Localization properties for nonlinear equations involving monotone operators

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Abstract

Using monotonicity methods, the Lagrange multiplier rule and some variational arguments, we consider a type of localization results pertaining to the existence of critical points to action functionals on a closed ball. A variant of the Schechter critical point theorem on a ball in Hilbert and Banach spaces is obtained. Applications to nonlinear Dirichlet problem and to partial difference equations are given in the final part of this paper.

Keywords: Schechter critical point theorem; localization; nonlinear operator; quasilinear Dirichlet problem; difference equation.

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1 Introduction

In this paper, we are concerned with a critical point theorem for differentiable functionals acting on a closed ball in an infinite dimensional space. Problems of this type are in relationship with classical energy actions for Dirichlet boundary value problems. Our main purpose is to formulate conditions under which a differentiable functional has a minimum which is a critical point. Moreover, there are provided some sufficient conditions for the existence of the critical point whose variational nature is not known. The starting point in our investigations is the Schechter critical point theorem on a ball in a Hilbert space, see [15, 16]. The proof of this abstract result combines pseudogradients and deformation methods in the sources mentioned. We also refer to [14], where the arguments leading to the proof of the Schechter Theorem rely on the Bishop-Phelps principle. Our aim is to consider such type of functionals which best suit the boundary value problems. A feature of this paper is that the proof exploits the Karush-Kuhn-Tucker necessary optimally conditions and does not require a special type of the Palais-Smale condition. Such an idea allows us to go further and consider the Schechter type theorem in a Banach space, obtained firstly in [9] and investigated further with some applications in [10]. Although again we have some special abstract structure, we note that it corresponds to the one used in examples in [9]. We point out that we do not need to check the very tedious in practice type of Palais-Smale condition on a ball. Moreover, using monotonicity arguments, we obtain a more direct and intuitive formulation, almost similar to the Hilbert space setting and without complicated considerations concerning the usage of a duality mapping corresponding to a certain increasing function.

The paper is organized as follows. We first provide some auxiliary results. The next section is concerned with various versions of the Schechter critical point theorem in both Hilbert and Banach spaces. Following is the section which is concerned with critical points on a ball which need not to be minimizers. We apply our abstract results first to Dirichlet problems driven by the p-Laplacian and next to partial discrete equations which are considered in the form of algebraic equations.

2 Auxiliary results

We mainly follow a recent book on nonlinear analysis [13] which contains a comprehensive survey of all necessary tools. For some earlier background one may see also [6]. In the sequel, E is a real, separable and reflexive Banach space and $\langle \cdot, \cdot \rangle$ is a duality pairing between E^* and E. The operator $A : E \to E^*$ is called:
i) *uniformly monotone* if there exists an increasing function $\rho : [0, +\infty) \to [0, +\infty)$ such that $\rho(0) = 0$ and for all $u, v \in E$

$$\langle A(u) - A(v), u - v \rangle \geq \|u - v\| \rho(\|u - v\|);$$

ii) *strongly monotone* if $\rho(x) = x$ in the above;
iii) $d-$*monotone* if for some increasing function $\rho : [0, +\infty) \to \mathbb{R}$ it holds for

$u, v \in E$

$$\langle A(u) - A(v), u - v\rangle \geq (\rho(\|u\|) - \rho(\|v\|))(\|u\| - \|v\|); \qquad (1)$$

iv) *radially continuous* if for all $u, v \in E$ function

$$s \to \langle A(u + sv), v\rangle$$

is continuous on $[0, 1]$
v) *demicontinuous if* $u_n \to u_0$ *in* E *implies* $A(u_n) \rightharpoonup A(u_0)$ *in* E^*;
vi) *hemicontinuous* if for any $u, v, h \in E$ function

$$s \to \langle A(u + sv), h\rangle$$

is continuous $[0, 1]$;
vii) *coercive* when

$$\lim_{\|v\| \to \infty} \frac{\langle A(v), v\rangle}{\|v\|} \to +\infty;$$

viii) *satisfies condition (S)* if

$$u_n \rightharpoonup u_0 \textit{ in } E \textit{ and } \langle A(u_n) - A(u_0), u_n - u_0\rangle \to 0$$

imply $u_n \to u_0$ *in* E;
ix) *potential*, if there exists a Gâteaux differentiable functional $f : E \to \mathbb{R}$, called *the potential of* A, such that $f' = A$.

