ON PRESERVATION OF FUNCTIONS WITH EXPONENTIAL GROWTH BY CERTAIN EXPONENTIAL OPERATORS

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Abstract. In this study, our aim is to provide a modification of the so-called Ismail-May operators that preserve exponential functions $e^{Ax}, A \in \mathbb{R}$. In consonance to this, we begin with estimating the convergence rate of the operators in terms of usual and exponential modulus of continuity. We also provide a global approximation and a quantitative Voronovskaya result. Moreover, to validate the modification, we exhibit some graphical representations using Mathematica software to compare the original operator and its modification. We conclude that the modified operators not only preserve exponential functions but also provide faster rate of convergence when $A > 0$.

1. Introduction

In extension to the work on exponential operators by May [1], Ismail and May [2] showed that for a polynomial $p(x)$ of degree $n \in \mathbb{N}$, an approximation operator can be uniquely obtained by determining its unique kernel. As a consequence of this, besides recovering some well known operators such as Szász operators, Classical Bernstein operators, Post-Widder operators etc. for polynomials of degree at most two, they also constructed some new operators with cubic polynomials. For instance, if $p(x) = 2x^{3/2}$, the newly constructed operators are defined as

$$\mathcal{L}_\lambda (f;x) = e^{-\lambda\sqrt{x}} \left\{ f(0) + \lambda \int_0^\infty e^{-\lambda t/\sqrt{x}} t^{-1/2} \mathcal{I}_1 \left(2\lambda\sqrt{t}\right) f(t)\, dt \right\}$$

where $\mathcal{I}_\lambda(y)$ is a first kind modified Bessel function identified as

$$I_\lambda (y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(\lambda+k)!} \left(\frac{y}{2}\right)^{\lambda+2k}$$

These operators were further studied in detail in [3]. Again for $p(x) = x(1+x^2)$, the corresponding operators obtained are

$$\mathcal{R}_\lambda (f;x) = \sum_{k=0}^{\infty} r_{\lambda,k}(x) f\left(\frac{k}{\lambda+k}\right), \quad x \in (0,1)$$

where

$$r_{\lambda,k}(x) = \frac{\lambda(\lambda+k)^{k-1}}{k!} e^{-\lambda x} \left(xe^{-x}\right)^k.$$

Key words and phrases. Exponential operators, Ismail-May operators, King-type operators, Voronovskaya type theorems.

and $\lambda > 0$. These operators were further studied in detail by ([4], [5]). Another such operator for $p(x) = x^3$ is defined by

$$\mathcal{P}_n(f,x) = \int_0^\infty k_n(x,t)f(t)dt, x \in (0,\infty) \tag{1.1}$$

whose kernel is defined as

$$k_n(x,t) = \left(\frac{n}{2\pi}\right)^{1/2} e^{n/x} t^{-3/2} exp\left(-\frac{nt}{2x^2} - \frac{n}{2t}\right).$$

These operators were studied further in detail by Gupta [6]. All the three approximation processes cited above are examples of exponential operators as they satisfy the normalization condition $W_\lambda(1,x) = \int\limits_{-\infty}^{\infty} S(\lambda,x,t)dt = 1$ and the partial differential equation

$$\frac{\partial}{\partial x} S(\lambda,x,t) = \frac{\lambda(t-x)}{p(x)} S(\lambda,x,t),$$

where, $S(\lambda,x,t) \geq 0$ is the kernel of the operators and λ, x belong to any subset of $\mathbb{R}$.

In past years, there have been several modifications of operators to enhance their convergence and error estimation process (see [7],[8],[9]). In 2003, King [10] presented a sequence of linear positive operators which approximated each continuous function on $[0,1]$ while preserving the test function x^2. This remarkable approach has been since applied by many researchers to propose good modifications and fulfil the need to achieve better approximation. For example, Duman and Özarslan [11] gave a modification of classical Szász operators to provide a better error estimation. Bodur et al. [12] introduced a general class of Baskakov–Szász–Stancu operators preserving exponential functions. Readers can refer to the articles [[13], [14],[15],[16]] for more such interesting papers related to this approach.

Instigated by the above-mentioned researches, we propose to construct a modification of the operators (1.1) which reproduce exponential functions. We begin with the following form of the operators (1.1), for functions $f \in C(\mathbb{R}^+)$ where $\mathbb{R}^+ = (0,\infty)$, we consider

$$\mathcal{B}_n(f;x) = \int_0^\infty l_n(x,t)f(t)dt, \quad x \in \mathbb{R}^+ \tag{1.2}$$

where

$$l_n(x,t) = \sqrt{\frac{n}{2\pi}} e^{n/\sigma_n(x)} t^{-3/2} \exp\left(-\frac{nt}{2(\sigma_n(x))^2} - \frac{n}{2t}\right)$$

Using calculation analogous to that given in [6], we can evaluate $\mathcal{B}_n(e^{at};x)$ as:

$$\mathcal{B}_n\left(e^{at};x\right) = \exp\left(\frac{n}{\sigma_n(x)}\left(1 - \sqrt{\frac{n - 2a(\sigma_n(x))^2}{n}}\right)\right), \tag{1.3}$$

which is the moment generating function of our proposed operators (1.2). This is used to find the moments and central moments throughout this paper. Now based on the above form of the operators, we divide our paper into three major sections. The first section defines the form of operators (1.2) that preserves constants and exponential function e^{-x}. We prove our main results that involve global approximation, Voronovskaya type asymptotic result and its quantitative version in this section. The second section deals with the form of proposed operators that preserve the exponential function $e^{Ax}, A \in \mathbb{R}$ and prove an improved Voronovskaya theorem for functions with exponential growth. Finally in the last section, we provide some graphical representations in support of our results using mathematica software and conclude that our modified operators along with preserving exponential functions, provide faster rate of convergence.

