Well-posedness and asymptotic regularity for the generalized MHD-Boussinesq equations*

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Abstract The generalized MHD-Boussinesq equations are studied in this paper. The well-posedness of the strong solutions for the generalized MHD-Boussinesq equations is proved. Asymptotic regularity for the generalized MHD-Boussinesq equations is proved in $H^{\frac{9}{2}} \times H^{\frac{9}{2}} \times H^{\frac{9}{2}}$. The higher regularity for the generalized MHD-Boussinesq equations is proved in $H^a \times H^a \times H^a$ for $a \geq \frac{9}{2}$.

Key words MHD-Boussinesq equations; Well-posedness; Strong solution; Regularity
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1 Introduction

In this paper, we consider the following generalized MHD-Boussinesq equations:

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u + (u\cdot\nabla)u + \nabla p = (b\cdot\nabla)b + \theta e_3 + f(x), \quad x \in \mathbb{T}^3,\ t>0, \\ \partial_t b + \eta(-\Delta)^\alpha b + (u\cdot\nabla)b = (b\cdot\nabla)u + g(x), \\ \partial_t \theta + \kappa(-\Delta)^\alpha \theta + (u\cdot\nabla)\theta = 0, \\ \nabla\cdot u = 0, \qquad \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.1}$$

Here, $\mathbb{T}^3 = [-\pi,\pi]^3$ is endowed with the periodic boundary condition and $\alpha = \frac{5}{4}$. u is the solenoidal velocity field, b is the magnetic filed, θ is the temperature and p is the pressure. $e_3 = (0,0,1)$ is the unit vector in the direction of gravity. ν is the kinematic viscosity, η is the magnetic diffusivity and κ is the thermal diffusivity. For simplicity, we set $\nu = \eta = \kappa = 1$. C is a positive constant which may be different from line to line.

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The MHD-Boussinesq equations are related with the Boussinesq equations of fluid and Maxwell's equations of electromagnetism. The system (1.1) has been investigated in [1,13,14]. Recently, the Prodi-Serrin-type global regularity for the 3D MHD-Boussinesq equations was proved without thermal diffusion in [8]. By using the Fourier localization technique, the well-posedness for the 3D MHD-Boussinesq equations with the temperature-dependent viscosity was proved in [17]. The existence and uniqueness of strong solutions and smooth solutions for the 3D modified MHD-Boussinesq equations without heat diffusion were proved in [12]. Li in [10] has proved the global weak solutions for the 3D inviscid Boussinesq equations with the magnetic field by using the convex integration method.

When the term θ absent, the 3D generalized MHD-Boussinesq equations reduce to the 3D generalized magnetohydrodynamic equations. Some regularity criteria for the 3D magnetohydrodynamic equations were obtained in [2,5,6,15,16]. When the term b absent, the 3D generalized MHD-Boussinesq equations reduce to the 3D generalized Boussinesq equations. Global well-posedness for the 3D generalized Navier-Stokes-Boussinesq equations was proved by using the Fourier localization technique in [7]. When the terms u and θ absent, the 3D generalized MHD-Boussinesq equations reduce to the 3D generalized Navier-Stokes equations. The existence of inertial manifolds for the hyperviscous Navier-Stokes equations for $\alpha \geq \frac{3}{2}$ by using the spatial averaging method in [3]. Meanwhile, Li and Sun in [9] proved the existence of inertial manifolds for the hyperviscous Navier-Stokes equations for $\alpha \geq \frac{5}{4}$ by using the extend slightly the spatial averaging method. The well-posedness of strong solutions for the hyperviscous magneto-micropolar equation was proved in [11].

In order to get the well-posedness and regularity for the generalized MHD-Boussinesq equations, we should overcome the main difficult for the estimations of the nonlinear terms $(u \cdot \nabla)u$, $(u \cdot \nabla)b$, $(b \cdot \nabla)u$, $(b \cdot \nabla)b$ and $(u \cdot \nabla)\theta$. Based on [3, 9], by using the delicate estimates, we improve their results and get the better estimations. Moreover, we will prove the regularity of the system (1.1) in $H^{\frac{9}{2}} \times H^{\frac{9}{2}} \times H^{\frac{9}{2}}$ and $H^a \times H^a \times H^a$.

The paper is divided into sections as follows. In section 2, we give some prepared works and the main results. In section 3, the existence and uniqueness of the strong solutions for the generalized MHD-Boussinesq equations are proved. In section 4, the regularity for the generalized MHD-Boussinesq equations is proved in $H^{\frac{9}{2}} \times H^{\frac{9}{2}} \times H^{\frac{9}{2}}$. In section 5, the higher regularity for the generalized MHD-Boussinesq equations is proved in $H^a \times H^a \times H^a$.

2 Preliminaries

First, we give the following abstract model:

$$\begin{cases} \partial_t u + A^\alpha u + B(u,u) - B(b,b) - \theta e_3 = f, \\ \partial_t b + A^\alpha b + B(u,b) - B(b,u) = g, \\ \partial_t \theta + A^\alpha \theta + B(u,\theta) = 0, \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0, \quad \theta|_{t=0} = \theta_0. \end{cases} \tag{2.1}$$

Here, $||\cdot||$ is the norm of $L^2(\mathbb{T}^3)$, $||\cdot||_{L^p}$ is the norm of $L^p(\mathbb{T}^3)$, $||\cdot||_{H^s}$ is the norm of $H^s(\mathbb{T}^3)$, here $H^s(\mathbb{T}^3) = W^{s,2}(\mathbb{T}^3)$. Let

$$\begin{aligned} H_1 &= \{u \in (L^2(\mathbb{T}^3))^3 : \int_{\mathbb{T}^3} u dx = 0, \nabla \cdot u = 0\}, \\ H_2 &= \{b \in (L^2(\mathbb{T}^3))^3 : \int_{\mathbb{T}^3} b dx = 0, \nabla \cdot b = 0\}, \\ H_3 &= \{\theta \in L^2(\mathbb{T}^3) : \int_{\mathbb{T}^3} \theta dx = 0\}, \end{aligned}$$

here, $(u_0, b_0, \theta_0) \in H_1 \times H_2 \times H_3 = H$. Let $P : (L^2(\mathbb{T}^3))^3 \times (L^2(\mathbb{T}^3))^3 \times L^2(\mathbb{T}^3) \to H := P((L^2(\mathbb{T}^3))^3 \times (L^2(\mathbb{T}^3))^3 \times L^2(\mathbb{T}^3))$ be the Helmholtz-Leray orthogonal projection operator. We have $Av = -P\Delta v = -\Delta v$ for any $v \in D(A)$. The Sobolev space and norm are defined by $H^s = D(A^{\frac{s}{2}}), s \in \mathbb{R}$, and $||\cdot||_{H^s} = ||A^{\frac{s}{2}} \cdot ||$. By virtue of the Parseval equality [4], we have

$$||u||^2_{H^s} = \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} |j|^{2s} |\hat{u}_j|^2, \quad u \in H^s.$$

For any $w_1, w_2 \in H^1$, the bilinear form is defined by

$$B(w_1, w_2) = P((w_1 \cdot \nabla) w_2).$$

Nextly, we give the following main results.

Theorem 2.1. Assume that $f \in H_1, g \in H_2$, $u_0 \in H^{\frac{5}{4}}$, $b_0 \in H^{\frac{5}{4}}$ and $\theta_0 \in H^{\frac{5}{4}}$, the system (2.1) has a unique strong solution such that $u \in L^\infty(0,T; H^{\frac{5}{4}}) \cap L^2(0,T; H^{\frac{5}{2}})$, $b \in L^\infty(0,T; H^{\frac{5}{4}}) \cap L^2(0,T; H^{\frac{5}{2}})$ and $\theta \in L^\infty(0,T; H^{\frac{5}{4}}) \cap L^2(0,T; H^{\frac{5}{2}})$ for $T > 0$.

Theorem 2.2. Assume that $f \in H^2, g \in H^2$, the dynamics of system (2.1) has an absorbing ball in $H^{\frac{9}{2}} \times H^{\frac{9}{2}} \times H^{\frac{9}{2}}$.

Theorem 2.3. Assume that $f \in H^{a-\frac{5}{2}}, g \in H^{a-\frac{5}{2}}$ for $a \geq \frac{9}{2}$, the dynamics of system (2.1) has an absorbing ball in $H^a \times H^a \times H^a$.

3 Well-posedness

In this section, we will prove the existence and uniqueness for the strong solutions of system (2.1). Meanwhile, we will give the proof of Theorem 2.1.

Proof of the Theorem 2.1. Multiplying the third equation for system (2.1) by θ, integrating the result on $\mathbb{T}^3$, then we get

$$\frac{1}{2}\frac{d}{dt}||\theta||^2 + ||\theta||^2_{H^{\frac{5}{4}}} = 0. \tag{3.1}$$

Moreover,

$$||\theta||^2 \leq ||\theta_0||^2. \tag{3.2}$$

Multiplying the first equation for system (2.1) by u, the second equation for system (2.1) by b, respectively. Integrating their results on $\mathbb{T}^3$ and adding up their results, then we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u||^2 + ||b||^2) + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} = (f, u) + (g, b) + (\theta e_3, u) \\
&\leq \frac{1}{4}(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}}) + C(||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}}) + C||u||_{H^{\frac{5}{4}}}||\theta|| \\
&\leq \frac{1}{2}(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}}) + C(1 + ||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}}).
\end{aligned} \tag{3.3}$$

Adding up (3.1) and (3.3), it yields

$$\begin{aligned}
&\frac{d}{dt}(||u||^2 + ||b||^2 + ||\theta||^2) + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} \\
&\leq C(1 + ||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}}).
\end{aligned} \tag{3.4}$$

Then there exists a positive constant γ such that

$$\begin{aligned}
&\frac{d}{dt}(||u||^2 + ||b||^2 + ||\theta||^2) + \gamma(||u||^2 + ||b||^2 + ||\theta||^2) \\
&\leq C(1 + ||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}}).
\end{aligned} \tag{3.5}$$

Then we get for any $t \geq 0$

$$\begin{aligned}
||u||^2 + ||b||^2 + ||\theta||^2 \leq\ & e^{-\gamma t}(||u_0||^2 + ||b_0||^2 + ||\theta_0||^2) \\
&+ C(1 + ||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}})(1 - e^{-\gamma t}),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\int_0^t (||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})ds \leq\ & ||u_0||^2 + ||b_0||^2 + ||\theta_0||^2 \\
&+ Ct(1 + ||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}}).
\end{aligned} \tag{3.7}$$