For monotone operators the above continuity notions are equivalent. When E is additionally strictly convex, then a $d-$monotone operator is strictly monotone. While a uniformly monotone operator necessarily satisfies condition (S), a $d-$monotone operator does so in case E is uniformly convex. A $d-$monotone operator is coercive when $\rho(\|u\|) \to \infty$ as $\|u\| \to \infty$, while a uniformly monotone operator is obviously coercive. For a radially continuous potential operator $A : E \to E^*$ we have

$$f(v) = f(0) + \int_0^1 \langle A(sv), v\rangle\, ds$$

for $v \in E$. When A *is potential and monotone its potential* is weakly lower semincontinous and A is necessarily demicontinous.

Theorem 1 (The Minty-Browder Theorem) *Assume that $A : E \to E^*$ is radially continuous, strictly monotone and coercive. Then A is invertible and $A^{-1} : E^* \to E$ is strictly monotone, bounded and demicontinuous. If additionally A satisfies property (S) then A^{-1} is continuous.*

We will need also a Schechter Critical Point Theorem in a version given in [14]. Let E be a real Hilbert space. In what follows B_R is always a closed ball centered at 0 with radius R.

Theorem 2 *Assume that $J : B_R \to \mathbb{R}$ is a C^1 functional which is bounded from below. There exists a sequence $(u_n) \subset B_R$ such that*

$$J(u_n) \to \inf_{u \in B_R} J(u)$$

and one of the following conditions holds:
either

$$J'(u_n) \to 0$$

or

$$J'(u_n) - \frac{\left(J'(u_n), u_n\right)}{R^2} u_n \to 0, \ \|u_n\| = R, \ \text{and} \ \left(J'(u_n), u_n\right) \le 0 \text{ for all } n \in \mathbb{N}.$$

If in addition $(J(u), u) \ge -a > -\infty$ for all $u \in \partial B_R$ and if J satisfies the Palais-Smale condition meaning that any of the above sequences admits a convergent sequence and a boundary condition

$$J'(u) + \mu u \ne 0 \text{ for } u \in \partial B_R \text{ and } \mu > 0$$

holds, then there exists $u_0 \in B_R$ with

$$J(u_0) = \inf_{u \in B_R} J(u) \ \text{ and } J'(u_0) = 0.$$

Moreover, we will require the Lagrange Multiplier Rule in the form of Karush-Kuhn-Tucker providing necessary optimality conditions given after [7] in a form which we require. Let $f : E \to \mathbb{R}$ be a given functional and let $g : E \to \mathbb{R}$ be a constraint. Define

$$S = \{x : g(x) \le 0\}.$$

Theorem 3 *Assume that u_0 a minimizer of f over S. Let f and g be Fréchet differentiable at u_0. Then there are nonnegative real numbers μ_0, μ such that*

$$\mu_0 f^{'}(u_0) + \mu g^{'}(u_0) = 0.$$

Remark 4 *In the above the last equality is understood in the sense of space E^*. When the Slater constraint qualitifaction holds, i.e. there is some x_0 that $g(x_0) < 0$, for example when S is a ball, one can assert that $\mu_0 > 0$, namely one can put $\mu_0 = 1$.*

3 On the Schechter Critical Point Theorem

We investigate here what type of assumptions that should be imposed on a functional J so that to have a version of Schecheter Critical Point Theorem, see Theorem 2, without requiring the special Palais-Smale compactness condition. Moreover, we can easily generalize our result to hold in a Banach space without assumptions other than those on the relevant duality mapping. Nevertheless a special structure on the functional is still required. Note that in [9], [10] the Authors develop the Schechter type critical point theorems in a more general setting while the applications mentioned are such which comply with what we suggest. However, in contrast to the sources mentioned, and in contrast to [14], our approach is not suitable for annular domains due to the weak compactness which is required here.

3.1 Critical point theorem on a closed ball in a Hilbert space

Here is the result for the Hilbert space which is necessarily different from the Schechter Theorem as far as the assumptions are concerned. Namely, in the original formulation of the Schechter Theorem it is not required that the functional is weakly l.s.c. In this section we assume that E is a Hilbert space with a scalar product $(\cdot, \cdot)$.

Theorem 5 (Schechter type theorem for minima) *Assume that J is a C^1 functional which is additionally sequentially weakly l.s.c. Moreover, assume that*

$$J^{'}(u) + \mu u \neq 0 \text{ for } \|u\| = R \text{ and } \mu > 0. \tag{2}$$

Then there is $u^* \in B_R$ with

$$J(u^*) = \inf_{B_R} J \text{ and } J^{'}(u^*) = 0.$$

Proof. Since B_R is weakly compact and J is weakly l.s.c. it admits a minimizer u^* over B_R. Observe that minimization of J over a ball B_R is equivalent to the following nonlinear programming problem:

$$\text{minimize } J$$

subject to

$$(u, u) \leq R^2.$$

Note that the Slater constraint qualification condition is satisfied since the ball has nonempty interior. Hence according to Theorem 3 and the Remark which follows it, the argument u_0 of a minimum J over a ball B_R satisfies that there is some $\mu \geq 0$ for which it holds:

$$J^{'}(u^*) + \mu \cdot 2u^* = 0 \text{ and } \mu \left((u^*, u^*) - R^2 \right) = 0.$$