2. Preservation for e^{-x}

We begin with our proposed operators (1.2). Assuming that they preserve the exponential function e^{-x}, we can write $\mathcal{B}_n(e^{-t};x) = e^{-x}$ and therefore making use of Eqn. (1.3), we get

$$e^{-x} = \exp\left(\frac{n}{\sigma_n(x)}\left(1 - \sqrt{1 + \frac{2(\sigma_n(x))^2}{n}}\right)\right).$$

Comparing exponents on either sides of the above equation and with easy manipulations, we obtain

$$\sigma_n(x) = \frac{2nx}{2n - x^2}.$$

Thus our proposed operators can be rewritten in the following form:

$$\mathcal{B}_n(f;x) = \int_0^\infty l_n(x,t) f(t)dt, \quad x \in \mathbb{R}^+ \tag{2.1}$$

where

$$l_n(x,t) = \left(\frac{n}{2\pi}\right)^{1/2} e^{(n/x - x/2)} t^{-3/2} exp\left(-\frac{\left(2n - x^2\right)^2 t}{8nx^2} - \frac{n}{2t}\right).$$

Lemma 2.1. *For all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, we have*

$$\mathcal{B}_n(e^{\gamma t};x) = \exp\left(\frac{2n - x^2}{2x}\left(1 - \sqrt{1 - \frac{8nx^2\gamma}{\left(2n - x^2\right)^2}}\right)\right).$$

which is also the moment generating function of the operators (2.1).

Lemma 2.2. *For the operators (2.1), if $e_v(t) = t^v$, $v = 0, 1, 2, ..,$ then the moments are as follows:*

$$\begin{aligned}
&\mathcal{B}_n(e_0;x) = 1;\\
&\mathcal{B}_n(e_1;x) = \sigma_n(x);\\
&\mathcal{B}_n(e_2;x) = \sigma_n^2(x) + \frac{\sigma_n^3(x)}{n};\\
&\mathcal{B}_n(e_3;x) = \sigma_n^3(x) + \frac{3\sigma_n^4(x)}{n} + \frac{3\sigma_n^5(x)}{n^2};\\
&\mathcal{B}_n(e_4;x) = \sigma_n^4(x) + \frac{6\sigma_n^5(x)}{n} + \frac{15\sigma_n^6(x)}{n^2} + \frac{15\sigma_n^7(x)}{n^3}.
\end{aligned}$$

Proof. In view of moment generating function given in Lemma 2.1, the $r^{th}-$ moment of operators (2.1) is given by-

$$\mathcal{B}_n^{[r]}(x)=\left[\frac{\partial^r}{\partial\gamma^r}\left\{\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1-\frac{8nx^2\gamma}{(2n-x^2)^2}}\right)\right)\right\}\right]_{\gamma=0}. \tag{2.2}$$

The expansion of Eqn. (2.2) in terms of γ calculated using Mathematica Software is as follows:

$$\begin{aligned}1&+\frac{2nx\gamma}{2n-x^2}+\frac{2n^2\left(2nx^2-x^4+2x^3\right)\gamma^2}{(2n-x^2)^3}\\&+\frac{4\left(4n^5x^3-4n^4x^5+12n^4x^4+n^3x^7-6n^3x^6+12n^3x^5\right)\gamma^3}{3\left(2n-x^2\right)^5}\\&+\frac{2\begin{pmatrix}8n^7x^4-12n^6x^6+48n^6x^5+6n^5x^8-48n^5x^7\\+120n^5x^6-n^4x^{10}+12n^4x^9-60n^4x^8+120n^4x^7\end{pmatrix}\gamma^4}{3(2n-x^2)^7}+O\left(\gamma^5\right)\end{aligned}$$

Thus the r^{th}-moment of the operators (2.1) can be obtained by evaluating r^{th}-partial differentiation with respect to γ of the above expansion at $\gamma=0$. □

Lemma 2.3. *Let $\eta_{n,m}(x)=\mathcal{B}_n\left((t-x)^m;x\right), m=1,2,$ denote the central moments of operators (2.1), then*

$$\eta_{n,1}(x)=\sigma_n(x)-x,$$
$$\eta_{n,2}(x)=\left(\sigma_n(x)-x\right)^2+\frac{\sigma_n^3(x)}{n}.$$

Proof. Using the property of change of origin of moment generating functions

$$e^{-\gamma x}\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1-\frac{8nx^2\gamma}{(2n-x^2)^2}}\right)\right),$$

Expanding this in terms of γ, we get

$$\begin{aligned}1&-\frac{x^3\gamma}{x^2-2n}+\frac{\left(8n^2x^3+2nx^6-x^8\right)\gamma^2}{2\left(2n-x^2\right)^3}\\&+\frac{\left(48n^3x^6+96n^3x^5+4n^2x^9-24n^2x^8-4nx^{11}+x^{13}\right)\gamma^3}{6\left(2n-x^2\right)^5}\\&+\frac{\begin{pmatrix}384n^5x^6+192n^4x^9+576n^4x^8+1920n^4x^7+8n^3x^{12}\\-192n^3x^{11}-384n^3x^{10}-12n^2x^{14}+48n^2x^{13}+6nx^{16}-x^{18}\end{pmatrix}\gamma^4}{24(2n-x^2)^7}+O\left(\gamma^5\right).\end{aligned}$$