Multiplying the first equation for system (2.1) by $A^{\frac{5}{4}}u$, the second equation for system (2.1) by $A^{\frac{5}{4}}b$ and the third equation for system (2.1) by $A^{\frac{5}{4}}\theta$, respectively. Integrating their results on $\mathbb{T}^3$ and adding up their results, then we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}}) + ||u||^2_{H^{\frac{5}{2}}} + ||b||^2_{H^{\frac{5}{2}}} + ||\theta||^2_{H^{\frac{5}{2}}} \\
&\leq \left|\int_{\mathbb{T}^3}(u\cdot\nabla)uA^{\frac{5}{4}}udx\right| + \left|\int_{\mathbb{T}^3}(b\cdot\nabla)bA^{\frac{5}{4}}udx\right| + \left|\int_{\mathbb{T}^3}\theta e_3A^{\frac{5}{4}}udx\right| + \left|\int_{\mathbb{T}^3}f(x)A^{\frac{5}{4}}udx\right| \\
&+ \left|\int_{\mathbb{T}^3}(u\cdot\nabla)bA^{\frac{5}{4}}bdx\right| + \left|\int_{\mathbb{T}^3}(b\cdot\nabla)uA^{\frac{5}{4}}bdx\right| + \left|\int_{\mathbb{T}^3}g(x)A^{\frac{5}{4}}bdx\right| + \left|\int_{\mathbb{T}^3}(u\cdot\nabla)\theta A^{\frac{5}{4}}\theta dx\right| \\
&\leq ||u||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{5}{4}}u|| + ||b||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{5}{4}}u|| + ||\theta||||A^{\frac{5}{4}}u|| + ||f||||A^{\frac{5}{4}}u|| \\
&+ ||u||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{5}{4}}b|| + ||b||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{5}{4}}b|| + ||g||||A^{\frac{5}{4}}b|| + ||u||_{L^{12}}||\nabla\theta||_{L^{\frac{12}{5}}}||A^{\frac{5}{4}}\theta|| \\
&\leq C||u||^2_{H^{\frac{5}{4}}}||u||_{H^{\frac{5}{2}}} + C||b||^2_{H^{\frac{5}{4}}}||u||_{H^{\frac{5}{2}}} + C||\theta||||u||_{H^{\frac{5}{2}}} + C||f||||u||_{H^{\frac{5}{2}}} \\
&+ C||u||_{H^{\frac{5}{4}}}||b||_{H^{\frac{5}{4}}}||b||_{H^{\frac{5}{2}}} + C||g||||b||_{H^{\frac{5}{2}}} + C||u||_{H^{\frac{5}{4}}}||\theta||_{H^{\frac{5}{4}}}||\theta||_{H^{\frac{5}{2}}} \\
&\leq \frac{1}{2}(||u||^2_{H^{\frac{5}{2}}} + ||b||^2_{H^{\frac{5}{2}}} + ||\theta||^2_{H^{\frac{5}{2}}}) + C(||\theta||^2 + ||f||^2 + ||g||^2) \\
&+ C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}}).
\end{aligned} \tag{3.8}$$

Then we have

$$\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}}) + ||u||^2_{H^{\frac{5}{2}}} + ||b||^2_{H^{\frac{5}{2}}} + ||\theta||^2_{H^{\frac{5}{2}}} \\
&\leq C(||\theta||^2 + ||f||^2 + ||g||^2) \\
&+ C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}}).
\end{aligned} \tag{3.9}$$

Since $\int_0^t(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})ds \leq C(t)$, applying the Gronwall inequality, we get

$$||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + \int_0^t(||u||^2_{H^{\frac{5}{2}}} + ||b||^2_{H^{\frac{5}{2}}} + ||\theta||^2_{H^{\frac{5}{2}}})ds \leq C(t). \tag{3.10}$$

Applying the Galerkin method and compactness argument, we get the existence of strong solution for the system (2.1). Nextly, we will prove the uniqueness of the system (2.1). Assume (u_1, b_1, θ_1) and (u_2, b_2, θ_2) are two solutions of the system (2.1). Let $\bar{u} = u_1 - u_2$, $\bar{b} = b_1 - b_2$ and $\bar{\theta} = \theta_1 - \theta_2$. Then we get the following equality:

$$\begin{cases}
\partial_t\bar{u} + (-\Delta)^\alpha\bar{u} + (u_1\cdot\nabla)\bar{u} + (\bar{u}\cdot\nabla)u_2 + \nabla(p_1 - p_2) \\
= (b_1\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)b_2 + (\theta_1 - \theta_2)e_3, \\
\partial_t\bar{b} + (-\Delta)^\alpha\bar{b} + (u_1\cdot\nabla)\bar{b} + (\bar{u}\cdot\nabla)b_2 = (b_1\cdot\nabla)\bar{u} + (\bar{b}\cdot\nabla)u_2, \\
\partial_t\bar{\theta} + (-\Delta)^\alpha\bar{\theta} + (u_1\cdot\nabla)\bar{\theta} + (\bar{u}\cdot\nabla)\theta_2 = 0, \\
\nabla\cdot\bar{u} = 0, \quad \nabla\cdot\bar{b} = 0.
\end{cases} \tag{3.11}$$

Here, $\alpha = \frac{5}{4}$. Testing the first equation of system (3.11) by $\bar{u}$, the second equation of system (3.11) by $\bar{b}$ and the third equation of system (3.11) by $\bar{\theta}$, respectively. Summing up their results, we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\bar{u}||^2 + ||\bar{b}||^2 + ||\bar{\theta}||^2) + ||\bar{u}||^2_{H^{\frac{5}{4}}} + ||\bar{b}||^2_{H^{\frac{5}{4}}} + ||\bar{\theta}||^2_{H^{\frac{5}{4}}} \\
&\leq \left|\int_{\mathbb{T}^3}(\bar{u}\cdot\nabla)u_2\bar{u}dx\right| + \left|\int_{\mathbb{T}^3}(\bar{b}\cdot\nabla)b_2\bar{u}dx\right| + \left|\int_{\mathbb{T}^3}(\theta_1-\theta_2)\bar{u}dx\right| \\
&+ \left|\int_{\mathbb{T}^3}(\bar{u}\cdot\nabla)b_2\bar{b}dx\right| + \left|\int_{\mathbb{T}^3}(\bar{b}\cdot\nabla)u_2\bar{b}dx\right| + \left|\int_{\mathbb{T}^3}(\bar{u}\cdot\nabla)\theta_2\bar{\theta}dx\right| \\
&\leq ||\bar{u}||_{L^{12}}||\nabla u_2||_{L^{\frac{12}{5}}}||\bar{u}|| + ||\bar{b}||_{L^{12}}||\nabla b_2||_{L^{\frac{12}{5}}}||\bar{u}|| + ||\bar{\theta}||||\bar{u}|| \\
&+ ||\bar{u}||_{L^{12}}||\nabla b_2||_{L^{\frac{12}{5}}}||\bar{b}|| + ||\bar{b}||_{L^{12}}||\nabla u_2||_{L^{\frac{12}{5}}}||\bar{b}|| + ||\bar{u}||_{L^{12}}||\nabla \theta_2||_{L^{\frac{12}{5}}}||\bar{\theta}|| \\
&\leq C||\bar{u}||_{H^{\frac{5}{4}}}||u_2||_{H^{\frac{5}{4}}}||\bar{u}|| + C||\bar{b}||_{H^{\frac{5}{4}}}||b_2||_{H^{\frac{5}{4}}}||\bar{u}|| + ||\bar{\theta}||||\bar{u}|| \\
&+ C||\bar{u}||_{H^{\frac{5}{4}}}||b_2||_{H^{\frac{5}{4}}}||\bar{b}|| + C||\bar{b}||_{H^{\frac{5}{4}}}||u_2||_{H^{\frac{5}{4}}}||\bar{b}|| + C||\bar{u}||_{H^{\frac{5}{4}}}||\theta_2||_{H^{\frac{5}{4}}}||\bar{\theta}|| \\
&\leq \frac{1}{2}(||\bar{u}||^2_{H^{\frac{5}{4}}} + ||\bar{b}||^2_{H^{\frac{5}{4}}} + ||\bar{\theta}||^2_{H^{\frac{5}{4}}}) + \frac{1}{2}||\bar{\theta}||^2 + \frac{1}{2}||\bar{u}||^2 \\
&+ C(||u_2||^2_{H^{\frac{5}{4}}} + ||b_2||^2_{H^{\frac{5}{4}}} + ||\theta_2||^2_{H^{\frac{5}{4}}})(||\bar{u}||^2 + ||\bar{b}||^2 + ||\bar{\theta}||^2). \qquad (3.12)
\end{aligned}
$$

It yields

$$
\begin{aligned}
&\frac{d}{dt}(||\bar{u}||^2 + ||\bar{b}||^2 + ||\bar{\theta}||^2) + ||\bar{u}||^2_{H^{\frac{5}{4}}} + ||\bar{b}||^2_{H^{\frac{5}{4}}} + ||\bar{\theta}||^2_{H^{\frac{5}{4}}} \\
&\leq C(||\bar{\theta}||^2 + ||\bar{u}||^2) + C(||u_2||^2_{H^{\frac{5}{4}}} + ||b_2||^2_{H^{\frac{5}{4}}} + ||\theta_2||^2_{H^{\frac{5}{4}}})(||\bar{u}||^2 + ||\bar{b}||^2 + ||\bar{\theta}||^2). \qquad (3.13)
\end{aligned}
$$

Applying the Gronwall inequality and (3.7), then we deduce for $t \geq 0$

$$
\begin{aligned}
&||\bar{u}(t)||^2 + ||\bar{b}(t)||^2 + ||\bar{\theta}(t)||^2 \\
&\leq (||\bar{u}(0)||^2 + ||\bar{b}(0)||^2 + ||\bar{\theta}(0)||^2)e^{C\int_0^t(1+||u_2||^2_{H^{\frac{5}{4}}}+||b_2||^2_{H^{\frac{5}{4}}}+||\theta_2||^2_{H^{\frac{5}{4}}})d\tau}. \qquad (3.14)
\end{aligned}
$$

If $\bar{u}(0) = u_1(0) - u_2(0) = 0$, $\bar{b}(0) = b_1(0) - b_2(0) = 0$ and $\bar{\theta}(0) = \theta_1(0) - \theta_2(0) = 0$, then we get $\bar{u}(t) = \bar{b}(t) = \bar{\theta}(t) = 0$. This completes the proof of Theorem 2.1.

4 Asymptotic regularity

In this subsection, we will prove that the system (2.1) has an absorbing ball in $H^{\frac{9}{2}} \times H^{\frac{9}{2}} \times H^{\frac{9}{2}}$. Nextly, we give the following some priori estimates.

Step 1. By (3.6), it is easy to get

$$
\limsup_{t\to+\infty}(||u(t)||^2 + ||b(t)||^2 + ||\theta(t)||^2) \leq C(1 + ||f||^2_{H^{-\frac{5}{4}}} + ||g||^2_{H^{-\frac{5}{4}}}). \qquad (4.1)
$$

There exists a positive constant t_0 such that

$$||u(t)||^2 + ||b(t)||^2 + ||\theta(t)||^2 \leq C, \quad \text{for} \quad \mathrm{t} \geq \mathrm{t}_0. \tag{4.2}$$

By virtue of the (3.4) and (4.2), we have

$$\int_t^{t+1} (||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}}) ds \leq C, \quad t \geq t_0. \tag{4.3}$$

Step 2. Testing the first equation of system (2.1) by Au, the second equation of system (2.1) by Ab and the third equation of system (2.1) by $A\theta$, respectively. Summing up their results, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u||^2_{H^1} + ||b||^2_{H^1} + ||\theta||^2_{H^1}) + ||u||^2_{H^{\frac{9}{4}}} + ||b||^2_{H^{\frac{9}{4}}} + ||\theta||^2_{H^{\frac{9}{4}}} \\
&\leq \left|\int_{\mathbb{T}^3} (u\cdot\nabla)uAudx\right| + \left|\int_{\mathbb{T}^3} (b\cdot\nabla)bAudx\right| + \left|\int_{\mathbb{T}^3} \theta Audx\right| + \left|\int_{\mathbb{T}^3} fAudx\right| \\
&+ \left|\int_{\mathbb{T}^3} (u\cdot\nabla)bAbdx\right| + \left|\int_{\mathbb{T}^3} (b\cdot\nabla)uAbdx\right| + \left|\int_{\mathbb{T}^3} gAbdx\right| + \left|\int_{\mathbb{T}^3} (u\cdot\nabla)\theta A\theta dx\right| \\
&\leq ||u||_{L^{12}}||\nabla u||||Au||_{L^{\frac{12}{5}}} + ||b||_{L^{12}}||\nabla b||||Au||_{L^{\frac{12}{5}}} + ||\theta||||Au|| + ||f||||Au|| + ||g||||Ab|| \\
&+ ||u||_{L^{12}}||\nabla b||||Ab||_{L^{\frac{12}{5}}} + ||b||_{L^{12}}||\nabla u||||Ab||_{L^{\frac{12}{5}}} + ||u||_{L^{12}}||\nabla\theta||||A\theta||_{L^{\frac{12}{5}}} \\
&\leq C||u||_{H^{\frac{5}{4}}}||u||_{H^1}||u||_{H^{\frac{9}{4}}} + C||b||_{H^{\frac{5}{4}}}||b||_{H^1}||u||_{H^{\frac{9}{4}}} + C||u||_{H^{\frac{5}{4}}}||b||_{H^1}||b||_{H^{\frac{9}{4}}} \\
&+ C||b||_{H^{\frac{5}{4}}}||u||_{H^1}||b||_{H^{\frac{9}{4}}} + C||u||_{H^{\frac{5}{4}}}||\theta||_{H^1}||\theta||_{H^{\frac{9}{4}}} + C||f||||u||^{\frac{1}{9}}||u||^{\frac{8}{9}}_{H^{\frac{9}{4}}} \\
&+ C||g||||b||^{\frac{1}{9}}||b||^{\frac{8}{9}}_{H^{\frac{9}{4}}} + C||\theta||||u||^{\frac{1}{9}}||u||^{\frac{8}{9}}_{H^{\frac{9}{4}}} \\
&\leq \frac{1}{2}(||u||^2_{H^{\frac{9}{4}}} + ||b||^2_{H^{\frac{9}{4}}} + ||\theta||^2_{H^{\frac{9}{4}}}) + C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}})(||u||^2_{H^1} + ||b||^2_{H^1} + ||\theta||^2_{H^1}) \\
&+ C||f||^2 + C||g||^2 + C(||u||^2 + ||b||^2 + ||\theta||^2).
\end{aligned} \tag{4.4}$$