Since $\mu \geq 0$ we can rewrite the above as (for some other μ)

$$J^{'}(u_0) + \mu u^* = 0 \text{ and } \mu \left((u^*, u^*) - R^2 \right) = 0.$$

We have two possibilities, either $\|u^*\| < R$ in which case $\mu = 0$ and thus $J^{'}(u^*) = 0$. Next case is $\|u^*\| = R$ which by (2) provides that μ cannot be positive. This means that $\mu = 0$ again and the assertion follows. ■

Remark 6 *Unfortunately the above mentioned scheme works for maxima but in a more restrictive case of weak upper semicontinuity which is rarely met in variational problems unless the setting is finite dimensional.*

3.2 Critical point theorem on a closed ball in a Banach space

We will provide a generalization of Theorem 5 to the case of functional defined on Banach spaces. One can either argue by introducing a duality mapping relative to a normalization functions or else by some auxiliary functional whose derivative shares the properties with the $p-$Laplacian understood as acting from $W_0^{1.p}$ into its dual for some $p \geq 2$. We assume in this section that E is a uniformly convex Banach space with a strictly convex dual and provide the relevant notions coined to this setting. A continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a normalization function if it is strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$. A duality mapping on E corresponding to a normalization function φ is an operator $A : E \to E^*$ such that for all $u \in E$ and $u^* = A(u)$

$$\|A(u)\|_* = \varphi(\|u\|), \ \langle u^*, u\rangle = \|u^*\|_* \|u\|.$$

Let us define a convex continuous functional $\psi : E \to \mathbb{R}$ by formula

$$\psi(u) = \int_0^{\|u\|} \varphi(t)\, dt.$$

We recall from [3] that $A : E \to E^*$ is:
(i) continuous and uniformly continuous on a unit ball;
(ii) $d-$monotone with respect to φ;
(iii) A is potential and

$$A(u) = \psi^{'}(u) \text{ for each } u \in E.$$

Remark 7 *Note that A^{-1} exists and it is continuous. Indeed, since E is uniformly continuous A satisfies condition (S) and since φ is increasing it is strictly monotone. Since also A is continuous, the assertion follows by Theorem 1.*

Theorem 8 *Assume that $A : E \to E^*$ is a duality mapping corresponding to a normalization function $\varphi(t) = t^{p-1}$ for some $p \geq 2$. Let $T : E \to E^*$ be potential with potential $\mathcal{T}$ and strongly continuous. Consider functional $J : E \to \mathbb{R}$ defined by $J(u) = \frac{1}{p}\|u\|^p + \mathcal{T}(u)$. Let condition*

$$J^{'}(u) + \mu A(u) \neq 0 \text{ for } \|u\| = R \text{ and } \mu > 0 \tag{3}$$

be satisfied. Then there is $u^* \in B_R$ with

$$J(u^*) = \inf_{B_R} J \text{ and } J^{'}(u^*) = 0.$$

Proof. Since J is weakly l.s.c. by the Weierstrass Theorem it admits a minimizer u^* over B_R which satisfies the assertion of the Karush-Kuhn-Tucker

Theorem under Slater constraint qualification, i.e. there is some $\mu \geq 0$ such that

$$J^{'}(u^{*}) + \mu A(u^{*}) = 0 \text{ and } \|u^{*}\|^{p} - R^{p} = 0.$$

If we suppose that $\mu > 0$ we have contradiction with (3). ■

We note that the improvement over results from [14], [9], [10] is that we do not need to check the Palais-Smale type condition while obtaining the critical point, and next the special structure on the functional which we imposed is satisfied for all examples from the sources mentioned.

3.3 A variant of a critical point theorem on a ball

Assumption (3) is a demanding one. The meaning of it is described in [14] in Section 4.1. so there is no need to recall the discussion around this assumption. Now we observe the following result which provides sufficient condition for relation (3) to hold. In this section we let E be uniformly convex Banach space with a strictly convex dual and that:

A $A : E \to E^{*}$ is a duality mapping corresponding to a normalization function $\varphi(t) = t^{p-1}$ for some $p \geq 2$.

T $T : E \to E^{*}$ be is strongly continuous and potential with potential $\mathcal{T}$.