The coefficient of $\gamma^m/m!$ in the above expansion is the $m^{th}-$order central moment of operators (2.1) . □

Remark 2.1. *With simple calculations from Mathematica software, for adequately large n we have:*

(1) $\lim_{n\to\infty} n\eta_{n,1}(x)=\frac{x^3}{2}$,
(2) $\lim_{n\to\infty} n\eta_{n,2}(x)=x^3$,
(3) $\lim_{n\to\infty} n^2\eta_{n,4}(x)=3x^6$,

(4) $\lim\limits_{n\to\infty} n^2\mathcal{B}_n\left(\left(e^{-x}-e^{-t}\right)^4;x\right)=3e^{-4x}x^6$.

In [17], Holhoş defined modulus of continuity for exponential operators as:

$$\varpi\left(f,\delta\right)=\sup_{\left|e^{-x}-e^{-t}\right|\leq\delta}\left|f\left(x\right)-f\left(t\right)\right|,\quad x,t\geq 0.$$

and provided a quantitative result for sequence of linear positive operators on a class of real-valued continuous functions $\c{C}(\mathbb{R}^+)$. These functions $f(x)$ have finite limit at infinity and are endowed with Chebyshev norm.
The defined modulus of continuity ϖ possess the following property:

$$\left|f\left(t\right)-f\left(x\right)\right|\leq\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\varpi\left(f,\delta\right). \tag{2.3}$$

The result by Holhoş [17] is given as:

Theorem 2.1. [17] *If $\mathcal{Q}_n:\c{C}(\mathbb{R}^+)\to\c{C}(\mathbb{R}^+)$ satisfy the following inequality for $v=0,1,2$*

$$\left\|\mathcal{Q}_n\left(e^{-vt}\right)-e^{-vx}\right\|_\infty=\rho_v\left(n\right),$$

then for $f\in\c{C}(\mathbb{R}^+)$, we have

$$\left\|\mathcal{Q}_nf-f\right\|_\infty\leq\rho_0\left(n\right)\left\|f\right\|_\infty+\left(2+\rho_0\left(n\right)\right)\varpi\left(f,\sqrt{\rho_0\left(n\right)+2\rho_1\left(n\right)+\rho_2\left(n\right)}\right).$$

Theorem 2.2. *The sequence of modified exponential operators $\mathcal{B}_n:\c{C}(\mathbb{R}^+)\to\c{C}(\mathbb{R}^+)$ satisfy the following inequality for $f\in\c{C}(\mathbb{R}^+)$*

$$\left\|\mathcal{B}_n\left(f;x\right)-f(x)\right\|_\infty\leqslant 2\varpi\left(f,\sqrt{\rho_2\left(n\right)}\right),$$

where $\rho_2\left(n\right)$ tends to zero for adequately large n.

Proof. Since the operators preserve the constant as well as exponential function e^{-x}, so by Theorem 2.1, $\rho_0\left(n\right)=0$ and $\rho_1\left(n\right)=0$. We only need to evaluate $\rho_2\left(n\right)$. Next from Lemma 2.1, we have

$$\mathcal{B}_n\left(e^{-2t};x\right)=\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1+\frac{16nx^2}{\left(2n-x^2\right)^2}}\right)\right).$$

Consider a sequence of functions

$$f_n\left(x\right)=\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1+\frac{16nx^2}{\left(2n-x^2\right)^2}}\right)\right)-e^{-2x}.$$

As $f_n\left(x\right)$ vanishes at end points of $\mathbb{R}^+$, therefore there exists a point $\vartheta_n\in\mathbb{R}^+$ such that

$$\left\|f_n\right\|_\infty=f_n\left(\vartheta_n\right).$$

Also the derivative of the sequence of functions vanishes at ϑ_n i.e. ${f'}_n\left(\vartheta_n\right)=0$. Making use of Mathematica software, this gives

$$\frac{2n+\vartheta_n^2}{2\vartheta_n^2}\left(\frac{-1}{\sqrt{1+\frac{16n\vartheta_n^2}{\left(2n-\vartheta_n^2\right)^2}}}+1\right)\exp\left(\frac{2n-\vartheta_n^2}{2\vartheta_n}\left(1-\sqrt{1+\frac{16n\vartheta_n^2}{\left(2n-\vartheta_n^2\right)^2}}\right)\right)=2e^{-2\vartheta_n}.$$

Therefore, we have

$$\|f_n\|_\infty = \exp\left(\frac{2n-\vartheta_n^2}{2\vartheta_n}\left(1-\sqrt{1+\frac{16n\vartheta_n^2}{\left(2n-\vartheta_n^2\right)^2}}\right)\right)$$
$$\times\left(2-\frac{2n+\vartheta_n^2}{2\vartheta_n^2}\left(\frac{-1}{\sqrt{1+\frac{16n\vartheta_n^2}{(2n-\vartheta_n^2)^2}}}+1\right)\right)=\rho_2(n)\to 0,$$

as $n\to\infty$. Thus in view of Theorem 2.1, we get the required result. □

Let $C_k(\mathbb{R}^+)$ denote the space of all real valued continuous and bounded functions equipped with the Chebyshev norm and let us consider the following K-functional:

$$K_2\left(f,\delta\right)=\inf_{g\in C_\kappa^2(\mathbb{R}^+)}\left\{\|f-g\|+\delta\left\|g''\right\|,\delta>0\right\},$$

where $C_\kappa^2\left(\mathbb{R}^+\right)=\left\{g\in C_\kappa\left(\mathbb{R}^+\right):g',g''\in C_\kappa\left(\mathbb{R}^+\right)\right\}.$