Then we get

$$\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^1} + ||b||^2_{H^1} + ||\theta||^2_{H^1}) + ||u||^2_{H^{\frac{9}{4}}} + ||b||^2_{H^{\frac{9}{4}}} + ||\theta||^2_{H^{\frac{9}{4}}} \\
&\leq C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}})(||u||^2_{H^1} + ||b||^2_{H^1} + ||\theta||^2_{H^1}) \\
&+ C(||f||^2 + ||g||^2) + C(||u||^2 + ||b||^2 + ||\theta||^2).
\end{aligned} \tag{4.5}$$

By (4.3) and (4.2), we get

$$\begin{aligned}
&\int_t^{t+1} (||u||^2_{H^1} + ||b||^2_{H^1} + ||\theta||^2_{H^1}) ds \\
&\leq \int_t^{t+1} (||u||^{\frac{8}{5}}_{H^{\frac{5}{4}}}||u||^{\frac{2}{5}} + ||b||^{\frac{8}{5}}_{H^{\frac{5}{4}}}||b||^{\frac{2}{5}} + ||\theta||^{\frac{8}{5}}_{H^{\frac{5}{4}}}||\theta||^{\frac{2}{5}}) ds
\end{aligned}$$

$$\leq C\int_t^{t+1}(||u||^2_{H^{\frac{5}{4}}}+||u||^2+||b||^2_{H^{\frac{5}{4}}}+||b||^2+||\theta||^2_{H^{\frac{5}{4}}}+||\theta||^2)ds\leq C. \tag{4.6}$$

Applying the uniform Gronwall lemma, it yields

$$\begin{aligned}&||u(t+1)||^2_{H^1}+||b(t+1)||^2_{H^1}+||\theta(t+1)||^2_{H^1}\\ &\leq e^{\int_t^{t+1}C(||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}})ds}(\int_t^{t+1}(||u||^2_{H^1}+||b||^2_{H^1}+||\theta||^2_{H^1})ds\\ &+C(||f||^2+||g||^2)+C\int_t^{t+1}(||u||^2+||b||^2+||\theta||^2)ds)\leq C. \end{aligned}\tag{4.7}$$

Moreover, it yields

$$||u(t)||^2_{H^1}+||b(t)||^2_{H^1}+||\theta(t)||^2_{H^1}\leq C,\quad t\geq t_0+1. \tag{4.8}$$

Step 3. By (4.3) and (3.9) and uniform Gronwall lemma, we get

$$||u(t)||^2_{H^{\frac{5}{4}}}+||b(t)||^2_{H^{\frac{5}{4}}}+||\theta(t)||^2_{H^{\frac{5}{4}}}\leq C,\quad t\geq t_0+1, \tag{4.9}$$

and

$$\int_t^{t+1}(||u(s)||^2_{H^{\frac{5}{2}}}+||b(s)||^2_{H^{\frac{5}{2}}}+||\theta(s)||^2_{H^{\frac{5}{2}}})ds\leq C. \tag{4.10}$$

Step 4. Testing the first equation of system (2.1) by A^2u, the second equation of system (2.1) by A^2b and the third equation of system (2.1) by $A^2\theta$, respectively. Summing up their results, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2})+||u||^2_{H^{\frac{13}{4}}}+||b||^2_{H^{\frac{13}{4}}}+||\theta||^2_{H^{\frac{13}{4}}}\\ &\leq\left|\int_{\mathbb{T}^3}(u\cdot\nabla)uA^2udx\right|+\left|\int_{\mathbb{T}^3}(b\cdot\nabla)bA^2udx\right|+\left|\int_{\mathbb{T}^3}\theta A^2udx\right|+\left|\int_{\mathbb{T}^3}fA^2udx\right|\\ &+\left|\int_{\mathbb{T}^3}(u\cdot\nabla)bA^2bdx\right|+\left|\int_{\mathbb{T}^3}(b\cdot\nabla)uA^2bdx\right|+\left|\int_{\mathbb{T}^3}gA^2bdx\right|+\left|\int_{\mathbb{T}^3}(u\cdot\nabla)\theta A^2\theta dx\right|\\ &\leq C||A^{\frac{3}{8}}u||_{L^3}||\nabla u||_{L^6}||A^{\frac{13}{8}}u||+C||u||_{L^{12}}||A^{\frac{7}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u||+C||A^{\frac{3}{8}}b||_{L^3}||\nabla b||_{L^6}||A^{\frac{13}{8}}u||\\ &+C||b||_{L^{12}}||A^{\frac{7}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u||+C||A^{\frac{3}{8}}\theta||||A^{\frac{13}{8}}u||+C||A^{\frac{3}{8}}f||||A^{\frac{13}{8}}u||\\ &+C||A^{\frac{3}{8}}u||_{L^3}||\nabla b||_{L^6}||A^{\frac{13}{8}}b||+C||u||_{L^{12}}||A^{\frac{7}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b||+C||A^{\frac{3}{8}}b||_{L^3}||\nabla u||_{L^6}||A^{\frac{13}{8}}b||\\ &+C||b||_{L^{12}}||A^{\frac{7}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b||+C||A^{\frac{3}{8}}g||||A^{\frac{13}{8}}b||\\ &+C||A^{\frac{3}{8}}u||_{L^3}||\nabla\theta||_{L^6}||A^{\frac{13}{8}}\theta||+C||u||_{L^{12}}||A^{\frac{7}{8}}\theta||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}\theta||\\ &\leq C||u||_{H^{\frac{5}{4}}}||u||_{H^2}||u||_{H^{\frac{13}{4}}}+C||b||_{H^{\frac{5}{4}}}||b||_{H^2}||u||_{H^{\frac{13}{4}}}+C||u||_{H^{\frac{5}{4}}}||b||_{H^2}||b||_{H^{\frac{13}{4}}}\\ &+C||b||_{H^{\frac{5}{4}}}||u||_{H^2}||b||_{H^{\frac{13}{4}}}+C||u||_{H^{\frac{5}{4}}}||\theta||_{H^2}||\theta||_{H^{\frac{13}{4}}}+C||A^{\frac{3}{8}}f||||u||_{H^{\frac{13}{4}}}\end{aligned}$$

$$+C||A^{\frac{3}{8}}g||||b||_{H^{\frac{13}{4}}}+C||\theta||_{H^2}||u||_{H^{\frac{13}{4}}}$$
$$\le\frac{1}{2}(||u||^2_{H^{\frac{13}{4}}}+||b||^2_{H^{\frac{13}{4}}}+||\theta||^2_{H^{\frac{13}{4}}})+C(||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}})(||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2})$$
$$+C||A^{\frac{3}{8}}f||^2+C||A^{\frac{3}{8}}g||^2+C||\theta||^2_{H^2}. \tag{4.11}$$

We get

$$\frac{d}{dt}(||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2})+||u||^2_{H^{\frac{13}{4}}}+||b||^2_{H^{\frac{13}{4}}}+||\theta||^2_{H^{\frac{13}{4}}}$$
$$\le C(||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}})(||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2})$$
$$+C||A^{\frac{3}{8}}f||^2+C||A^{\frac{3}{8}}g||^2+C||\theta||^2_{H^2}. \tag{4.12}$$

By (4.3) and (4.10), we get

$$\int_t^{t+1}(||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2})ds$$
$$\le C\int_t^{t+1}(||u||^{\frac{4}{5}}_{H^{\frac{5}{4}}}||u||^{\frac{6}{5}}_{H^{\frac{5}{2}}}+||b||^{\frac{4}{5}}_{H^{\frac{5}{4}}}||b||^{\frac{6}{5}}_{H^{\frac{5}{2}}}+||\theta||^{\frac{4}{5}}_{H^{\frac{5}{4}}}||\theta||^{\frac{6}{5}}_{H^{\frac{5}{2}}})ds$$
$$\le C\int_t^{t+1}(||u||^2_{H^{\frac{5}{4}}}+||u||^2_{H^{\frac{5}{2}}}+||b||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{2}}}+||\theta||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{2}}})ds\le C. \tag{4.13}$$

By (4.3) and (4.13), applying the uniform Gronwall lemma, it yields

$$||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2}\le C,\quad t\ge t_0+1. \tag{4.14}$$

Step 5. Applying ∂_t to the first equation of system (2.1), testing by u_t. Applying ∂_t to the second equation of system (2.1), testing by b_t. Applying ∂_t to the third equation of system (2.1), testing by θ_t. Summing up their results, then we have

$$\frac{1}{2}\frac{d}{dt}(||u_t||^2+||b_t||^2+||\theta_t||^2)+||u_t||^2_{H^{\frac{5}{4}}}+||b_t||^2_{H^{\frac{5}{4}}}+||\theta_t||^2_{H^{\frac{5}{4}}}$$
$$\le\left|\int_{\mathbb{T}^3}(u_t\nabla u)u_tdx\right|+\left|\int_{\mathbb{T}^3}(b_t\nabla b)u_tdx\right|+\left|\int_{\mathbb{T}^3}\theta_tu_tdx\right|+\left|\int_{\mathbb{T}^3}(u_t\nabla b)b_tdx\right|$$
$$+\left|\int_{\mathbb{T}^3}(b_t\nabla u)b_tdx\right|+\left|\int_{\mathbb{T}^3}(u_t\nabla\theta)\theta_tdx\right|$$
$$\le||u_t||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||u_t||+||b_t||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||u_t||+\frac{1}{2}(||\theta_t||^2+||u_t||^2)$$
$$+||u_t||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||b_t||+||b_t||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||b_t||+||u_t||_{L^{12}}||\nabla\theta||_{L^{\frac{12}{5}}}||\theta_t||$$
$$\le C||u_t||_{H^{\frac{5}{4}}}||u||_{H^{\frac{5}{4}}}||u_t||+C||b_t||_{H^{\frac{5}{4}}}||b||_{H^{\frac{5}{4}}}||u_t||+\frac{1}{2}(||\theta_t||^2+||u_t||^2)$$
$$+C||u_t||_{H^{\frac{5}{4}}}||b||_{H^{\frac{5}{4}}}||b_t||+C||b_t||_{H^{\frac{5}{4}}}||u||_{H^{\frac{5}{4}}}||b_t||+C||u_t||_{H^{\frac{5}{4}}}||\theta||_{H^{\frac{5}{4}}}||\theta_t||$$
$$\le\frac{1}{2}(||u_t||^2_{H^{\frac{5}{4}}}+||b_t||^2_{H^{\frac{5}{4}}}+||\theta_t||^2_{H^{\frac{5}{4}}})+\frac{1}{2}(||\theta_t||^2+||u_t||^2)$$

$$+ C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u_t||^2 + ||b_t||^2 + ||\theta_t||^2). \tag{4.15}$$