Theorem 9 *Assume conditions* ***A*** *and* ***T****. Consider functional $J : E \to \mathbb{R}$ defined by $J(u) = \frac{1}{p}\|u\|^{p} + \mathcal{T}(u)$. Assume that for any $u \in B_R$ relation*

$$A(v) + T(u) = 0$$

implies that $v \in B_R$. Then there is $u^{*} \in B_R$ with

$$J(u^{*}) = \inf_{B_R} J \text{ and } J^{'}(u^{*}) = 0.$$

Proof. As in the proof of Theorem 8 we see that there is $u^{*} \in B_R$ with $J(u^{*}) = \inf_{B_R} J$ and such that (after writing the derivative of J explicitly)

$$(1+\mu) A(u^{*}) + T(u^{*}) = 0 \text{ for } \mu(\|u^{*}\|^{p} - R^{p}) = 0 \text{ and } \mu \geq 0. \tag{4}$$

Suppose that $\mu > 0$ which happens only when $\|u^{*}\|^{p} - R^{p} = 0$. By Remark 7 it follows that for $u^{*} \in B_R$ there is exactly one v^{*} such that $A(v^{*}) + T(u^{*}) = 0$. This means from (4) that

$$(1+\mu) A(u^{*}) = A(v^{*})$$

and further after calculations of norms

$$\|v^*\|^{p-1} = (1+\mu)\|u^*\|^{p-1} = (1+\mu) R^{p-1} > R^{p-1}.$$

This means that $v^* \notin B_R$ which is impossible. Therefore $\mu = 0$ and the assertion follows. ■

Remark 10 *Observe that assumption that for any $u \in B_R$ relation*

$$A(v) + T(u) = 0$$

implies that $v \in B_R$ provides a sufficient condition for assumption (3) to hold, i.e. for the following condition to be satisfied:

$$(1+\mu) A(u) + T(u) \neq 0 \text{ for } \mu (\|u\|^p - R^p) = 0 \text{ and } \mu \geq 0.$$

We can provide some additional version of Theorem 9 in which the most demanding assumption of invariance of the set B_R under operator $A^{-1}T$ is transferred to a type of nonlinear eigenvalue problem.

Theorem 11 *Assume conditions **A** and **T**. Consider functional $J : E \to \mathbb{R}$ defined by*

$$J(u) = \frac{1}{p}\|u\|^p + \lambda \mathcal{T}(u),$$

where $\lambda \in \mathbb{R}$. Let B_R be any fixed closed ball. There exists some $\lambda^* < 0$ such that for all $\lambda \in [\lambda^*, 0)$ there is $u^* \in B_R$ with

$$J(u^*) = \inf_{B_R} J \text{ and } J'(u^*) = 0.$$

Proof. Put

$$m := \sup_{u \in B_R} \|T(u)\|.$$

Number m is finite. Supposing to the contrary we have a sequence $(u_n) \subset B_R$ such that

$$\lim_{n\to\infty} \|T(u_n)\| = +\infty.$$

We can assume (u_n) to be weakly convergent to some u_0. Thus since T is strongly continuous, we reach a contradiction. We define

$$\lambda^* := -\frac{R^{p-1}}{m}$$

and fix a negative $\lambda \geq \lambda^*$. Let $u \in B_R$ be fixed and let v be the unique solution to $A(v) = -\lambda T(u)$ which exists by Remark 7. Now we see that

$$\|v\|^p = \langle A(v), v\rangle = -\lambda \langle T(u), v\rangle \leq -\lambda^* m \|v\| \leq R^{p-1} \|v\|$$

and the assertion holds. ■

We have the following generalization of the above:

Theorem 12 *Assume conditions* ***A*** *and* ***T****. Assume that $B : E \to E^*$ is strictly monotone, continuous and potential with potential $\mathcal{B}$. Consider functional $J : E \to \mathbb{R}$ defined by $J(u) = \mathcal{B}(u) + \mathcal{T}(u)$. Let for any $u \in B_R$ relation*

$$B(v) + T(u) = 0$$

implies that $v \in B_R$. Then there is $u^* \in B_R$ with

$$J(u^*) = \inf_{B_R} J \text{ and } J^{'}(u^*) = 0.$$

Proof. Arguing as in the proof of Theorem 9 we see that there are an element $u^* \in B_R$ and multiplier $\mu > 0$ such that

$$B(u^*) + T(u^*) + \mu A(u^*) = 0 \text{ for } \mu(\|u^*\|^p - R^p) = 0 \text{ and } \mu \geq 0. \tag{5}$$