Theorem 2.3. *Let $f\in C_\kappa(\mathbb{R}^+)$. We define auxiliary operators*

$$\hat{\mathcal{T}}_n\left(f;x\right)=\mathcal{B}_n(f;x)-f\left(\sigma_n\left(x\right)\right)+f(x), \tag{2.4}$$

then, there exists a constant $R>0$ such that

$$\left|\mathcal{B}_n(f;x)-f(x)\right|\leqslant R\omega_2(f,\sqrt{\delta})+\omega(f,\sigma_n\left(x\right)-x),$$

where

$$\delta=\eta_{n,2}\left(x\right)+\left(\sigma_n\left(x\right)-x\right)^2.$$

Proof. Let $g\in C_\kappa^2(\mathbb{R}^+)$ and $x,t\in\mathbb{R}^+$, then by application of Taylor's expansion, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Using Eqn. 2.4 and the fact that $\hat{\mathcal{T}}_n\left((t-x);x\right)=0$, we have

$$\begin{aligned}\left|\hat{\mathcal{T}}_n(g;x)-g(x)\right| &= \left|\hat{\mathcal{T}}_n\left(\int_x^t(t-u)g''(u)du;x\right)\right|\\ &\leqslant \left|\mathcal{B}_n\left(\int_x^t(t-u)g''(u)du,x\right)\right|+\left|\int_x^{\sigma_n(x)}\left(\sigma_n\left(x\right)-u\right)g''(u)du\right|\\ &\leqslant \left(\eta_{n,2}\left(x\right)+\left(\sigma_n\left(x\right)-x\right)^2\right)\left\|g''\right\|. \end{aligned}\tag{2.5}$$

Also, we have

$$\left|\mathcal{B}_n\left(f;x\right)\right|\leqslant\|f\|. \tag{2.6}$$

Combining equations (2.4), (2.5) and (2.6), we get

$$\begin{aligned}\left|\mathcal{B}_n\left(f;x\right)-f(x)\right| &\leqslant \left|\hat{\mathcal{T}}_n(f-g);x)-(f-g)(x)\right|+\left|\hat{\mathcal{T}}_n(g;x)-g(x)\right|+\left|f(x)-f\left(\sigma_n\left(x\right)\right)\right|\\ &\leqslant 4\left\|f-g\right\|+\left(\eta_{n,2}\left(x\right)+\left(\sigma_n\left(x\right)-x\right)^2\right)\left\|g''\right\|+\omega\left(f,\sigma_n\left(x\right)-x\right).\end{aligned}$$

Taking infimum over all $g\in C_\kappa^2(\mathbb{R}^+)$ and using the relation given in [18], $K_2\left(f,\delta\right)\leqslant C\omega_2\left(f,\sqrt{\delta}\right)$, $\delta>0$, we get the desired result. □

Theorem 2.4. *Let $f \in C_k(\mathbb{R}^+)$ with continuous first and second derivative exist. Then for $x \in \mathbb{R}^+$, the following inequality holds:*

$$\begin{aligned}&\left| n\left[\mathcal{B}_n\left(f;x\right)-f\left(x\right)\right]-\frac{x^3}{2}\left(f'\left(x\right)+f''\left(x\right)\right)\right|\\&\leq\left|u_n\left(x\right)\right|\left|f'\left(x\right)\right|+\left|v_n\left(x\right)\right|\left|f''\left(x\right)\right|+\frac{x^3}{2}\varpi\left(f'',\frac{1}{\sqrt{n}}\right)\left(1+3x^3e^{-2x}\right).\end{aligned}$$

where $u_n\left(x\right)=n\eta_{n,1}\left(x\right)-\frac{x^3}{2}$ *and* $v_n\left(x\right)=\frac{1}{2}\left(n\eta_{n,2}\left(x\right)-x^3\right)n\eta_{n,2}\left(x\right)-x^3$.

Proof. By the Taylor's expansion, we have

$$f\left(t\right)=\sum_{i=0}^{2}\left(t-x\right)^i\frac{f^{(i)}\left(x\right)}{i!}+\Theta\left(t,x\right)\left(t-x\right)^2, \tag{2.7}$$

where $\Theta\left(t,x\right)$ is a continuous function given by:

$$\Theta\left(t,x\right)=\frac{f''\left(\Im\right)-f''\left(x\right)}{2},\quad \Im\in(x,t).$$

Applying the operator $\mathcal{B}_n$ to the inequality (2.7), we can write

$$\mathcal{B}_n\left(f;x\right)-\sum_{i=0}^{2}\eta_{n,i}\left(x\right)\frac{f^{(i)}\left(x\right)}{i!}=\mathcal{B}_n\left(\Theta\left(t,x\right)\left(t-x\right)^2;x\right).$$

