{q_1} t d_{q_2} s \right. \\
& \times \left. \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sc + (1-s)d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p d_{q_1} t d_{q_2} s \right]^{\frac{1}{p}} \\
& \leq A_6^{1-\frac{1}{p}} A_6^{1-\frac{1}{p}} \left[A_3(q_1) \left\{ A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right. \\
& \quad \left. + A_4(q_1) \left\{ A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right\} \right]^{\frac{1}{p}}.
\end{aligned}$$

From (4.19)-(4.23), we obtain desired inequality and proof is ended. \square

5. ADDITIONAL RESULTS

In this section, we present some results without proof since the proofs are similar to ones given in Section 3.

Lemma 4. *Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. Then following identity holds for $q_1 q_2$ -integrals:*

$$\begin{aligned}
(5.1) \quad & {}_a^d \mathcal{I}_{q_1, q_2}(F) \\
& = (b-a)(d-c) \int_0^1 \int_0^1 \Lambda_{q_1}(t) \Lambda_{q_2}(s) \frac{{}^a \partial_{q_1, q_2}^2 F(tb + (1-t)a, sc + (1-s)d)}{{}^a \partial_{q_1} t {}^d \partial_{q_2} s} d_{q_1} t d_{q_2} s,
\end{aligned}$$

where $q_1, q_2 \in (0, 1)$, Λ_{q_1} and Λ_{q_2} are defined as in Lemma 3 and

$$\begin{aligned}
{}_a^d \mathcal{I}_{q_1, q_2}(F) & = \frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9} \\
& \quad + \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{36} \\
& \quad - \frac{1}{6(b-a)} \int_a^b \left[F(x, c) + 4F\left(x, \frac{c+d}{2}\right) + F(x, d) \right] {}_a d_{q_1} x \\
& \quad - \frac{1}{6(d-c)} \int_c^d \left[F(a, y) + 4F\left(\frac{a+b}{2}, y\right) + F(b, y) \right] {}^d d_{q_2} y \\
& \quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) {}_a d_{q_1} x {}^d d_{q_2} y.
\end{aligned}$$

and $0 < q_1, q_2 < 1$.

Proof. If the strategy which was used in the proof of Lemma 3 are applied by taking into account the definition of $\frac{{}^a \partial_{q_1, q_2}^2 F(t, s)}{{}^a \partial_{q_1} t {}^d \partial_{q_2} s}$, the desired inequality (5.1) can be attained. \square

Remark 3. *If we choose $q_1, q_2 \rightarrow 1^-$ in Lemma 4, then the identity (5.1) reduces to identity (3.10).*

Now we use Lemma 4 to find some new quantum estimates. We first examine a new result for functions whose partially $q_1 q_2$ -derivatives in modulus are convex in the following theorem.

Theorem 10. *Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^a \partial_{q_1, q_2}^2 F(t, s)}{{}^a \partial_{q_1} t {}^d \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. Then we have*

following inequality provided that $\left| \frac{{}^d\partial_{q_1,q_2}^2 F(t,s)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right|$ is convex on $[a, b] \times [c, d]$

$$\begin{aligned}
 (5.2) \quad \left| {}^d\mathcal{I}_{q_1,q_2}(F) \right| &\leq (b-a)(d-c) \left[(A_1(q_1) + A_3(q_1))(A_1(q_2) + A_3(q_2)) \left| \frac{{}^d\partial_{q_1,q_2} F(a,c)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right. \\
 &\quad + (A_1(q_1) + A_3(q_1))(A_2(q_2) + A_4(q_2)) \left| \frac{{}^d\partial_{q_1,q_2} F(a,d)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right| \\
 &\quad + (A_2(q_1) + A_4(q_1))(A_1(q_2) + A_3(q_2)) \left| \frac{{}^d\partial_{q_1,q_2} F(b,c)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right| \\
 &\quad \left. + (A_2(q_1) + A_4(q_1))(A_2(q_2) + A_4(q_2)) \left| \frac{{}^d\partial_{q_1,q_2} F(b,d)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right| \right],
 \end{aligned}$$

where $q_1, q_2 \in (0, 1)$.