Again we suppose that $\mu > 0$ and that $v^* \in B_R$ is such that

$$B(v^*) + T(u^*) = 0.$$

Note that $\|u^*\|^p = R^p = \langle A(u^*), u^*\rangle$. We assume that $v^* \neq u^*$ since otherwise there is nothing to be proved. We then have from the first relation in (5) by a direct calculation

$$\begin{aligned} &\langle B(u^*) - B(v^*), u^* - v^*\rangle = -\mu \langle A(u^*), u^* - v^*\rangle = \\ &-\mu R^p + \mu \langle A(u^*), v^*\rangle. \end{aligned} \tag{6}$$

Using the estimation that

$$\langle A(u^*), v^*\rangle \leq \|A(u^*)\|_* \|v\| = \|u^*\|^{p-1} \|v^*\| = R^{p-1} \|v^*\|$$

and the strict monotonicity of B we obtain from (6) that

$$0 < \langle B(u^*) - B(v^*), u^* - v^*\rangle \leq -\mu R^p + \mu R^{p-1} \|v^*\|.$$

Hence $\|v^*\| > R$ which is a contradiction. ■

Theorem 11 also has its more general version.

Theorem 13 *Assume conditions **A** and **T**. Assume that $B : E \to E^*$ is $d-$monotone with respect to an increasing coercive function $\rho : [0, +\infty) \to [0, +\infty)$, continuous and potential with potential $\mathcal{B}$, additionally let $B(0) = 0$. Consider functional $J : E \to \mathbb{R}$ defined by $J(u) = \mathcal{B}(u) + \lambda \mathcal{T}(u)$, where $\lambda \in \mathbb{R}$. Let B_R be any fixed closed ball. There exists some $\lambda^* < 0$ such that for all $\lambda \in [\lambda^*, 0)$ there is $u^* \in B_R$ with*

$$J(u^*) = \inf_{B_R} J \text{ and } J^{'}(u^*) = 0.$$

Proof. Since B is $d-$monotone and $B(0) = 0$ it holds that

$$\langle B(u), u \rangle \geq \rho(\|u\|) \|u\|. \tag{7}$$

Since E is strictly convex it follows that B is strictly monotone and by (7) it follows by assumption on ρ that it is coercive. Put

$$\lambda^* = -\frac{\rho(R)}{m}$$

and fix $\lambda \in [\lambda^*, 0)$. Then by Theorem 1 equation

$$B(v) + \lambda T(u) = 0$$

has exactly one solution $v \in E$ for each $u \in B_R$. Moreover, we see using (7) that

$$\rho(\|v\|) \leq -\lambda m,$$

where we keep notation from Theorem 12. Hence by a direct calculation we see that $\|v\| \leq R$. ■

Remark 14 *In case we take E to be just a separable, reflexive Banach space, it suffice to impose that B is uniformly monotone, continuous and that $B(0) = 0$.*

3.4 Some remarks on a finite dimensional setting

In a finite dimensional case we have a following simple two critical point theorem with possible applications to discrete problems, for example algebraic equations. Let E be a finite dimensional Euclidean space and assume that J is a C^1 functional.

Theorem 15 *Let conditions*

$$J^{'}(u) - \mu_1 u \neq 0,\ J^{'}(u) + \mu_2 u \neq 0 \text{ for } \|u\| = R \text{ and } \mu_1,\ \mu_2 > 0$$

be satisfied. Then there are distinct $u^*, v^* \in B_R$ *with*

$$J(u^*) = \sup_{B_R} J \text{ and } J^{'}(u^*) = 0$$

and also

$$J(v^*) = \inf_{B_R} J \text{ and } J^{'}(v^*) = 0.$$

Note that if u^* and v^* as above coincide, then J is constant on B_R, so the assertion holds anyway. Moreover, combining coercivity or anti-coercivity with the above mentioned critical point theorems we obtain the following two critical point theorem if we recall that a coercive C^1 functional on a finite dimensional space necessarily has a global minimizer, while an anti-coercive C^1 functional has a global maximizer.

Theorem 16 *i) Assume that* J *is coercive and that condition*

$$J^{'}(u) - \mu u \neq 0, \text{ for } \|u\| = R \text{ and } \mu > 0$$

is satisfied. Then there are distinct $u^* \in B_R, v^* \in E$ *with*

$$J(u^*) = \sup_{B_R} J,\ J^{'}(u^*) = 0$$

and

$$J(v^*) = \inf_{E} J\ ,\ J^{'}(v^*) = 0.$$

ii) Assume that J *is anti-coercive and that condition*

$$J^{'}(u) + \mu u \neq 0, \text{ for } \|u\| = R \text{ and } \mu > 0$$

is satisfied. Then there are distinct $u^* \in E, v^* \in B_R$ *with*

$$J(u^*) = \sup_{E} J,\ J^{'}(u^*) = 0$$

and

$$J(v^*) = \inf_{B_R} J\ ,\ J^{'}(v^*) = 0.$$

4 On some other localization result

Now we provide some comments to the abstract existence result from [4], see Theorem 1. Instead to what is done in [4], we assume that equation