Therefore using Remark 2.1, we get

$$\begin{aligned}&\left|n\left[\mathcal{B}_n\left(f;x\right)-f\left(x\right)\right]-\frac{x^3}{2}\left(f'\left(x\right)+f''\left(x\right)\right)\right|\\&\leq\left|n\eta_{n,1}\left(x\right)-\frac{x^3}{2}\right|\left|f'\left(x\right)\right|+\frac{1}{2}\left|n\eta_{n,2}\left(x\right)-x^3\right|\left|f''\left(x\right)\right|+\left|n\mathcal{B}_n\left(\Theta\left(t,x\right)\left(t-x\right)^2;x\right)\right|\\&\leq\left|u_n\left(x\right)\right|\left|f'\left(x\right)\right|+\left|v_n\left(x\right)\right|\left|f''\left(x\right)\right|+\left|n\mathcal{B}_n\left(\Theta\left(t,x\right)\left(t-x\right)^2;x\right)\right|,\end{aligned} \tag{2.8}$$

where $u_n\left(x\right)=n\eta_{n,1}\left(x\right)-\frac{x^3}{2}\to 0$ and $v_n\left(x\right)=\frac{1}{2}\left(n\eta_{n,2}\left(x\right)-x^3\right)n\eta_{n,2}\left(x\right)-x^3\to 0$ in accordance with Lemma 2.2, for adequately large n.
Using the Property 2.3 of modulus of continuity defined by Holhoş [17], we get

$$\left|\Theta\left(t,x\right)\right|\leq\frac{1}{2}\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\varpi\left(f'',\delta\right).$$

Hence, after applying Cauchy-Schwarz inequality to the last part of Eqn. (2.8), we get

$$\begin{aligned}n\mathcal{B}_n\left(\left|\Theta\left(t,x\right)\right|\left(t-x\right)^2;x\right)&\leq\frac{n}{2}\varpi\left(f'',\delta\right)\eta_{n,2}\left(x\right)\\&\quad+\frac{n}{2\delta^2}\varpi\left(f'',\delta\right)\sqrt{\mathcal{B}_n\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{\eta_{n,4}\left(x\right)}.\end{aligned}$$

Choosing $\delta=n^{-1/2}$,

$$\begin{aligned}&n\mathcal{B}_n\left(\left|\Theta\left(t,x\right)\right|\left(t-x\right)^2;x\right)\\&\leq\frac{1}{2}\varpi\left(f'',\frac{1}{\sqrt{n}}\right)\left[n\eta_{n,2}\left(x\right)+\sqrt{n^2\mathcal{B}_n\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{n^2\eta_{n,4}\left(x\right)}\right].\end{aligned}$$

In view of Eqn. (2.8) and Remark 2.1, we obtain the desired result. □

Corollary 2.1. *Let $f, f', f'' \in Ç(\mathbb{R}^+)$, then for $x \in \mathbb{R}^+$ we have*

$$\lim_{n\to\infty} n\left[\mathcal{B}_n(f;x) - f(x)\right] = \frac{x^3}{2}\left[f'(x) + f''(x)\right]$$

3. The case of e^{Ax}, $A \in \mathbb{R}$

In this section, we present a more general form of the operators (1.2) that reproduces both constants and exponential functions of the form e^{Ax}, $A \in \mathbb{R}$. We observe that the modified operators possess faster and better rate of convergence as compared to the original operators (1.2) for $A > 0$. To endorse the assertion made, we exhibit some graphical representations with the aid of numerical examples and compare the rate of convergence of both original and the modified operators.

Taking into consideration operators (1.2) again and assuming they reproduce functions of the form e^{Ax}, i.e $\mathcal{B}_n(e^{At};x) = e^{Ax}$, we obtain

$$\sigma_n(x) = \frac{2nx}{2n + Ax^2}.$$

Operators (1.2) therefore now take the following form:

$$\mathcal{B}_n(f;x) = \int_0^\infty l_n(x,t)f(t)dt, \quad x \in \mathbb{R}^+ \tag{3.1}$$

where

$$l_n(x,t) = \left(\frac{n}{2\pi}\right)^{1/2} e^{(n/x + Ax/2)} t^{-3/2} \exp\left(-\frac{\left(2n + Ax^2\right)^2}{8nx^2} - \frac{n}{2t}\right).$$

Lemma 3.1. *For all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, we have*

$$\mathcal{B}_n^A\left(e^{At};x\right) = \exp\left(\frac{2n + Ax^2}{2x}\left(1 - \sqrt{1 - \frac{8nAx^2}{\left(2n + Ax^2\right)^2}}\right)\right),$$

$$\mathcal{B}_n^A\left(te^{At};x\right) = \frac{2nx}{\sqrt{\left(2n + Ax^2\right)^2 - 8nAx^2}} \exp\left(\frac{2n + Ax^2}{2x}\left(1 - \sqrt{1 - \frac{8nAx^2}{\left(2n + Ax^2\right)^2}}\right)\right),$$

$$\mathcal{B}_n^A\left(t^2e^{At};x\right) = \frac{4n^2}{\left(\left(2n + Ax^2\right)^2 - 8nAx^2\right)}\left[\frac{2x^3}{\sqrt{\left(2n + Ax^2\right)^2 - 8nAx^2}} + x^2\right] \times \exp\left(\frac{2n + Ax^2}{2x}\left(1 - \sqrt{1 - \frac{8nAx^2}{\left(2n + Ax^2\right)^2}}\right)\right).$$

Proof. The quantities $\mathcal{B}_n^A\left(te^{At};x\right)$ and $\mathcal{B}_n^A\left(t^2e^{At};x\right)$ are obtained simply by successively partially differentiating $\mathcal{B}_n^A\left(e^{At};x\right)$ with respect to A on both sides. $\square$