Moreover,

$$\begin{aligned}
&\frac{d}{dt}(||u_t||^2 + ||b_t||^2 + ||\theta_t||^2) + ||u_t||^2_{H^{\frac{5}{4}}} + ||b_t||^2_{H^{\frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{5}{4}}} \\
&\le ||\theta_t||^2 + ||u_t||^2 + C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u_t||^2 + ||b_t||^2 + ||\theta_t||^2).
\end{aligned} \tag{4.16}$$

Nextly, we get the following estimates

$$\begin{aligned}
&||u_t|| + ||b_t|| + ||\theta_t|| \\
&\le ||A^{\frac{5}{4}}u|| + ||A^{\frac{5}{4}}b|| + ||A^{\frac{5}{4}}\theta|| + ||B(u,u)|| + ||B(b,b)|| + ||B(u,b)|| + ||B(b,u)|| \\
&+ ||B(u,\theta)|| + ||\theta|| + ||f|| + ||g|| \\
&\le ||u||_{H^{\frac{5}{2}}} + ||b||_{H^{\frac{5}{2}}} + ||\theta||_{H^{\frac{5}{2}}} + ||u||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}} + ||b||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}} + ||u||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}} \\
&+ ||b||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}} + ||u||_{L^{12}}||\nabla \theta||_{L^{\frac{12}{5}}} + ||\theta|| + ||f|| + ||g|| \\
&\le ||u||_{H^{\frac{5}{2}}} + ||b||_{H^{\frac{5}{2}}} + ||\theta||_{H^{\frac{5}{2}}} + ||\theta|| + ||f|| + ||g|| \\
&+ C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||u||_{H^{\frac{5}{4}}}||b||_{H^{\frac{5}{4}}} + ||u||_{H^{\frac{5}{4}}}||\theta||_{H^{\frac{5}{4}}}) \\
&\le ||u||_{H^{\frac{5}{2}}} + ||b||_{H^{\frac{5}{2}}} + ||\theta||_{H^{\frac{5}{2}}} + ||\theta|| + ||f|| + ||g|| \\
&+ C(||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}}).
\end{aligned} \tag{4.17}$$

By (4.9) and (4.10), we get

$$\begin{aligned}
&||u_t||^2_{L^2(t,t+1;H)} + ||b_t||^2_{L^2(t,t+1;H)} + ||\theta_t||^2_{L^2(t,t+1;H)} \\
&\le C\int_t^{t+1}(||u||^2_{H^{\frac{5}{2}}} + ||b||^2_{H^{\frac{5}{2}}} + ||\theta||^2_{H^{\frac{5}{2}}} + ||\theta||^2 + ||f||^2 + ||g||^2 \\
&+ ||u||^4_{H^{\frac{5}{4}}} + ||b||^4_{H^{\frac{5}{4}}} + ||\theta||^4_{H^{\frac{5}{4}}})ds \le C.
\end{aligned} \tag{4.18}$$

For (4.16), applying the uniform Gronwall lemma and (4.3) and (4.9) and (4.18), we get

$$||u_t||^2 + ||b_t||^2 + ||\theta_t||^2 \le C, \quad t \ge t_0 + 1. \tag{4.19}$$

Step 6. Applying ∂_t to the first equation of system (2.1), testing by Au_t. Meanwhile, applying ∂_t to the second equation of system (2.1), testing by Ab_t. Applying ∂_t to the third equation of system (2.1), testing by $A\theta_t$. Summing up their results, then we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_t||^2_{H^1} + ||b_t||^2_{H^1} + ||\theta_t||^2_{H^1}) + ||u_t||^2_{H^{\frac{9}{4}}} + ||b_t||^2_{H^{\frac{9}{4}}} + ||\theta_t||^2_{H^{\frac{9}{4}}} \\
&\le \left|\int_{\mathbb{T}^3}(u_t\nabla u)Au_t dx\right| + \left|\int_{\mathbb{T}^3}(u\nabla u_t)Au_t dx\right| + \left|\int_{\mathbb{T}^3}(b_t\nabla b)Au_t dx\right| + \left|\int_{\mathbb{T}^3}(b\nabla b_t)Au_t dx\right| \\
&+ \left|\int_{\mathbb{T}^3}\theta_t Au_t dx\right| + \left|\int_{\mathbb{T}^3}(u_t\nabla b)Ab_t dx\right| + \left|\int_{\mathbb{T}^3}(u\nabla b_t)Ab_t dx\right| + \left|\int_{\mathbb{T}^3}(b_t\nabla u)Ab_t dx\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{\mathbb{T}^3}(b\nabla u_t)Ab_t dx\right|+\left|\int_{\mathbb{T}^3}(u_t\nabla\theta)A\theta_t dx\right|+\left|\int_{\mathbb{T}^3}(u\nabla\theta_t)A\theta_t dx\right|\\
&\le ||u_t||_{L^6}||\nabla u||_{L^{\frac{12}{5}}}||Au_t||_{L^{\frac{12}{5}}}+||u||_{L^{12}}||\nabla u_t||||Au_t||_{L^{\frac{12}{5}}}+||b_t||_{L^6}||\nabla b||_{L^{\frac{12}{5}}}||Au_t||_{L^{\frac{12}{5}}}\\
&+||b||_{L^{12}}||\nabla b_t||||Au_t||_{L^{\frac{12}{5}}}+||\theta_t||||Au_t||+||u_t||_{L^6}||\nabla b||_{L^{\frac{12}{5}}}||Ab_t||_{L^{\frac{12}{5}}}\\
&+||u||_{L^{12}}||\nabla b_t||||Ab_t||_{L^{\frac{12}{5}}}+||b_t||_{L^6}||\nabla u||_{L^{\frac{12}{5}}}||Ab_t||_{L^{\frac{12}{5}}}+||b||_{L^{12}}||\nabla u_t||||Ab_t||_{L^{\frac{12}{5}}}\\
&+||u_t||_{L^6}||\nabla\theta||_{L^{\frac{12}{5}}}||A\theta_t||_{L^{\frac{12}{5}}}+||u||_{L^{12}}||\nabla\theta_t||||A\theta_t||_{L^{\frac{12}{5}}}\\
&\le C||u_t||_{H^1}||u||_{H^{\frac{5}{4}}}||u_t||_{H^{\frac{9}{4}}}+C||b||_{H^{\frac{5}{4}}}||b_t||_{H^1}||u_t||_{H^{\frac{9}{4}}}+C||\theta_t||^2+C||u_t||^{\frac{2}{9}}||u_t||_{H^{\frac{9}{4}}}^{\frac{16}{9}}\\
&+C||u_t||_{H^1}||b||_{H^{\frac{5}{4}}}||b_t||_{H^{\frac{9}{4}}}+C||u||_{H^{\frac{5}{4}}}||b_t||_{H^1}||b_t||_{H^{\frac{9}{4}}}\\
&+C||u_t||_{H^1}||\theta||_{H^{\frac{5}{4}}}||\theta_t||_{H^{\frac{9}{4}}}+C||u||_{H^{\frac{5}{4}}}||\theta_t||_{H^1}||\theta_t||_{H^{\frac{9}{4}}}\\
&\le \frac{1}{2}(||u_t||_{H^{\frac{9}{4}}}^2+||b_t||_{H^{\frac{9}{4}}}^2+||\theta_t||_{H^{\frac{9}{4}}}^2)+C||\theta_t||^2+C||u_t||^2\\
&+C(||u||_{H^{\frac{5}{4}}}^2+||b||_{H^{\frac{5}{4}}}^2+||\theta||_{H^{\frac{5}{4}}}^2)(||u_t||_{H^1}^2+||b_t||_{H^1}^2+||\theta_t||_{H^1}^2). \qquad (4.20)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{d}{dt}(||u_t||_{H^1}^2+||b_t||_{H^1}^2+||\theta_t||_{H^1}^2)+||u_t||_{H^{\frac{9}{4}}}^2+||b_t||_{H^{\frac{9}{4}}}^2+||\theta_t||_{H^{\frac{9}{4}}}^2\\
&\le C(||\theta_t||^2+||u_t||^2)\\
&+C(||u||_{H^{\frac{5}{4}}}^2+||b||_{H^{\frac{5}{4}}}^2+||\theta||_{H^{\frac{5}{4}}}^2)(||u_t||_{H^1}^2+||b_t||_{H^1}^2+||\theta_t||_{H^1}^2). \qquad (4.21)
\end{aligned}
$$

We integrate (4.16) on $[t,t+1]$ to get

$$
\int_t^{t+1}(||u_t||_{H^{\frac{5}{4}}}^2+||b_t||_{H^{\frac{5}{4}}}^2+||\theta_t||_{H^{\frac{5}{4}}}^2)ds\le C. \qquad (4.22)
$$

Moreover,

$$
\begin{aligned}
&\int_t^{t+1}(||u_t||_{H^1}^2+||b_t||_{H^1}^2+||\theta_t||_{H^1}^2)ds\\
&\le C\int_t^{t+1}(||u_t||_{H^{\frac{5}{4}}}^{\frac{8}{5}}||u_t||^{\frac{2}{5}}+||b_t||_{H^{\frac{5}{4}}}^{\frac{8}{5}}||b_t||^{\frac{2}{5}}+||\theta_t||_{H^{\frac{5}{4}}}^{\frac{8}{5}}||\theta_t||^{\frac{2}{5}})ds\\
&\le C\int_t^{t+1}(||u_t||^2+||u_t||_{H^{\frac{5}{4}}}^2+||b_t||^2+||b_t||_{H^{\frac{5}{4}}}^2+||\theta_t||^2+||\theta_t||_{H^{\frac{5}{4}}}^2)ds\le C. \qquad (4.23)
\end{aligned}
$$

By (4.3), (4.19) and (4.23), applying the uniform Gronwall lemma, we have

$$
||u_t||_{H^1}^2+||b_t||_{H^1}^2+||\theta_t||_{H^1}^2\le C,\quad t\ge t_0+1. \qquad (4.24)
$$

Step 7. Taking ∂_t to the first equation of system (2.1) and the scalar product with A^2u_t. Similarly, taking ∂_t to the second equation of system (2.1) and the scalar product with