$$Au = f$$

is uniquely solvable. This allows us to provide a very simple and direct proof of the existence tool and next to investigate the case of potential operators involved in the equation. The results here are somewhat counterparts of those from 3.3 but work for not necessarily potential problems and do not require the operator to be continuous. On the other hand we require condition (S) to be satisfied. Moreover, the main existence tool used is different which is now the *Schauder Theorem* and which was previously the *Weierstrass Theorem* and at least in the abstract formulation we do need to consider equation on a ball. We assume if not said otherwise that E is a strictly convex reflexive Banach space. The assumptions are:

A1 $A : E \to E^*$ is $d-$monotone with respect to increasing coercive function $\rho : [0, +\infty) \to [0, +\infty)$, radially continuous and satisfies condition (S).

T1 $T : E \to E^*$ is strongly continuous.

Theorem 17 *Assume that condtions **A1** and **T1** are satisfied. Let $M \subset E$ be a nonempty, convex and closed set. We assume additionally that*

$$A(v) = T(u) \;\; \text{and } u \in M$$

imply that $v \in M$. Then equation

$$A(u) = T(u) \tag{8}$$

has a solution in M.

Proof. Since A is $d-$monotone and E is strictly convex, it follows that A is strictly monotone. Since A is hemicontinuous and since by relation $\lim_{x\to\infty} \rho(x) = +\infty$ it is coercive, we see that equation $A(v) = T(u)$ for any fixed $u \in M$ has exactly one solution v by Theorem 1. Consider the mapping $S : M \to M$ defined by the following formula

$$S := A^{-1}T.$$

By Theorem 1 it follows that A^{-1} is strictly monotone, bounded and by the condition (S) it is also continuous. Since T is strongly continuous it follows that S is continuous and $A^{-1}T(M)$ is relatively compact. Using the Schauder Fixed Point Theorem we obtain a fixed point to S and then the assertion readily follows. ■

Remark 18 *About A we can assume that it is uniformly monotone and radially continuous instead of the above mentioned assumptions.*

When M is a ball we obtain what follows:

Theorem 19 *Assume that condtions **A1** and **T1** are satisfied. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ equation*

$$A(u) = \lambda T(u)$$

has a solution in B_R.

Proof. We put
$$\lambda^* := \frac{\rho(r)}{m}$$
and apply Theorem 17 with calculation similar to those applied in Theorem 13. ■

Assuming that both A and T are potential, with potentials $\mathcal{A}$ and $\mathcal{T}$, respectively, we see that equation (8) provides critical points to the Euler action functional $J : E \to \mathbb{R}$ defined by $J(u) = \mathcal{A}(u) - \mathcal{T}(u)$. Recalling that a potential monotone operator is already demicontinuous (and thus radially continuous), we have the following result about the critical point of the functional which need not be C^1.

Theorem 20 *Assume that condtions **A1** and **T1** are satisfied and that operators A and T are potential. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ functional*
$$J(u) = \mathcal{A}(u) - \lambda\mathcal{T}(u)$$
has at least one critical point in B_R.

5 Applications

In this section we provide application of Theorem 13 noting that examples pertaining the usage of other results follow within the same pattern. We provide application for two kinds of problems: the continuous and the discrete one. The former is the classical Dirichlet problem, while the latter concerns the so called algebraic equations.

5.1 Application to partial differential equations

Let $N \geq 3$ be a natural number and let

$$2 \leq p < N,\ p^* = \frac{Np}{N-p},\ \frac{1}{p} + \frac{1}{p'} = 1.$$

$\Omega \subset \mathbb{R}^N$ is a bounded region with locally Lipschitz boundary. Let $q \in (1, p^*)$. From the Rellich-Kondrashov Theorem, see [5], see also [12], we know that the embedding

$$W_0^{1,p}(\Omega) \subset L^q(\Omega)$$

is compact. We may consider the following classical Dirichlet problem driven by the $p-$Laplace operator

$$\begin{aligned} &-div\left(|\nabla u(y)|^{p-2} \nabla u(y)\right) + \lambda f(y, u(y)) = h(y), \\ &u(y)|_{\partial\Omega} = 0, \end{aligned} \tag{9}$$

where

F1 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is* a *Caratheodory function with* $f(y,0) = 0$ *for a.e.* $y \in \Omega$, $h \in L^q(\Omega)$, $h(y) \neq 0$ for *a.e.* $y \in \Omega$;