Lemma 3.2. *For $b_r(x) = x^r, r \in \mathbb{N} \cup \{0\}$, the operators (3.1) hold the following moments:*

i) $\mathcal{B}_n^A(b_0;x) = 1$;
ii) $\mathcal{B}_n^A(b_1;x) = \frac{2nx}{2n+Ax^2}$;
iii) $\mathcal{B}_n^A(b_2;x) = \frac{4n^2\left(Ax^4+2nx^2+2x^3\right)}{\left(Ax^2+2n\right)^3}$;
iv) $\mathcal{B}_n^A(b_3;x) = \frac{8n^3\left(A^2x^7+4Anx^5+6Ax^6+4n^2x^3+12nx^4+12x^5\right)}{\left(2n+Ax^2\right)^5}$;

v) $\mathcal{B}_n^A(b_4;x) = \dfrac{16n^4\begin{pmatrix} A^3x^{10}+6A^2nx^8+12A^2x^9+12An^2x^6+48Anx^7 \\ +60Ax^8+8n^3x^4+48n^2x^5+120nx^6+120x^7 \end{pmatrix}}{(2n+Ax^2)^7}$.

Lemma 3.3. *Let $\eta_{n,m}^A(x) = \mathcal{B}_n^A((t-x)^m;x), m=1,2,$ denote the central moments of operators (3.1), then it can be verified:*

(1) $\eta_{n,1}^A(x) = -\frac{Ax^3}{2n+Ax^2}$,

(2) $\eta_{n,2}^A(x) = \frac{(8n^2x^3+2A^2nx^6+A^3x^8)}{(2n+Ax^2)^3}$.

In order to prove our next theorem, let us define a space $\mathcal{S}$ of all functions having exponential growth of order A endowed with norm:

$$\|f\|_A = \sup_{x\in\mathbb{R}^+}\left|f(x)e^{-Ax}\right| < \infty.$$

Let for some $0\le\alpha<1$, $Lip(\alpha,A)$ be the space containing all those functions f which satisfy $\omega^*(f,\delta,A)\le M\delta^\alpha$, where ω^* is the first order modulus of continuity defined in [19] as:

$$\omega^*(f,\delta,A) \le \sup_{h<\delta, x\in\mathbb{R}^+}|f(x)-f(x+h)|e^{-Ax}.$$

and for every positive number $h>0$ and $k\in\mathbb{N}$ has the following property:

$$\omega^*(f,kh,A)\le k.e^{A(k-1)h}.\omega_1(f,h,A) \tag{3.2}$$

Theorem 3.1. *Let $\mathcal{B}_n^A : S\to Ç(\mathbb{R}^+)$. If $f\in C_\kappa^2(\mathbb{R}^+)\cap\mathcal{S}$ and $f''\in Lip(\alpha,A)$, then for fixed $x\in\mathbb{R}^+$ and $n>2Ax$, we have*

$$\left|\mathcal{B}_n^A(f;x)-f(x)-\eta_{n,1}^A(x)f'(x)-\eta_{n,2}^A(x)\frac{f''(x)}{2}\right|$$
$$\le \frac{1}{2}\omega^*\left(f'',\sqrt{\frac{\eta_{n,4}^A(x)}{\eta_{n,2}^A(x)}},A\right)\left[2e^{2Ax}+M(A,x)+\sqrt{M(2A,x)}\right]\eta_{n,2}^A(x),$$

where $M(A,x) = \frac{(2+Ax^2)^2}{(2-Ax^2)^3}e^{2Ax}$ is a constant independent of n but dependent on A and x.

Proof. By Taylor's Expansion, we have

$$f(t) = f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+\Theta_2(t,x), \tag{3.3}$$

where

$$\Theta_2(t,x) = \frac{f''(\tau)-f''(x)}{2}(t-x)^2.$$

such that τ lies between x and t and $\Theta_2(t,x)$ is a continuous function which vanishes as t approaches x.

Applying the operator $\mathcal{B}_n^A$ on Eqn. (3.3) and in view of Lemma 3.3, we have

$$\left|\mathcal{B}_n^A(f;x)-f(x)-\eta_{n,1}^A(x)f'(x)-\eta_{n,2}^A(x)f''(x)\right| \le \mathcal{B}_n^A(|\Theta_2(t,x)|;x). \tag{3.4}$$

Using the Property 3.2 of exponential modulus of continuity, with simple manipulations we have the relation

$$\mathcal{B}_n^A(|\Theta_2(t,x)|;x) \le \frac{\omega^*(f'',h,A)}{2}\left[\mathcal{B}_n^A\left(\left(e^{2Ax}+e^{At}\right).\left(|t-x|^2+\frac{|t-x|^3}{h}\right);x\right)\right]. \tag{3.5}$$