$A^2 b_t$. Taking ∂_t to the third equation of system (2.1) and the scalar product with $A^2\theta_t$. Summing up their results, then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_t||_{H^2}^2+||b_t||_{H^2}^2+||\theta_t||_{H^2}^2)+||u_t||_{H^{\frac{13}{4}}}^2+||b_t||_{H^{\frac{13}{4}}}^2+||\theta_t||_{H^{\frac{13}{4}}}^2\\
&\le\left|\int_{\mathbb{T}^3}(u_t\nabla u)A^2u_tdx\right|+\left|\int_{\mathbb{T}^3}(u\nabla u_t)A^2u_tdx\right|+\left|\int_{\mathbb{T}^3}(b_t\nabla b)A^2u_tdx\right|+\left|\int_{\mathbb{T}^3}(b\nabla b_t)A^2u_tdx\right|\\
&+\left|\int_{\mathbb{T}^3}\theta_tA^2u_tdx\right|+\left|\int_{\mathbb{T}^3}(u_t\nabla b)A^2b_tdx\right|+\left|\int_{\mathbb{T}^3}(u\nabla b_t)A^2b_tdx\right|+\left|\int_{\mathbb{T}^3}(b_t\nabla u)A^2b_tdx\right|\\
&+\left|\int_{\mathbb{T}^3}(b\nabla u_t)A^2b_tdx\right|+\left|\int_{\mathbb{T}^3}(u_t\nabla\theta)A^2\theta_tdx\right|+\left|\int_{\mathbb{T}^3}(u\nabla\theta_t)A^2\theta_tdx\right|\\
&\le C||A^{\frac{3}{8}}u_t||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u_t||+C||u_t||_{L^{12}}||A^{\frac{7}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u_t||\\
&+C||A^{\frac{3}{8}}u||_{L^3}||\nabla u_t||_{L^6}||A^{\frac{13}{8}}u_t||+C||u||_{L^{12}}||A^{\frac{7}{8}}u_t||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u_t||\\
&+C||A^{\frac{3}{8}}b_t||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u_t||+C||b_t||_{L^{12}}||A^{\frac{7}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u_t||\\
&+C||A^{\frac{3}{8}}b||_{L^3}||\nabla b_t||_{L^6}||A^{\frac{13}{8}}u_t||+C||b||_{L^{12}}||A^{\frac{7}{8}}b_t||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}u_t||+C||A^{\frac{3}{8}}\theta_t||||A^{\frac{13}{8}}u_t||\\
&+C||A^{\frac{3}{8}}u_t||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b_t||+C||u_t||_{L^{12}}||A^{\frac{7}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b_t||\\
&+C||A^{\frac{3}{8}}u||_{L^3}||\nabla b_t||_{L^6}||A^{\frac{13}{8}}b_t||+C||u||_{L^{12}}||A^{\frac{7}{8}}b_t||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b_t||\\
&+C||A^{\frac{3}{8}}b_t||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b_t||+C||b_t||_{L^{12}}||A^{\frac{7}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b_t||\\
&+C||A^{\frac{3}{8}}b||_{L^3}||\nabla u_t||_{L^6}||A^{\frac{13}{8}}b_t||+C||b||_{L^{12}}||A^{\frac{7}{8}}u_t||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}b_t||\\
&+C||A^{\frac{3}{8}}u_t||_{L^{12}}||\nabla\theta||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}\theta_t||+C||u_t||_{L^{12}}||A^{\frac{7}{8}}\theta||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}\theta_t||\\
&+C||A^{\frac{3}{8}}u||_{L^3}||\nabla\theta_t||_{L^6}||A^{\frac{13}{8}}\theta_t||+C||u||_{L^{12}}||A^{\frac{7}{8}}\theta_t||_{L^{\frac{12}{5}}}||A^{\frac{13}{8}}\theta_t||\\
&\le C||u||_{H^{\frac{5}{4}}}||u_t||_{H^2}||u_t||_{H^{\frac{13}{4}}}+C||u||_{H^2}||u_t||_{H^{\frac{5}{4}}}||u_t||_{H^{\frac{13}{4}}}\\
&+C||b||_{H^{\frac{5}{4}}}||b_t||_{H^2}||u_t||_{H^{\frac{13}{4}}}+C||b||_{H^2}||b_t||_{H^{\frac{5}{4}}}||u_t||_{H^{\frac{13}{4}}}\\
&+C||\theta_t||_{H^{\frac{3}{4}}}||u_t||_{H^{\frac{13}{4}}}+C||b||_{H^{\frac{5}{4}}}||u_t||_{H^2}||b_t||_{H^{\frac{13}{4}}}+C||b||_{H^2}||u_t||_{H^{\frac{5}{4}}}||b_t||_{H^{\frac{13}{4}}}\\
&+C||u||_{H^{\frac{5}{4}}}||b_t||_{H^2}||b_t||_{H^{\frac{13}{4}}}+C||u||_{H^2}||b_t||_{H^{\frac{5}{4}}}||b_t||_{H^{\frac{13}{4}}}\\
&+C||\theta||_{H^{\frac{5}{4}}}||u_t||_{H^2}||\theta_t||_{H^{\frac{13}{4}}}+C||\theta||_{H^2}||u_t||_{H^{\frac{5}{4}}}||\theta_t||_{H^{\frac{13}{4}}}+C||u||_{H^{\frac{5}{4}}}||\theta_t||_{H^2}||\theta_t||_{H^{\frac{13}{4}}}\\
&\le\frac{1}{2}(||u_t||_{H^{\frac{13}{4}}}^2+||b_t||_{H^{\frac{13}{4}}}^2+||\theta_t||_{H^{\frac{13}{4}}}^2)+C(||\theta_t||_{H^2}^2+||\theta_t||^2)\\
&+C(||u||_{H^{\frac{5}{4}}}^2+||b||_{H^{\frac{5}{4}}}^2+||\theta||_{H^{\frac{5}{4}}}^2)(||u_t||_{H^2}^2+||b_t||_{H^2}^2+||\theta_t||_{H^2}^2)\\
&+C(||u||_{H^2}^2+||b||_{H^2}^2+||\theta||_{H^2}^2)(||u_t||_{H^{\frac{5}{4}}}^2+||b_t||_{H^{\frac{5}{4}}}^2)\\
&\le\frac{1}{2}(||u_t||_{H^{\frac{13}{4}}}^2+||b_t||_{H^{\frac{13}{4}}}^2+||\theta_t||_{H^{\frac{13}{4}}}^2)+C(1+||u||_{H^{\frac{5}{4}}}^2+||b||_{H^{\frac{5}{4}}}^2+||\theta||_{H^{\frac{5}{4}}}^2\\
&+||u||_{H^2}^2+||b||_{H^2}^2+||\theta||_{H^2}^2)(||u_t||_{H^2}^2+||b_t||_{H^2}^2+||\theta_t||_{H^2}^2).
\end{aligned}
\tag{4.25}
$$

Then we get

$$
\begin{aligned}
&\frac{d}{dt}(||u_t||^2_{H^2}+||b_t||^2_{H^2}+||\theta_t||^2_{H^2})+||u_t||^2_{H^{\frac{13}{4}}}+||b_t||^2_{H^{\frac{13}{4}}}+||\theta_t||^2_{H^{\frac{13}{4}}}\\
&\leq C(1+||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{4}}}+||u||^2_{H^2}+||b||^2_{H^2}+||\theta||^2_{H^2})\\
&(||u_t||^2_{H^2}+||b_t||^2_{H^2}+||\theta_t||^2_{H^2}).
\end{aligned} \tag{4.26}
$$

We integrate (4.21) on $[t,t+1]$ to get

$$
\int_t^{t+1}(||u_t||^2_{H^{\frac{9}{4}}}+||b_t||^2_{H^{\frac{9}{4}}}+||\theta_t||^2_{H^{\frac{9}{4}}})ds\leq C. \tag{4.27}
$$

Moreover,

$$
\begin{aligned}
&\int_t^{t+1}(||u_t||^2_{H^2}+||b_t||^2_{H^2}+||\theta_t||^2_{H^2})ds\\
&\leq C\int_t^{t+1}(||u_t||^{\frac{16}{9}}_{H^{\frac{9}{4}}}||u_t||^{\frac{2}{9}}+||b_t||^{\frac{16}{9}}_{H^{\frac{9}{4}}}||b_t||^{\frac{2}{9}}+||\theta_t||^{\frac{16}{9}}_{H^{\frac{9}{4}}}||\theta_t||^{\frac{2}{9}})ds\\
&\leq C\int_t^{t+1}(||u_t||^2_{H^{\frac{9}{4}}}+||u_t||^2+||b_t||^2_{H^{\frac{9}{4}}}+||b_t||^2+||\theta_t||^2_{H^{\frac{9}{4}}}+||\theta_t||^2)ds\leq C.
\end{aligned} \tag{4.28}
$$

Applying the uniform Gronwall lemma and (4.3), (4.13), (4.14) and (4.28), we get

$$
||u_t||^2_{H^2}+||b_t||^2_{H^2}+||\theta_t||^2_{H^2}\leq C,\quad t\geq t_0+1. \tag{4.29}
$$

Step 8. Proof of the Theorem 2.2. Finally, we will get that the system (2.1) has an absorbing ball in $H^{\frac{9}{2}}\times H^{\frac{9}{2}}\times H^{\frac{9}{2}}$.

$$
\begin{aligned}
&||u||_{H^{\frac{9}{2}}}+||b||_{H^{\frac{9}{2}}}+||\theta||_{H^{\frac{9}{2}}}=||A^{\frac{9}{4}}u||+||A^{\frac{9}{4}}b||+||A^{\frac{9}{4}}\theta||\\
&\leq ||u_t||_{H^2}+||b_t||_{H^2}+||\theta_t||_{H^2}+||B(u,u)||_{H^2}+||B(b,b)||_{H^2}+||B(u,b)||_{H^2}\\
&+||B(b,u)||_{H^2}+||B(u,\theta)||_{H^2}+||\theta e_3||_{H^2}+||f||_{H^2}+||g||_{H^2}\\
&\leq ||u_t||_{H^2}+||b_t||_{H^2}+||\theta_t||_{H^2}+C||u||_{H^2}||\nabla u||_{H^2}+C||b||_{H^2}||\nabla b||_{H^2}+C||u||_{H^2}||\nabla b||_{H^2}\\
&+C||b||_{H^2}||\nabla u||_{H^2}+C||u||_{H^2}||\nabla\theta||_{H^2}+||\theta||_{H^2}+||f||_{H^2}+||g||_{H^2}\\
&\leq ||u_t||_{H^2}+||b_t||_{H^2}+||\theta_t||_{H^2}+||f||_{H^2}+||g||_{H^2}+C||\theta||_{H^2}\\
&+C(||u||^{\frac{8}{5}}_{H^2}||u||^{\frac{2}{5}}_{H^{\frac{9}{2}}}+||b||^{\frac{8}{5}}_{H^2}||b||^{\frac{2}{5}}_{H^{\frac{9}{2}}}+||b||_{H^2}||u||^{\frac{3}{5}}_{H^2}||u||^{\frac{2}{5}}_{H^{\frac{9}{2}}}+||u||_{H^2}||b||^{\frac{3}{5}}_{H^2}||b||^{\frac{2}{5}}_{H^{\frac{9}{2}}}\\
&+||u||_{H^2}||\theta||^{\frac{3}{5}}_{H^2}||\theta||^{\frac{2}{5}}_{H^{\frac{9}{2}}})\\
&\leq ||u_t||_{H^2}+||b_t||_{H^2}+||\theta_t||_{H^2}+||f||_{H^2}+||g||_{H^2}+C||\theta||_{H^2}\\
&+\frac{1}{2}(||u||_{H^{\frac{9}{2}}}+||b||_{H^{\frac{9}{2}}}+||\theta||_{H^{\frac{9}{2}}})\\
&+C(||u||^{\frac{8}{3}}_{H^2}+||b||^{\frac{8}{3}}_{H^2}+||b||^{\frac{5}{3}}_{H^2}||u||_{H^2}+||u||^{\frac{5}{3}}_{H^2}||b||_{H^2}+||u||^{\frac{5}{3}}_{H^2}||\theta||_{H^2}).
\end{aligned} \tag{4.30}
$$

We get

$$
\begin{aligned}
&||u||_{H^{\frac{9}{2}}}+||b||_{H^{\frac{9}{2}}}+||\theta||_{H^{\frac{9}{2}}}\\
&\le 2(||u_t||_{H^2}+||b_t||_{H^2}+||\theta_t||_{H^2}+||f||_{H^2}+||g||_{H^2})+C||\theta||_{H^2}\\
&+C(||u||_{H^2}^{\frac{8}{3}}+||b||_{H^2}^{\frac{8}{3}}+||b||_{H^2}^{\frac{5}{3}}||u||_{H^2}+||u||_{H^2}^{\frac{5}{3}}||b||_{H^2}+||u||_{H^2}^{\frac{5}{3}}||\theta||_{H^2}).
\end{aligned}
\tag{4.31}
$$

By virtue of (4.14) and (4.29), then we have

$$
||u||_{H^{\frac{9}{2}}}+||b||_{H^{\frac{9}{2}}}+||\theta||_{H^{\frac{9}{2}}}\le C,\quad t\ge t_0+1. \tag{4.32}
$$

5 Higher regularity

In this subsection, we will prove that the system (2.1) has an absorbing ball in $H^a\times H^a\times H^a$. We get the following some priori estimates.