F2 *there exist constants* $\beta_1, \eta \in (1, p^*)$, $\beta_2 \geq 0$ *such that for all* $v \in \mathbb{R}$ *and a.e.* $y \in \Omega$

$$|f(y,v)| \leq \beta_1 |v|^\eta + \beta_2;$$

Denote by $E = W_0^{1,p}(\Omega)$ the usual Sobolev spaces which is a uniformly convex, separable real Banach space, see for example [3], when endowed with a norm

$$\|u\| := \|\nabla u\|_{L^p(\Omega)} := \sqrt[p]{\int_\Omega |\nabla u(y)|^p \, dy}.$$

The dual to E is the following $E^* = W_0^{1,-p'}(\Omega)$. Function $u \in E$ solves (9) in a weak sense, i.e.

$$\int_\Omega |\nabla u(y)|^{p-2} \nabla u(y) \nabla v(y)\, dy + \lambda \int_\Omega f(y, u(y)) v(y)\, dy = \int_\Omega h(y) v(y)\, dy$$

for all $v \in E$.

The above definition of a weak solution clearly suggests the abstract operator formulation which we should employ. We define operators $A, T : E \to E^*$ by

$$\langle Au, v \rangle = \int_\Omega |\nabla u(y)|^{p-2} \nabla u(y) \nabla v(y)\, dy \tag{10}$$

and

$$\langle Tu, v \rangle = \int_\Omega f(y, u(y)) v(y)\, dy$$

for $u, v \in E$. We also define functional $h^* \in E^*$ by

$$h^*(v) = \int_\Omega h(y) v(y)\, dy.$$

By a direct calculation we see that

Lemma 21 *Then operator A defined by (10) is uniformly monotone with with respect to $\rho(x) = x^{p-1}$, potential and continuous.*

The potential $\mathcal{A} : W_0^{1,p}(\Omega) \to \mathbb{R}$ of A is defined by $\mathcal{A}(u) = \frac{1}{p}\|u\|^p$. Define $F(y, u) := \int_0^u f(y, s) ds$ for a.e. $y \in \Omega$ and $u \in \mathbb{R}$. About operator $T : E \to E^*$ we have the following result:

Lemma 22 *Assume that **F1**, **F2** are satisfied. Operator T is well defined, potential with potential*

$$\mathcal{T}(u) = \int_\Omega F(y, u(y))\, ds$$

and strongly continuous.

Proof. We define Niemytskij operator N_f associated to f which according to the Krasnoselskii Theorem is continuous from $L^p(\Omega)$ to $L^\eta(\Omega)$. Note that any function $w \in L^\eta(\Omega)$ defines a continuous functional w^* on E given by

$$w^*(v) = \int_\Omega w(y) v(y)\, dy.$$

Given a compact embedding of E into $L^{\eta}(\Omega)$ since $\eta \in (1, p^*)$ we see that T is strongly continuous. ■

From the above lemmas it follows that $J : E \to \mathbb{R}$ defined by

$$J(u) = \mathcal{A}(u) + \lambda \mathcal{T}(u) - h^*(u)$$

is a classical Euler action functional corresponding to (9) which now has the following equivalent form

$$A(u) + \lambda T(u) = h^* \tag{11}$$

understood in the sense of space E^*. We have the following:

Proposition 23 *Let $R > 0$ be fixed. Assume **F1**, **F2** are satisfied. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$problem (9) has a non-trivial weak solution $u_0 \in W_0^{1,p}(\Omega)$, $\|u_0\| \leq R$, having the property that*

$$\lim_{n\to\infty} J(u_n) = \inf_{\|u\|\leq R} J(u) = J(u_0) \text{ and } J'(u_0) = 0.$$

Proof. By the above and the remarks proceeding the formulation we see that Theorem 13 applies. ■

For the reference towards general mathematical theory of nonlinear problems described by elliptic partial differential equations we refer to a comprehensive introduction from [12].

5.2 Application to the partial difference equations

Difference equations, considered by variational approaches, attracted some attention as of late. There are very many results in this area concerning both existence and multiplicity of results. We can mention the following works without being exhaustive at any level: [1], [2], [17]. We consider the following system

$$[u(i+1,j)-2u(i,j)+u(i-1,j)]+[u(i,j+1)-2u(i,j)+u(i,j-1)]$$
$$+\lambda f((i,j),u(i,j))=0,$$
$$\text{for all } i\in\{1,..,m\}, j\in\{1,...,n\},$$
$$u(i,0)=u(i,n+1)=0 \text{ for all } i\in\{1,..,m\},$$
$$u(0,j)=u(m+1,j)=0 \text{ for all } j\in\{1,...,n\} \tag{12}$$

which may be viewed as the discrete counterpart of the problem

$$\begin{aligned}&\tfrac{\partial^2 u}{\partial x^2}+\tfrac{\partial^2 u}{\partial y^2}+\lambda f((x,y),u(x,y))=0,\\ &u(x,0)=u(x,n+1)=0, \text{ for all } x\in(0,m+1),\\ &u(0,y)=u(m+1,y)=0 \text{ for all } y\in(0,n+1).\end{aligned} \tag{13}$$