Taking x fixed and $n > 2Ax$, we have

$$\begin{aligned}
\mathcal{B}_n^A\left((t-x)^2 e^{At};x\right) &= \left[\left(\frac{2x^3}{\sqrt{(2n+Ax^2)^2-8nAx^2}}+x^2\right)\frac{4n^2}{(2n+Ax^2)^2-8nAx^2}\right.\\
&\quad\left.-\frac{4x^2n}{\sqrt{(2n+Ax^2)^2-8nAx^2}}+x^2\right]e^{Ax}\\
&\le \frac{1}{\left(1-\frac{8nAx^2}{(2n+Ax^2)^2}\right)^{3/2}}\left[\frac{8n^2x^3}{(2n+Ax^2)^3}+x^2\left(\sqrt{1-\frac{8nAx^2}{(2n+Ax^2)^2}}\right.\right.\\
&\quad\left.\left.-4n\left(1-\frac{8nAx^2}{(2n+Ax^2)^2}\right)+1\right)\right]e^{Ax}\\
&\le \frac{1}{\left(1-\frac{8nAx^2}{(2n+Ax^2)^2}\right)^{3/2}}\left[\frac{8n^2x^3+2nA^2x^6+A^3x^8}{(2n+Ax^2)^3}+x^2\right.\\
&\quad\left.\times\left(1-4n\left(1-\frac{8nAx^2}{(2n+Ax^2)^2}\right)+1\right)\right]e^{Ax}\\
&\le \frac{1}{\left(1-\frac{8nAx^2}{(2n+Ax^2)^2}\right)^{3/2}}\left[\frac{8n^2x^3+2nA^2x^6+A^3x^8}{(2n+Ax^2)^3}+2x^2\right]e^{Ax}\\
&\le \frac{(2+Ax^2)^2}{(2-Ax^2)^3}e^{2Ax}\eta_{n,2}^A(x)\le M(A,x)\,\eta_{n,2}^A(x),
\end{aligned}\tag{3.6}$$

where

$$M(A,x)=\frac{(2+Ax^2)^2}{(2-Ax^2)^3}e^{2Ax}.$$

Moreover using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\mathcal{B}_n^A\left(|t-x|^3 e^{At};x\right) &\le \sqrt{\mathcal{B}_n^A\left((t-x)^2 e^{2At};x\right)}.\sqrt{\eta_{n,4}^A(x)}\\
&\le \sqrt{M(2A,x)\,\eta_{n,2}^A(x)}.\sqrt{\eta_{n,4}^A(x)}.
\end{aligned}\tag{3.7}$$

Combining Eqns.(3.4), (3.6) and (3.7) and substituting $h=\sqrt{\frac{\eta_{n,4}^A(x)}{\eta_{n,2}^A(x)}}$ in Eqn. (3.5), we get

$$\mathcal{B}_n^A\left(|\Theta_2(t,x)|\,;x\right)\le\frac{1}{2}\omega^*\left(f'',\sqrt{\frac{\eta_{n,4}^A(x)}{\eta_{n,2}^A(x)}},A\right)\left[2e^{2Ax}+M(A,x)+\sqrt{M(2A,x)}\right]\eta_{n,2}^A(x),$$

and hence the theorem. □

Remark 3.1. *One can easily observe in the above theorem,*

- *Second central moment $\eta_{n,4}^A(x)$ of the proposed operators (3.1) is smaller for $A > 0$ and $x > \frac{1}{2A}$ as compared to that of original operators (1.1),*

- *For* $A > 0$, *the ratio* $h = \sqrt{\frac{\eta_{n,4}^{A}(x)}{\eta_{n,2}^{A}(x)}}$ *is higher of original operators as compared to that for our modified operators.*
- *In addition, the constant* $M(A, x)$ *which is independent of* n *is also significantly reduced for our modified exponential operators if we take* $A > 0$.

Thus judging on the basis of above mentioned rationales, we can say that Theorem 3.1 is an improved version of [6, Theorem 4] *for* $A > 0$.

Corollary 3.1. *Let* $f, f'' \in \mathcal{S}$ *and* $A > 0$, *then for any* $x \in \mathbb{R}^+$, *we have*

$$\lim_{n\to\infty} n\left[\mathcal{B}_n^A\left(f;x\right) - f\left(x\right)\right] = \frac{x^3}{2}\left[-Af'\left(x\right) + f''\left(x\right)\right].$$

Remark 3.2. *The advantage of Corollary 3.1 over Corollary 2.1 is in the fact that latter is defined for a larger function space* $\mathcal{S}$ *while the former is only for* $\c{C}(\mathbb{R}^+)$.

4. Conclusion

We now conclude that our proposed operators (3.1) is an improved approximation operator which not only preserves constant and exponential functions e^{Ax} and but in fact also provides faster convergence and better approximation for some functions in comparison to the original exponential operators for $A > 0$. Here we have shown properties which are superior to that of original operators and work for a much wider function spaces. To highlight our statements, we exhibit some figures based on numerical examples to show a faster rate of convergence for our modified operators and also its comparison with the original operators (1.1) for arbitrarily chosen values of n and $A > 0$.

5. Graphical Comparisons

Example 5.1. *Let* $f(x) = 5x[sinh(x)]$. *Then we have the following graphical representations where our function* $f(x)$ *is represented in purple color throughout.*

a) *Figure 1 exhibits the comparison between the modified operator* $\mathcal{B}_n^A$ *(Green), and the original operator* $\mathcal{P}_n$ *(Red) for* $n = 10, A = 1$.
b) *Figure 2 exhibits the comparison between the original operator* $\mathcal{P}_n$ *(Cyan) and the modified operator* $\mathcal{B}_n^A$ *(Brown) for* $n = 50, A = 1$.
c) *Figure 3 shows the rate of convergence of the modified operators* $\mathcal{B}_n^A$ *for* $A = 1$ *and* $n = 10$ *(Green),* $n = 25$ *(Orange) and* $n = 50$ *(Cyan), towards the function* $f(x)$. *The graph clearly shows faster rate of convergence.*

After analyzing Figure 1 and Figure 2, it can be easily concluded that operators $\mathcal{B}_n^A$ *provide faster rate of convergence and therefore better approximation as compared to the operators* $\mathcal{P}_n$ *for* $A > 0$.