Theorem 11. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^d\partial_{q_1,q_2}^2 F(t,s)}{{}_b\partial_{q_1} t {}^d\partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. If $\left| \frac{{}^b, {}^d\partial_{q_1,q_2}^2 F(t,s)}{{}_b\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p$ is convex on $[a, b] \times [c, d]$ for some $p > 1$ and $\frac{1}{r} + \frac{1}{p} = 1$. Then we have following inequality

$$\begin{aligned}
 (5.3) \quad &\left| {}^d\mathcal{I}_{q_1,q_2}(F) \right| \\
 &\leq (b-a)(d-c) \left(\int_0^1 \int_0^1 |\Lambda_{q_1}(t) \Lambda_{q_2}(s)|^r d_{q_1} t d_{q_2} s \right)^{\frac{1}{r}} \\
 &\quad \left[\frac{1}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^d\partial_{q_1,q_2} F(a,c)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + \frac{q_2}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^d\partial_{q_1,q_2} F(a,d)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right. \\
 &\quad \left. + \frac{q_1}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^d\partial_{q_1,q_2} F(b,c)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + \frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^d\partial_{q_1,q_2} F(b,d)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right]^{\frac{1}{p}},
 \end{aligned}$$

where $0 < q_1, q_2 < 1$.

Theorem 12. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^d\partial_{q_1,q_2}^2 F(t,s)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. If $\left| \frac{{}^d\partial_{q_1,q_2}^2 F(t,s)}{{}_a\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p$ is convex on $[a, b] \times [c, d]$ for some $p \geq 1$. Then we have following inequality

$$\begin{aligned}
 (5.4) \quad &\left| {}^d\mathcal{I}_{q_1,q_2}(F) \right| \\
 &\leq (b-a)(d-c) \left[A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \right. \\
 &\quad \times \left\{ A_1(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d\partial_{q_1,q_2}^2 F(b,c)}{{}_b\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d\partial_{q_1,q_2}^2 F(b,d)}{{}_b\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right. \\
 &\quad \left. + A_2(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d\partial_{q_1,q_2}^2 F(a,c)}{{}_b\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d\partial_{q_1,q_2}^2 F(a,d)}{{}_b\partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right\} \right]^{\frac{1}{p}}
 \end{aligned}$$

$$\begin{aligned}
& + A_5^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_1(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& \left. + A_2(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right\}^{\frac{1}{p}} \\
& + A_6^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_3(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& \left. + A_4(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right\}^{\frac{1}{p}} \\
& + A_6^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_3(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\
& \left. + A_4(q_1) \left(A_3(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right\}^{\frac{1}{p}} \Bigg]
\end{aligned}$$

where $0 < q_1, q_2 < 1$.

Lemma 5. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. Then following identity holds for $q_1 q_2$ -integrals:

$$\begin{aligned}
(5.5) \quad & {}^b \mathcal{I}_{q_1, q_2}(F) \\
& = (b-a)(d-c) \int_0^1 \int_0^1 \Lambda_{q_1}(t) \Lambda_{q_2}(s) \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(ta + (1-t)b, sd + (1-s)c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} {}^b d_{q_1} t {}^d d_{q_2} s,
\end{aligned}$$

where $q_1, q_2 \in (0, 1)$, Λ_{q_1} and Λ_{q_2} are defined as in Lemma 3 and

$$\begin{aligned}
{}^b \mathcal{I}_{q_1, q_2}(F) & = \frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9} \\
& + \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{36} \\
& - \frac{1}{6(b-a)} \int_a^b \left[F(x, c) + 4F\left(x, \frac{c+d}{2}\right) + F(x, d) \right] {}^b d_{q_1} x \\
& - \frac{1}{6(d-c)} \int_c^d \left[F(a, y) + 4F\left(\frac{a+b}{2}, y\right) + F(b, y) \right] {}^d d_{q_2} y \\
& + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) {}^b d_{q_1} x {}^d d_{q_2} y.
\end{aligned}$$

Proof. If the strategy which was used in the proof of Lemma 3 are applied by taking into account the definition of $\frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s}$, the desired inequality (5.5) can be attained. \square

Remark 4. If we choose $q_1, q_2 \rightarrow 1^-$ in Lemma 5, then the identity (5.5) reduces to identity (3.10).

Now we use Lemma 5 to find some new quantum estimates. We first examine a new result for functions whose partially $q_1 q_2$ -derivatives in modulus are convex in the following theorem.

Theorem 13. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^b \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. Then we have following inequality provided that $\left| \frac{{}^b \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|$ is convex on $[a, b] \times [c, d]$

$$(5.6) \quad \left| {}^b \mathcal{I}_{q_1, q_2}(F) \right| \leq (b-a)(d-c) \left[(A_1(q_1) + A_3(q_1))(A_1(q_2) + A_3(q_2)) \left| \frac{{}^b \partial_{q_1, q_2} F(a, c)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right| \right. \\ \left. + (A_1(q_1) + A_3(q_1))(A_2(q_2) + A_4(q_2)) \left| \frac{{}^b \partial_{q_1, q_2} F(a, d)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right| \right. \\ \left. + (A_2(q_1) + A_4(q_1))(A_1(q_2) + A_3(q_2)) \left| \frac{{}^b \partial_{q_1, q_2} F(b, c)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right| \right. \\ \left. + (A_2(q_1) + A_4(q_1))(A_2(q_2) + A_4(q_2)) \left| \frac{{}^b \partial_{q_1, q_2} F(b, d)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right| \right],$$

where $q_1, q_2 \in (0, 1)$.