Step 9 Testing the first equation of system (2.1) by $A^{a-\frac{5}{2}}u$, the second equation of system (2.1) by $A^{a-\frac{5}{2}}b$ and the third equation of system (2.1) by $A^{a-\frac{5}{2}}\theta$, respectively. Summing up their results, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u||^2_{H^{a-\frac{5}{2}}}+||b||^2_{H^{a-\frac{5}{2}}}+||\theta||^2_{H^{a-\frac{5}{2}}})+||u||^2_{H^{a-\frac{5}{4}}}+||b||^2_{H^{a-\frac{5}{4}}}+||\theta||^2_{H^{a-\frac{5}{4}}}\\
&\le\left|\int_{\mathbb{T}^3}(u\cdot\nabla)uA^{a-\frac{5}{2}}udx\right|+\left|\int_{\mathbb{T}^3}(b\cdot\nabla)bA^{a-\frac{5}{2}}udx\right|+\left|\int_{\mathbb{T}^3}\theta A^{a-\frac{5}{2}}udx\right|+\left|\int_{\mathbb{T}^3}fA^{a-\frac{5}{2}}udx\right|\\
&+\left|\int_{\mathbb{T}^3}(u\cdot\nabla)bA^{a-\frac{5}{2}}bdx\right|+\left|\int_{\mathbb{T}^3}(b\cdot\nabla)uA^{a-\frac{5}{2}}bdx\right|+\left|\int_{\mathbb{T}^3}gA^{a-\frac{5}{2}}bdx\right|+\left|\int_{\mathbb{T}^3}(u\cdot\nabla)\theta A^{a-\frac{5}{2}}\theta dx\right|\\
&\le C||A^{\frac{a}{2}-\frac{15}{8}}u||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u||+C||u||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}b||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u||+C||b||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}\theta||||A^{\frac{a}{2}-\frac{5}{8}}u||+C||A^{\frac{a}{2}-\frac{15}{8}}f||||A^{\frac{a}{2}-\frac{5}{8}}u||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}u||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b||+C||u||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}b||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b||+C||b||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}g||||A^{\frac{a}{2}-\frac{5}{8}}b||+C||A^{\frac{a}{2}-\frac{15}{8}}u||_{L^{12}}||\nabla\theta||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}\theta||\\
&+C||u||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}\theta||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}\theta||\\
&\le C||u||_{H^{\frac{5}{4}}}||u||_{H^{a-\frac{5}{2}}}||u||_{H^{a-\frac{5}{4}}}+C||b||_{H^{\frac{5}{4}}}||b||_{H^{a-\frac{5}{2}}}||u||_{H^{a-\frac{5}{4}}}\\
&+C||u||_{H^{\frac{5}{4}}}||b||_{H^{a-\frac{5}{2}}}||b||_{H^{a-\frac{5}{4}}}+C||b||_{H^{\frac{5}{4}}}||u||_{H^{a-\frac{5}{2}}}||b||_{H^{a-\frac{5}{4}}}\\
&+C||u||_{H^{\frac{5}{4}}}||\theta||_{H^{a-\frac{5}{2}}}||\theta||_{H^{a-\frac{5}{4}}}+C||\theta||_{H^{\frac{5}{4}}}||u||_{H^{a-\frac{5}{2}}}||\theta||_{H^{a-\frac{5}{4}}}\\
&+C||\theta||_{H^{a-\frac{15}{4}}}||u||_{H^{a-\frac{5}{4}}}+C||f||_{H^{a-\frac{15}{4}}}||u||_{H^{a-\frac{5}{4}}}+C||g||_{H^{a-\frac{15}{4}}}||\theta||_{H^{a-\frac{5}{4}}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}(||u||^2_{H^{a-\frac{5}{4}}} + ||b||^2_{H^{a-\frac{5}{4}}} + ||\theta||^2_{H^{a-\frac{5}{4}}}) + C(||f||^2_{H^{a-\frac{15}{4}}} + ||g||^2_{H^{a-\frac{15}{4}}} + ||\theta||^2) \\
&+ C(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{a-\frac{5}{2}}} + ||b||^2_{H^{a-\frac{5}{2}}} + ||\theta||^2_{H^{a-\frac{5}{2}}}).
\end{aligned} \tag{5.1}
$$

Moreover,

$$
\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^{a-\frac{5}{2}}} + ||b||^2_{H^{a-\frac{5}{2}}} + ||\theta||^2_{H^{a-\frac{5}{2}}}) + ||u||^2_{H^{a-\frac{5}{4}}} + ||b||^2_{H^{a-\frac{5}{4}}} + ||\theta||^2_{H^{a-\frac{5}{4}}} \\
&\leq C(||f||^2_{H^{a-\frac{15}{4}}} + ||g||^2_{H^{a-\frac{15}{4}}} + ||\theta||^2) \\
&+ C(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{a-\frac{5}{2}}} + ||b||^2_{H^{a-\frac{5}{2}}} + ||\theta||^2_{H^{a-\frac{5}{2}}}).
\end{aligned} \tag{5.2}
$$

By virtue of (4.2), we have

$$
\int_t^{t+1} (||f||^2_{H^{a-\frac{15}{4}}} + ||g||^2_{H^{a-\frac{15}{4}}} + ||\theta||^2) ds \leq C. \tag{5.3}
$$

We integrate (4.12) on $[t, t+1]$ to get

$$
\int_t^{t+1} (||u(s)||^2_{H^{\frac{13}{4}}} + ||b(s)||^2_{H^{\frac{13}{4}}} + ||\theta(s)||^2_{H^{\frac{13}{4}}}) ds \leq C. \tag{5.4}
$$

Let $a = \frac{23}{4}$, we have

$$
\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^{\frac{13}{4}}} + ||b||^2_{H^{\frac{13}{4}}} + ||\theta||^2_{H^{\frac{13}{4}}}) + ||u||^2_{H^{\frac{9}{2}}} + ||b||^2_{H^{\frac{9}{2}}} + ||\theta||^2_{H^{\frac{9}{2}}} \\
&\leq C(||f||^2_{H^2} + ||g||^2_{H^2} + ||\theta||^2) \\
&+ C(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{\frac{13}{4}}} + ||b||^2_{H^{\frac{13}{4}}} + ||\theta||^2_{H^{\frac{13}{4}}}).
\end{aligned} \tag{5.5}
$$

Applying the uniform Gronwall lemma and (4.3) and (5.4), we get

$$
||u(t)||^2_{H^{\frac{13}{4}}} + ||b(t)||^2_{H^{\frac{13}{4}}} + ||\theta(t)||^2_{H^{\frac{13}{4}}} \leq C, \quad t \geq t_0 + 1. \tag{5.6}
$$

Integrating (5.5) on $[t, t+1]$, we get

$$
\int_t^{t+1} (||u(s)||^2_{H^{\frac{9}{2}}} + ||b(s)||^2_{H^{\frac{9}{2}}} + ||\theta(s)||^2_{H^{\frac{9}{2}}}) ds \leq C. \tag{5.7}
$$

Let $a = \frac{23}{4} + \frac{5}{4} = 7$, we have

$$
\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^{\frac{9}{2}}} + ||b||^2_{H^{\frac{9}{2}}} + ||\theta||^2_{H^{\frac{9}{2}}}) + ||u||^2_{H^{\frac{23}{4}}} + ||b||^2_{H^{\frac{23}{4}}} + ||\theta||^2_{H^{\frac{23}{4}}} \\
&\leq C(||f||^2_{H^{\frac{13}{4}}} + ||g||^2_{H^{\frac{13}{4}}} + ||\theta||^2) \\
&+ C(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{\frac{9}{2}}} + ||b||^2_{H^{\frac{9}{2}}} + ||\theta||^2_{H^{\frac{9}{2}}}).
\end{aligned} \tag{5.8}
$$

Applying the uniform Gronwall lemma and (4.3) and (5.7), we have

$$||u(t)||^2_{H^{\frac{9}{2}}} + ||b(t)||^2_{H^{\frac{9}{2}}} + ||\theta(t)||^2_{H^{\frac{9}{2}}} \leq C, \quad t \geq t_0 + 2. \tag{5.9}$$

Integrating (5.8) on $[t, t+1]$, we get

$$\int_t^{t+1} (||u(s)||^2_{H^{\frac{23}{4}}} + ||b(s)||^2_{H^{\frac{23}{4}}} + ||\theta(s)||^2_{H^{\frac{23}{4}}}) ds \leq C. \tag{5.10}$$

Let $a = \frac{23}{4} + 2 \times \frac{5}{4} = \frac{33}{4}$, we have

$$\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^{\frac{23}{4}}} + ||b||^2_{H^{\frac{23}{4}}} + ||\theta||^2_{H^{\frac{23}{4}}}) + ||u||^2_{H^7} + ||b||^2_{H^7} + ||\theta||^2_{H^7} \\
&\leq C(||f||^2_{H^{\frac{9}{2}}} + ||g||^2_{H^{\frac{9}{2}}} + ||\theta||^2) \\
&+ C(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{\frac{23}{4}}} + ||b||^2_{H^{\frac{23}{4}}} + ||\theta||^2_{H^{\frac{23}{4}}}).
\end{aligned} \tag{5.11}$$

Applying the uniform Gronwall lemma and (4.3) and (5.10), we have

$$||u(t)||^2_{H^{\frac{23}{4}}} + ||b(t)||^2_{H^{\frac{23}{4}}} + ||\theta(t)||^2_{H^{\frac{23}{4}}} \leq C, \quad t \geq t_0 + 3. \tag{5.12}$$

If $a = \frac{23}{4} + k \times \frac{5}{4}$, $k \in \mathbb{N}$, (5.2) holds. Then we get

$$||u||^2_{H^{\frac{13}{4}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{13}{4}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{13}{4}+k\times\frac{5}{4}}} \leq C, \quad t \geq t_0 + k + 1 \tag{5.13}$$

and

$$\int_t^{t+1} (||u||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}}) ds \leq C. \tag{5.14}$$

Let $a = \frac{23}{4} + (k+1) \times \frac{5}{4}$, $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{d}{dt}(||u||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}}) + ||u||^2_{H^{\frac{23}{4}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{23}{4}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{23}{4}+k\times\frac{5}{4}}} \\
&\leq C(||f||^2_{H^{\frac{13}{4}+k\times\frac{5}{4}}} + ||g||^2_{H^{\frac{13}{4}+k\times\frac{5}{4}}} + ||\theta||^2) \\
&+ C(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}})(||u||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}}).
\end{aligned} \tag{5.15}$$

Applying the uniform Gronwall lemma and (4.3) and (5.14), we have

$$||u(t)||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b(t)||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta(t)||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} \leq C, \quad t \geq t_0 + k + 2. \tag{5.16}$$

Since $k \in \mathbb{N}$ is arbitrarily, there exists a $t_0' > t_0$ such that

$$||u||^2_{H^{a-\frac{5}{2}}} + ||b||^2_{H^{a-\frac{5}{2}}} + ||\theta||^2_{H^{a-\frac{5}{2}}} \leq C, \quad t \geq t_0'. \tag{5.17}$$