Following some ideas from [8], we write (12) as a nonlinear system which we further investigate. Let

$$A:=\begin{bmatrix} L & -I_m & 0 & 0 & ... & 0 & 0 & 0 & 0\\ -I_m & L & -I_m & 0 & ... & 0 & 0 & 0 & 0\\ 0 & -I_m & L & -I_m & ... & 0 & 0 & 0 & 0\\ 0 & 0 & -I_m & L & ... & 0 & 0 & 0 & 0\\ ... & ... & ... & ... & ... & ... & ... & ... & ...\\ 0 & 0 & 0 & 0 & ... & L & -I_m & 0 & 0\\ 0 & 0 & 0 & 0 & ... & -I_m & L & -I_m & 0\\ 0 & 0 & 0 & 0 & ... & 0 & -I_m & L & -I_m\\ 0 & 0 & 0 & 0 & ... & 0 & 0 & -I_m & L \end{bmatrix}$$

where I_m is identity matrix of order m and L is $m \times m$ matrix defined by

$$L := \begin{bmatrix} 4 & -1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & -1 & 4 \end{bmatrix}.$$

Matrix A is positive definite, see [8]. Thus problem (12) can be rewritten as an algebraic system

$$Au = \lambda f(u), \tag{14}$$

with the following obvious definitions

$$u = (u(1,1), ..., u(m,1); u(1,2), ..., u(m,2); u(1,n), ..., u(m,n))^T,$$

$$f(u) := ((f((1,1), u(1,1)), ..., f((m,1), u(m,1)),$$

$$f((1,2), u(1,2)), ..., f((m,2), u(m,2)),$$

$$f((1,n), u(1,n)), ..., f((m,n), u(m,n))^T.$$

With f being continuous, solutions to (14) correspond in a one to one manner to critical points of a functional

$$J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

defined by

$$J(u) = \frac{1}{2}(u, Au) - \lambda \sum_{i=1}^{m} \sum_{j=1}^{n} F((i,j), u(i,j)),$$

where

$$F((i,j), u(i,j)) := \int_0^{u(i,j)} f((i,j), v)dv \text{ for all } i \in \{1, .., m\},\ j \in \{1, ..., n\}.$$

By $\alpha_1, \alpha_2, ..., \alpha_{mn}$ we denote the eigenvalues of A ordered as

$$0 < \alpha_1 < \alpha_2 < ... \leq \alpha_{mn}. \tag{15}$$

If on $\mathbb{R}^n \times \mathbb{R}^m$ we consider the usual Euclidean norm $\|\cdot\|$ and if we use Theorem 13 we obtain directly, recalling that a positive definite linear operator is strongly monotone, that

Theorem 24 *Fix some $R > 0$. Assume that $f((i,j),\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for all $i \in \{1,..,m\}$, $j \in \{1,...,n\}$. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ problem (12) has at least one solution u^* such that $\|u^*\| \leq R$ which is nontrivial provided for some (i,j) it holds $f((i,j),0) \neq 0$.*

Note that if we impose that the following holds

A and there exist constants $\mu > 2$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $d > 0$

$$F((i,j),x) \geq c_1|x|^{\mu} + c_2$$

for all $i \in \{1,..,m\}, j \in \{1,...,n\}$ and all $|x| \geq d$

then functional J is anticoercive for any $\lambda > 0$ if we recall (15) and perform direct calculations. A continuous anti-coercive functionals has necessarily an argument of a maximum which obeys the Fermat Rule when this functional is differentiable. This observation suggests the following result:

Theorem 25 *Fix some $R > 0$. Assume that $f((i,j),\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for all $i \in \{1,..,m\}$, $j \in \{1,...,n\}$ and that for some (i,j) it holds $f((i,j),0) \neq 0$. There exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ problem (12) has at least one solution u^*such that $\|u^*\| \leq R$ and*

$$J(u^*) = \inf_{\|u\| \leq R} J(u).$$

Moreover, there is another nontrivial solution $v^* \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$J(v^*) = \sup_{v \in \mathbb{R}^n \times \mathbb{R}^m} J(v).$$

Remark 26 *Algebraic equations serve as an example here. One can easily apply the methods developed here to anisotropic discrete problems, see for example paper [11] convering the existence of solutions by other types of variational approaches.*

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