Theorem 14. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^b \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. If $\left| \frac{{}^b \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|^p$ is convex on $[a, b] \times [c, d]$ for some $p > 1$ and $\frac{1}{r} + \frac{1}{p} = 1$. Then we have following inequality

$$(5.7) \quad \left| {}^b \mathcal{I}_{q_1, q_2}(F) \right| \leq (b-a)(d-c) \left(\int_0^1 \int_0^1 |\Lambda_{q_1}(t) \Lambda_{q_2}(s)|^r d_{q_1} t d_{q_2} s \right)^{\frac{1}{r}} \\ \left[\frac{1}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^b \partial_{q_1, q_2} F(a, c)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|^p + \frac{q_2}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^b \partial_{q_1, q_2} F(a, d)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|^p \right. \\ \left. + \frac{q_1}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^b \partial_{q_1, q_2} F(b, c)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|^p + \frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \left| \frac{{}^b \partial_{q_1, q_2} F(b, d)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|^p \right]^{\frac{1}{p}},$$

where $0 < q_1, q_2 < 1$.

Theorem 15. Let $F : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° such that partial $q_1 q_2$ -derivative $\frac{{}^b \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s}$ is continuous and integrable on $[a, b] \times [c, d] \subseteq \Delta^\circ$. If $\left| \frac{{}^b \partial_{q_1, q_2}^2 F(t, s)}{{}^b \partial_{q_1} t {}^c \partial_{q_2} s} \right|^p$ is convex on $[a, b] \times [c, d]$ for some $p \geq 1$. Then we have following inequality

$$(5.8) \quad \left| {}^b \mathcal{I}_{q_1, q_2}(F) \right| \leq (b-a)(d-c) \left[A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \right. \\ \times \left\{ A_1(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(a, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right. \\ \left. + A_2(q_1) \left(A_1(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, d)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{{}^b, {}^d \partial_{q_1, q_2}^2 F(b, c)}{{}^b \partial_{q_1} t {}^d \partial_{q_2} s} \right|^p \right) \right\} \right]^{\frac{1}{p}}$$

$$\begin{aligned}
& + A_5^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_1(q_1) \left(A_3(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(a, d)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(a, c)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right. \\
& \left. + A_2(q_1) \left(A_3(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(b, d)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(b, c)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right\}^{\frac{1}{p}} \\
& + A_6^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_3(q_1) \left(A_1(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(a, d)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(a, c)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right. \\
& \left. + A_4(q_1) \left(A_1(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(b, d)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_2(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(b, c)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right\}^{\frac{1}{p}} \\
& + A_6^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
& \times \left\{ A_3(q_1) \left(A_3(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(a, d)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(a, c)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right. \\
& \left. + A_4(q_1) \left(A_3(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(b, d)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p + A_4(q_2) \left| \frac{b, {}^d\partial_{q_1, q_2}^2 F(b, c)}{b \partial_{q_1} t {}^d\partial_{q_2} s} \right|^p \right) \right\}^{\frac{1}{p}} \Bigg]
\end{aligned}$$

where $0 < q_1, q_2 < 1$.

6. CONCLUSION

In this paper, Simpson's type inequalities for co-ordinated convex functions by applying the $q_1 q_2$ -integral are obtained. It is also showed that the results proved in this paper are potential generalisation of the existing comparable results in the literature. As future directions, one may finds the similar inequalities through different types of convexities.

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JIANGSU KEY LABORATORY FOR NSLSCS, SCHOOL OF MATHEMATICAL SCIENCES, NANJING NORMAL UNIVERSITY, NANJING, 210023, CHINA.

E-mail address: mahr.muhammad.aamir@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, DÜZCE-TURKEY

E-mail address: hsyn.budak@gmail.com

JIANGSU KEY LABORATORY FOR NSLSCS, SCHOOL OF MATHEMATICAL SCIENCES, NANJING NORMAL UNIVERSITY, NANJING, 210023, CHINA.

E-mail address: 05298@njnu.edu.cn

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, UNIVERSITY OF KAHRAMANMARAS SÜTÇÜ İMAM, 46000, KAHRAMANMARAS, TURKEY

E-mail address: hyildir@ksu.edu.tr