Step 10 Taking ∂_t to the first equation of system (2.1) and the scalar product with $A^{a-\frac{5}{2}} u_t$. Similarly, taking ∂_t to the second equation of system (2.1) and the scalar product with $A^{a-\frac{5}{2}}b_t$. Taking ∂_t to the third equation of system (2.1) and the scalar product with $A^{a-\frac{5}{2}}\theta_t$. Summing up their results, then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_t||^2_{H^{a-\frac{5}{2}}}+||b_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}})+||u_t||^2_{H^{a-\frac{5}{4}}}+||b_t||^2_{H^{a-\frac{5}{4}}}+||\theta_t||^2_{H^{a-\frac{5}{4}}}\\
&\leq\left|\int_{\mathbb{T}^3}(u_t\nabla u)A^{a-\frac{5}{2}}u_tdx\right|+\left|\int_{\mathbb{T}^3}(u\nabla u_t)A^{a-\frac{5}{2}}u_tdx\right|+\left|\int_{\mathbb{T}^3}(b_t\nabla b)A^{a-\frac{5}{2}}u_tdx\right|\\
&+\left|\int_{\mathbb{T}^3}(b\nabla b_t)A^{a-\frac{5}{2}}u_tdx\right|+\left|\int_{\mathbb{T}^3}\theta_tA^{a-\frac{5}{2}}u_tdx\right|+\left|\int_{\mathbb{T}^3}(u_t\nabla b)A^{a-\frac{5}{2}}b_tdx\right|\\
&+\left|\int_{\mathbb{T}^3}(u\nabla b_t)A^{a-\frac{5}{2}}b_tdx\right|+\left|\int_{\mathbb{T}^3}(b_t\nabla u)A^{a-\frac{5}{2}}b_tdx\right|+\left|\int_{\mathbb{T}^3}(b\nabla u_t)A^{a-\frac{5}{2}}b_tdx\right|\\
&+\left|\int_{\mathbb{T}^3}(u_t\nabla\theta)A^{a-\frac{5}{2}}\theta_tdx\right|+\left|\int_{\mathbb{T}^3}(u\nabla\theta_t)A^{a-\frac{5}{2}}\theta_tdx\right|\\
&\leq C||A^{\frac{a}{2}-\frac{15}{8}}u_t||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||+C||u_t||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}u||_{L^{12}}||\nabla u_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||+C||u||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}u_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}b_t||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||+C||b_t||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}b||_{L^{12}}||\nabla b_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||+C||b||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}b_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}u_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}u_t||_{L^{12}}||\nabla b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||+C||u_t||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}b||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}u||_{L^{12}}||\nabla b_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||+C||u||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}b_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}b_t||_{L^{12}}||\nabla u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||+C||b_t||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}u||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}b||_{L^{12}}||\nabla u_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||+C||b||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}u_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}b_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}u_t||_{L^{12}}||\nabla \theta||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}\theta_t||+C||u_t||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}\theta||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}\theta_t||\\
&+C||A^{\frac{a}{2}-\frac{15}{8}}u||_{L^{12}}||\nabla \theta_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}\theta_t||+C||u||_{L^{12}}||A^{\frac{a}{2}-\frac{11}{8}}\theta_t||_{L^{\frac{12}{5}}}||A^{\frac{a}{2}-\frac{5}{8}}\theta_t||\\
&+C||A^{\frac{a}{2}-\frac{5}{4}}\theta_t||||A^{\frac{a}{2}-\frac{5}{4}}u_t||\\
&\leq C||u||_{H^{\frac{5}{4}}}||u_t||_{H^{a-\frac{5}{2}}}||u_t||_{H^{a-\frac{5}{4}}}+C||u||_{H^{a-\frac{5}{2}}}||u_t||_{H^{\frac{5}{4}}}||u_t||_{H^{a-\frac{5}{4}}}\\
&+C||b||_{H^{\frac{5}{4}}}||b_t||_{H^{a-\frac{5}{2}}}||u_t||_{H^{a-\frac{5}{4}}}+C||b||_{H^{a-\frac{5}{2}}}||b_t||_{H^{\frac{5}{4}}}||u_t||_{H^{a-\frac{5}{4}}}\\
&+C||b||_{H^{\frac{5}{4}}}||u_t||_{H^{a-\frac{5}{2}}}||b_t||_{H^{a-\frac{5}{4}}}+C||b||_{H^{a-\frac{5}{2}}}||u_t||_{H^{\frac{5}{4}}}||b_t||_{H^{a-\frac{5}{4}}}\\
&+C||u||_{H^{a-\frac{5}{2}}}||b_t||_{H^{\frac{5}{4}}}||b_t||_{H^{a-\frac{5}{4}}}+C||u||_{H^{\frac{5}{4}}}||b_t||_{H^{a-\frac{5}{2}}}||b_t||_{H^{a-\frac{5}{4}}}\\
&+C||\theta||_{H^{\frac{5}{4}}}||u_t||_{H^{a-\frac{5}{2}}}||\theta_t||_{H^{a-\frac{5}{4}}}+C||\theta||_{H^{a-\frac{5}{2}}}||u_t||_{H^{\frac{5}{4}}}||\theta_t||_{H^{a-\frac{5}{4}}}\\
&+C||u||_{H^{a-\frac{5}{2}}}||\theta_t||_{H^{\frac{5}{4}}}||\theta_t||_{H^{a-\frac{5}{4}}}+C||u||_{H^{\frac{5}{4}}}||\theta_t||_{H^{a-\frac{5}{2}}}||\theta_t||_{H^{a-\frac{5}{4}}}\\
&+C||\theta_t||_{H^{a-\frac{5}{2}}}||u_t||_{H^{a-\frac{5}{2}}}\\
&\leq\frac{1}{2}(||u_t||^2_{H^{a-\frac{5}{4}}}+||b_t||^2_{H^{a-\frac{5}{4}}}+||\theta_t||^2_{H^{a-\frac{5}{4}}})+C(||u_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2)
\end{aligned}
$$

$$
\begin{aligned}
&+C(||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{4}}})(||u_t||^2_{H^{a-\frac{5}{2}}}+||b_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}})\\
&+C(||u||^2_{H^{a-\frac{5}{2}}}+||b||^2_{H^{a-\frac{5}{2}}}+||\theta||^2_{H^{a-\frac{5}{2}}})(||u_t||^2_{H^{\frac{5}{4}}}+||b_t||^2_{H^{\frac{5}{4}}}+||\theta_t||^2_{H^{\frac{5}{4}}})\\
&\le \frac{1}{2}(||u_t||^2_{H^{a-\frac{5}{4}}}+||b_t||^2_{H^{a-\frac{5}{4}}}+||\theta_t||^2_{H^{a-\frac{5}{4}}})+C(||u_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2)\\
&+C(||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{4}}}+||u||^2_{H^{a-\frac{5}{2}}}+||b||^2_{H^{a-\frac{5}{2}}}+||\theta||^2_{H^{a-\frac{5}{2}}})\\
&(||u_t||^2_{H^{a-\frac{5}{2}}}+||b_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}}).
\end{aligned}
\tag{5.18}
$$

Moreover, we have

$$
\begin{aligned}
&\frac{d}{dt}(||u_t||^2_{H^{a-\frac{5}{2}}}+||b_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}})+||u_t||^2_{H^{a-\frac{5}{4}}}+||b_t||^2_{H^{a-\frac{5}{4}}}+||\theta_t||^2_{H^{a-\frac{5}{4}}}\\
&\le C(||u_t||^2_{H^{a-\frac{5}{2}}}+||b_t||^2_{H^{a-\frac{5}{2}}}+||\theta_t||^2_{H^{a-\frac{5}{2}}})\\
&(1+||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{4}}}+||u||^2_{H^{a-\frac{5}{2}}}+||b||^2_{H^{a-\frac{5}{2}}}+||\theta||^2_{H^{a-\frac{5}{2}}}).
\end{aligned}
\tag{5.19}
$$

We integrate (4.26) on $[t, t+1]$ to get

$$
\int_t^{t+1}(||u_t||^2_{H^{\frac{13}{4}}}+||b_t||^2_{H^{\frac{13}{4}}}+||\theta_t||^2_{H^{\frac{13}{4}}})ds\le C.
\tag{5.20}
$$

Let $a=\frac{23}{4}$, we have

$$
\begin{aligned}
&\frac{d}{dt}(||u_t||^2_{H^{\frac{13}{4}}}+||b_t||^2_{H^{\frac{13}{4}}}+||\theta_t||^2_{H^{\frac{13}{4}}})+||u_t||^2_{H^{\frac{9}{2}}}+||b_t||^2_{H^{\frac{9}{2}}}+||\theta_t||^2_{H^{\frac{9}{2}}}\\
&\le C(||u_t||^2_{H^{\frac{13}{4}}}+||b_t||^2_{H^{\frac{13}{4}}}+||\theta_t||^2_{H^{\frac{13}{4}}})\\
&(1+||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{4}}}+||u||^2_{H^{\frac{13}{4}}}+||b||^2_{H^{\frac{13}{4}}}+||\theta||^2_{H^{\frac{13}{4}}}).
\end{aligned}
\tag{5.21}
$$

By using (4.3) and (5.4), we have

$$
\int_t^{t+1}(1+||u||^2_{H^{\frac{5}{4}}}+||b||^2_{H^{\frac{5}{4}}}+||\theta||^2_{H^{\frac{5}{4}}}+||u||^2_{H^{\frac{13}{4}}}+||b||^2_{H^{\frac{13}{4}}}+||\theta||^2_{H^{\frac{13}{4}}})ds\le C.
\tag{5.22}
$$

Applying the uniform Gronwall lemma and (5.20) and (5.22), we have

$$
||u_t||^2_{H^{\frac{13}{4}}}+||b_t||^2_{H^{\frac{13}{4}}}+||\theta_t||^2_{H^{\frac{13}{4}}}\le C,\quad t\ge t_0+1.
\tag{5.23}
$$

Integrating (5.21) on $[t, t+1]$ to get

$$
\int_t^{t+1}(||u_t||^2_{H^{\frac{9}{2}}}+||b_t||^2_{H^{\frac{9}{2}}}+||\theta_t||^2_{H^{\frac{9}{2}}})ds\le C.
\tag{5.24}
$$

Let $a=\frac{23}{4}+\frac{5}{4}=7$, we have

$$
\frac{d}{dt}(||u_t||^2_{H^{\frac{9}{2}}}+||b_t||^2_{H^{\frac{9}{2}}}+||\theta_t||^2_{H^{\frac{9}{2}}})+||u_t||^2_{H^{\frac{23}{4}}}+||b_t||^2_{H^{\frac{23}{4}}}+||\theta_t||^2_{H^{\frac{23}{4}}}
$$

$$\le C(||u_t||^2_{H^{\frac{9}{2}}} + ||b_t||^2_{H^{\frac{9}{2}}} + ||\theta_t||^2_{H^{\frac{9}{2}}})$$
$$(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + ||u||^2_{H^{\frac{9}{2}}} + ||b||^2_{H^{\frac{9}{2}}} + ||\theta||^2_{H^{\frac{9}{2}}}). \tag{5.25}$$

By using (4.3) and (5.7), we have

$$\int_t^{t+1} (1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + ||u||^2_{H^{\frac{9}{2}}} + ||b||^2_{H^{\frac{9}{2}}} + ||\theta||^2_{H^{\frac{9}{2}}})ds \le C. \tag{5.26}$$

Applying the uniform Gronwall lemma and (5.24) and (5.26), we have

$$||u_t||^2_{H^{\frac{9}{2}}} + ||b_t||^2_{H^{\frac{9}{2}}} + ||\theta_t||^2_{H^{\frac{9}{2}}} \le C, \quad t \ge t_0 + 2. \tag{5.27}$$

Integrating (5.25) on $[t, t+1]$ to get

$$\int_t^{t+1} (||u_t||^2_{H^{\frac{23}{4}}} + ||b_t||^2_{H^{\frac{23}{4}}} + ||\theta_t||^2_{H^{\frac{23}{4}}})ds \le C. \tag{5.28}$$

Let $a = \frac{23}{4} + 2 \times \frac{5}{4} = \frac{33}{4}$, we have

$$\frac{d}{dt}(||u_t||^2_{H^{\frac{23}{4}}} + ||b_t||^2_{H^{\frac{23}{4}}} + ||\theta_t||^2_{H^{\frac{23}{4}}}) + ||u_t||^2_{H^7} + ||b_t||^2_{H^7} + ||\theta_t||^2_{H^7}$$
$$\le C(||u_t||^2_{H^{\frac{23}{4}}} + ||b_t||^2_{H^{\frac{23}{4}}} + ||\theta_t||^2_{H^{\frac{23}{4}}})$$
$$(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + ||u||^2_{H^{\frac{23}{4}}} + ||b||^2_{H^{\frac{23}{4}}} + ||\theta||^2_{H^{\frac{23}{4}}}). \tag{5.29}$$