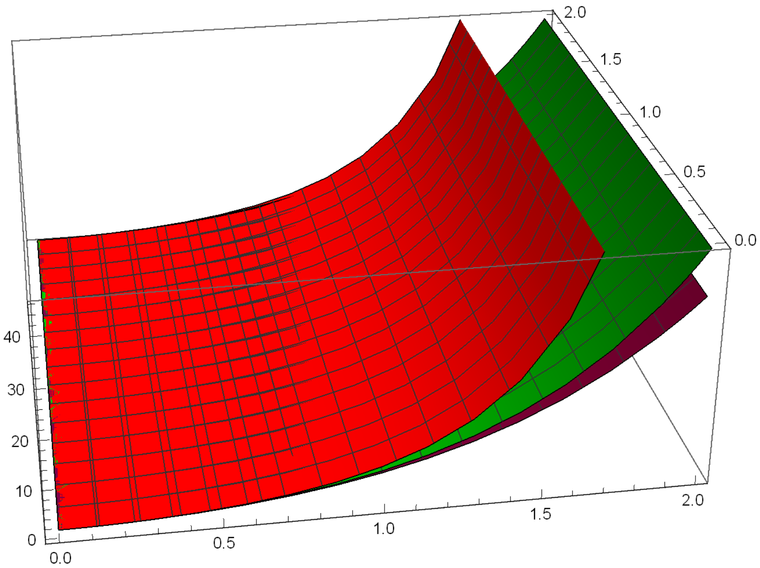

FIGURE 1. Comparison between operators $\mathcal{B}_{10}^{A}$ (Green), $\mathcal{P}_{10}$ (Red) towards function $f(x)$ (Purple) for $n = 10$ and $A = 1$.

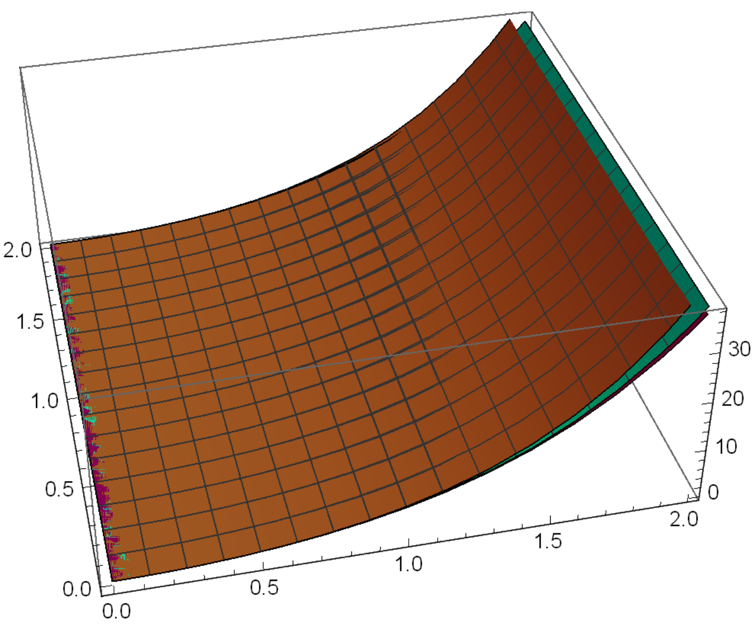

FIGURE 2. Comparison between convergence of operators $\mathcal{B}_{50}^{A}$ (Cyan), $\mathcal{P}_{50}$ (Brown) towards function $f(x)$ (Purple) for $n = 50$ and $A = 1$.

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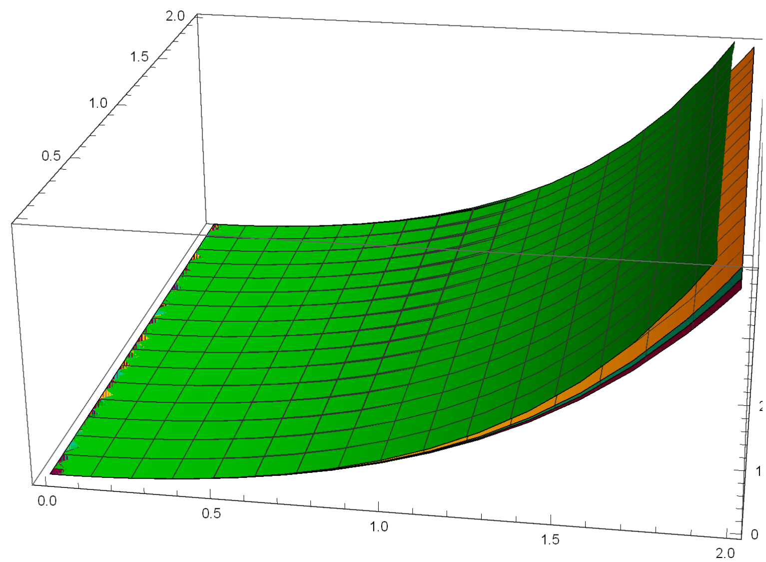

FIGURE 3. Convergence of $\mathcal{B}_n^A(f;x)$ for the function $f(x) = 5x[Sinh(x)]$ (Purple) is illustrated for $n = 10$ (Green), $n = 25$ (Orange), $n = 50$ (Cyan) for $A = 1$.

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