By using (4.3) and (5.10), we have

$$\int_t^{t+1} (1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + ||u||^2_{H^{\frac{23}{4}}} + ||b||^2_{H^{\frac{23}{4}}} + ||\theta||^2_{H^{\frac{23}{4}}})ds \le C. \tag{5.30}$$

Applying the uniform Gronwall lemma and (5.28) and (5.30), we have

$$||u_t||^2_{H^{\frac{23}{4}}} + ||b_t||^2_{H^{\frac{23}{4}}} + ||\theta_t||^2_{H^{\frac{23}{4}}} \le C, \quad t \ge t_0 + 3. \tag{5.31}$$

If $a = \frac{23}{4} + k \times \frac{5}{4}$, $k \in \mathbb{N}$, (5.19) holds. Then we get

$$||u_t||^2_{H^{\frac{13}{4} + k \times \frac{5}{4}}} + ||b_t||^2_{H^{\frac{13}{4} + k \times \frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{13}{4} + k \times \frac{5}{4}}} \le C, \quad t \ge t_0 + k + 1 \tag{5.32}$$

and

$$\int_t^{t+1} (||u_t||^2_{H^{\frac{9}{2} + k \times \frac{5}{4}}} + ||b_t||^2_{H^{\frac{9}{2} + k \times \frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{9}{2} + k \times \frac{5}{4}}})ds \le C. \tag{5.33}$$

Let $a = \frac{23}{4} + (k+1) \times \frac{5}{4}$, $k \in \mathbb{N}$, we have

$$\frac{d}{dt}(||u_t||^2_{H^{\frac{9}{2} + k \times \frac{5}{4}}} + ||b_t||^2_{H^{\frac{9}{2} + k \times \frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{9}{2} + k \times \frac{5}{4}}}) + ||u_t||^2_{H^{\frac{23}{4} + k \times \frac{5}{4}}} + ||b_t||^2_{H^{\frac{23}{4} + k \times \frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{23}{4} + k \times \frac{5}{4}}}$$

$$
\begin{aligned}
&\le C(||u_t||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b_t||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}})\\
&(1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + ||u||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}}).
\end{aligned}
\tag{5.34}
$$

By using (4.3) and (5.14), we have

$$
\int_t^{t+1} (1 + ||u||^2_{H^{\frac{5}{4}}} + ||b||^2_{H^{\frac{5}{4}}} + ||\theta||^2_{H^{\frac{5}{4}}} + ||u||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}})ds \le C.
\tag{5.35}
$$

Applying the uniform Gronwall lemma and (5.33) and (5.35), we have

$$
||u_t||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||b_t||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} + ||\theta_t||^2_{H^{\frac{9}{2}+k\times\frac{5}{4}}} \le C, \quad t \ge t_0 + k + 2.
\tag{5.36}
$$

Since $k \in \mathbb{N}$ is arbitrarily, there exists a $t_0' > t_0$ such that

$$
||u_t||^2_{H^{a-\frac{5}{2}}} + ||b_t||^2_{H^{a-\frac{5}{2}}} + ||\theta_t||^2_{H^{a-\frac{5}{2}}} \le C, \quad t \ge t_0'.
\tag{5.37}
$$

Step 11. Proof of the Theorem 2.3. We will prove that the system (2.1) has an absorbing ball in $H^a \times H^a \times H^a$ for $a \ge \frac{9}{2}$.

$$
\begin{aligned}
&||u||_{H^a} + ||b||_{H^a} + ||\theta||_{H^a} = ||A^{\frac{a}{2}}u|| + ||A^{\frac{a}{2}}b|| + ||A^{\frac{a}{2}}\theta||\\
&\le ||u_t||_{H^{a-\frac{5}{2}}} + ||b_t||_{H^{a-\frac{5}{2}}} + ||\theta_t||_{H^{a-\frac{5}{2}}} + ||B(u,u)||_{H^{a-\frac{5}{2}}} + ||B(b,b)||_{H^{a-\frac{5}{2}}} + ||B(u,b)||_{H^{a-\frac{5}{2}}}\\
&+ ||B(b,u)||_{H^{a-\frac{5}{2}}} + ||B(u,\theta)||_{H^{a-\frac{5}{2}}} + ||\theta e_3||_{H^{a-\frac{5}{2}}} + ||f||_{H^{a-\frac{5}{2}}} + ||g||_{H^{a-\frac{5}{2}}}\\
&\le ||u_t||_{H^{a-\frac{5}{2}}} + ||b_t||_{H^{a-\frac{5}{2}}} + ||\theta_t||_{H^{a-\frac{5}{2}}} + ||\theta||_{H^{a-\frac{5}{2}}} + ||f||_{H^{a-\frac{5}{2}}} + ||g||_{H^{a-\frac{5}{2}}}\\
&+ C(||u||_{H^{a-\frac{5}{2}}}||\nabla u||_{H^{a-\frac{5}{2}}} + ||b||_{H^{a-\frac{5}{2}}}||\nabla b||_{H^{a-\frac{5}{2}}} + ||u||_{H^{a-\frac{5}{2}}}||\nabla b||_{H^{a-\frac{5}{2}}} + ||b||_{H^{a-\frac{5}{2}}}||\nabla u||_{H^{a-\frac{5}{2}}}\\
&+ ||u||_{H^{a-\frac{5}{2}}}||\nabla\theta||_{H^{a-\frac{5}{2}}})\\
&\le ||u_t||_{H^{a-\frac{5}{2}}} + ||b_t||_{H^{a-\frac{5}{2}}} + ||\theta_t||_{H^{a-\frac{5}{2}}} + ||\theta||_{H^{a-\frac{5}{2}}} + ||f||_{H^{a-\frac{5}{2}}} + ||g||_{H^{a-\frac{5}{2}}}\\
&+ C(||u||^{\frac{8}{5}}_{H^{a-\frac{5}{2}}}||u||^{\frac{2}{5}}_{H^a} + ||b||^{\frac{8}{5}}_{H^{a-\frac{5}{2}}}||b||^{\frac{2}{5}}_{H^a} + ||u||_{H^{a-\frac{5}{2}}}||b||^{\frac{3}{5}}_{H^{a-\frac{5}{2}}}||b||^{\frac{2}{5}}_{H^a}\\
&+ ||b||_{H^{a-\frac{5}{2}}}||u||^{\frac{3}{5}}_{H^{a-\frac{5}{2}}}||u||^{\frac{2}{5}}_{H^a} + ||u||_{H^{a-\frac{5}{2}}}||\theta||^{\frac{3}{5}}_{H^{a-\frac{5}{2}}}||\theta||^{\frac{2}{5}}_{H^a})\\
&\le \frac{1}{2}(||u||_{H^a} + ||b||_{H^a} + ||\theta||_{H^a})\\
&+ ||u_t||_{H^{a-\frac{5}{2}}} + ||b_t||_{H^{a-\frac{5}{2}}} + ||\theta_t||_{H^{a-\frac{5}{2}}} + ||\theta||_{H^{a-\frac{5}{2}}} + ||f||_{H^{a-\frac{5}{2}}} + ||g||_{H^{a-\frac{5}{2}}}\\
&+ C(||u||^{\frac{8}{3}}_{H^{a-\frac{5}{2}}} + ||b||^{\frac{8}{3}}_{H^{a-\frac{5}{2}}} + ||\theta||^{\frac{8}{3}}_{H^{a-\frac{5}{2}}}).
\end{aligned}
\tag{5.38}
$$

Moreover,

$$
\begin{aligned}
&||u||_{H^a} + ||b||_{H^a} + ||\theta||_{H^a}\\
&\le 2(||u_t||_{H^{a-\frac{5}{2}}} + ||b_t||_{H^{a-\frac{5}{2}}} + ||\theta_t||_{H^{a-\frac{5}{2}}} + ||\theta||_{H^{a-\frac{5}{2}}} + ||f||_{H^{a-\frac{5}{2}}} + ||g||_{H^{a-\frac{5}{2}}})
\end{aligned}
$$

$$+ C(||u||^{\frac{8}{3}}_{H^{a-\frac{5}{2}}} + ||b||^{\frac{8}{3}}_{H^{a-\frac{5}{2}}} + ||\theta||^{\frac{8}{3}}_{H^{a-\frac{5}{2}}}). \tag{5.39}$$

By using the (5.17) and (5.37), we get

$$||u||_{H^a} + ||b||_{H^a} + ||\theta||_{H^a} \leq C, \quad t \geq t_0'. \tag{5.40}$$

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Data Availability Statements Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] L.C Berselli, J.S. Fan; Logarithmic and improved regularity criteria for the 3D nematic liquid crystals models, Boussinesq system, and MHD equations in a bounded domain, *Commun. Pure Appl. Anal.*, 14 (2015), 637-655.

[2] C.S. Cao, J.H. Wu; Two regularity criteria for the 3D MHD equations, *J. Differential Equations*, 248 (2010), 2263-2274.

[3] C.G. Gal, Y.Q. Guo; Inertial manifolds for the hyperviscous Navier-Stokes equations, *J. Differential Equations*, 265 (2018), 4335-4374.

[4] L. Grafakos; Classical Fourier analysis, Second edition. Springer, New York, 2008..

[5] C. He, Z.P. Xin; On the regularity of weak solutions to the magnetohydrodynamic equations, *J. Differential Equations*, 213 (2005), 235-254.

[6] C. He, Z.P. Xin; Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, *J. Funct. Anal.*, 227 (2005), 113-152.

[7] Q.S. Jiu, H. Yu; Global well-posedness for 3D generalized Navier-Stokes-Boussinesq equations, *Acta Math. Appl. Sin. Engl. Ser.*, 32 (2016), 1-16.

[8] A. Larios, P. Yuan; On the local well-posedness and a Prodi-Serrin-type regularity criterion of the three-dimensional MHD-Boussinesq system without thermal diffusion, *J. Differential Equations*, 263 (2017), 1419-1450.

[9] X.H. Li, C.Y. Sun; Inertial manifolds for the incompressible Navier-Stokes equations, *arXiv:1910.05939v1.*

[10] Y. Li; Global weak solutions to the three-dimensional inviscid Boussinesq system in the presence of magnetic field, *Z. Angew. Math. Phys.*, 70 (2019), Art. 172, 10 pp.

[11] H. Liu, C.F. Sun, J. Xin; Well-posedness for the hyperviscous magneto-micropolar equations, *Appl. Math. Lett.*, 107 (2020), 106403.

[12] H.M. Liu, D.F. Bian, X.K. Pu; Global well-posedness of the 3D Boussinesq-MHD system without heat diffusion, *Z. Angew. Math. Phys.*, 70 (2019), Art. 81, 19 pp.

[13] G. Mulone, S. Rionero; Necessary and sufficient conditions for nonlinear stability in the magnetic Bénard problem, *Arch. Ration. Mech. Anal.*, 166 (2003), 197-218.

[14] Y.M. Trève, O.P. Manley; Energy conserving Galerkin approximations for 2-D hydrodynamic and MHD Bénard convection, *Phys. D*, 4 (1982), 319-342.

[15] J.H. Wu; Generalized MHD equations, *J. Differential Equations*, 195 (2003), 284-312.

[16] J.H. Wu; Regularity criteria for the generalized MHD equations, *Comm. Partial Differential Equations*, 33 (2008), 285-306.

[17] X.P. Zhai, Z.M. Chen; Global well-posedness for the MHD-Boussinesq system with the temperature-dependent viscosity, *Nonlinear Anal. Real World Appl.*, 44 (2018), 260-282.