HYPONORMALITY ON AN ANNULUS WITH A WEIGHT

HOUCINE SADRAOUI AND BORHEN HALOUANI

Abstract. In this work we consider the hyponormality of Toeplitz operators on the Bergman space of the annulus with a logarithmic weight. We give necessary conditions when the symbol is of the form $\psi + \overline{\phi}$ where ψ and ϕ are analytic on the annulus $\{z \in \mathbb{C}, 1/2 < |z| < 1\}$.

1. Introduction

A bounded operator S on a Hibert space is hyponormal if $S^*S - SS^*$is positive. Hyponormality of Toeplitz operators on the Hardy space was considered by C.C.Cowen([2, 3]). The first work on hyponormality on the Bergman space can be found in [6]. An improvement of the necessary condition therein, using function theory, is due to P.Ahern and Z.Cuckovic[1]. A recent improvement of the necessary condition in a special case uses matrix theory and is due to Z.Cuckovic R.Curto and [4]. Sufficient conditions for hyponormality when the analytic part of the symbol is a monomial are given in[7] and have not been improved. All other results on hyponormality on the Bergman space deal mostly with the case of very specific symbols and use matrix computations. Some of these results can be found in [8]. In this work we consider hyponormality of Toeplitz operators on the Bergman space of an annulus with a logarithmic weight with fairly general harmonic symbols. We begin with definitions and notations. Set $A_{1/2} = \{v \in \mathbb{C}, 1/2 < |v| < 1\}$. The space L^2_l is the space of measurable functions f on $A_{1/2}$ such that $\int_{A_{1/2}} |f|^2 d\omega(v) < \infty$ where $d\omega(v) = \frac{8}{\pi(3-2\ln 2)} r|\log r| dr d\theta$. The subspace of L^2_l consisting of analytic functions is denoted by A^2_ω. If f is analytic on $A_{1/2}$ we have

$$f = \sum_{-\infty}^{-1} d_n v^n + \sum_{0}^{\infty} d_n v^n$$

and

$$||f||^2 = \frac{1}{3-2\ln 2} \sum_{0}^{\infty} \frac{2^{2n+2} - 1 - (2n+2)\ln 2}{2^{2n}(n+1)^2} |d_n|^2 + \frac{8(\ln 2)^2}{3-2\ln 2} |d_{-1}|^2 + \frac{4}{3-2\ln 2} \sum_{2}^{\infty} \frac{2^{2n-1}(n-1)\ln 2 - 2^{2n-2} + 1}{(n-1)^2} |d_{-n}|^2.$$

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The space A^2_v has the following orthonormal basis:

$$\{e_n, n \geq 0\} \cup \{e_{-1}\} \cup \{e_{-n}, n \geq 2\} = \{\frac{\sqrt{3-2\ln 2}\; 2^n(n+1)}{\sqrt{2^{2n+2}-1-(2n+2)\ln 2}} v^n, n \geq 0\}$$

$$\cup \{\frac{\sqrt{3-2\ln 2}}{2\sqrt{2}\ln 2}\frac{1}{v}\} \cup \{\frac{\sqrt{3-2\ln 2}}{2}\frac{n-1}{\sqrt{2^{2n-1}(n-1)\ln 2 - 2^{2n-2}+1}}\frac{1}{v^n}, n \geq 2\}.$$

For h bounded measurable on $A_{1/2}$ we define the Toeplitz operator B_h by $B_h(f) = Q(hf)$ where Q is the orthogonal projection of L^2_l onto A^2_ω. We also define Hankel operators by $H_h(f) = (I-Q)(hf)$. In this work the Toeplitz operators considered have a symbol of the form $\psi + \overline{\phi}$, where ψ and ϕ are bounded analytic on $A_{1/2}$. We give necessary conditions for the hyponormality of these operators. We start with some basic properties of these operators. These properties are easy to prove and the proof is omitted.

2. General properties of Toeplitz and Hankel operators

Lemma 2.1. *Let ψ and ϕ be bounded measurable on $A_{1/2}$. The following holds:*

(i) $B_{\psi+\phi} = B_\psi + B_\phi$.
(ii) $B^*_\psi = B_{\overline{\psi}}$.
(iii) $B_\psi B_\phi = B_{\phi\psi}$ *if and only if ϕ is analytic or ψ is conjugate analytic.*
(iv) $B^*_\psi B_\psi - B_\psi B^*_\psi = H^*_{\overline{\psi}} H_{\overline{\psi}}$ *if ψ is analytic.*

As in the case of the Bergman space of the unit disk[6], the following proposition gives equivalent forms of hyponormality.

Proposition 2.2. *For ψ_1 and ψ_2 bounded analytic on $A_{1/2}$ the following statements are equivalent*

(a) $B_{\psi_1+\overline{\psi_2}}$ *is hyponormal.*
(b) $B_{\overline{\psi_2}} B_{\psi_2} - B_{\psi_2} B_{\overline{\psi_2}} \leq B_{\overline{\psi_1}} B_{\psi_1} - B_{\psi_1} B_{\overline{\psi_1}}$.
(c) $H^*_{\overline{\psi_2}} H_{\overline{\psi_2}} \leq H^*_{\overline{\psi_1}} H_{\overline{\psi_1}}$.
(d) $H_{\psi_2} = T H_{\psi_1}$ *where T is bounded of norm less than 1.*

Computations involving the projection are given in the next lemma.

Lemma 2.3. *The orthogonal projection of L^2_l onto A^2_ω satisfies the following properties:*

(1) $Q(v^m \overline{v^n}) = \frac{2^{2m+2}-1-(2m+2)\ln 2}{2^{2m}(m+1)^2} \frac{2^{2(m-n)}(m-n+1)^2}{2^{2(m-n)+2}-1-(2(m-n)+2)\ln 2} v^{m-n}$ *if $m \geq n$.*
(2) $Q(v^m \overline{v^n}) = \frac{1}{4} \frac{2^{2m+2}-1-(2m+2)\ln 2}{2^{2m}(m+1)^2} \frac{(n-m-1)^2}{2^{2(n-m)-1}((n-m)-1)\ln 2 - 2^{2(n-m)-2}+1} \frac{1}{v^{n-m}}$ *if* $n-m \geq 2$.
(3) $Q(v^m \overline{v^{m+1}}) = \frac{1}{8(\ln 2)^2} \frac{2^{2m+2}-1-(2m+2)\ln 2}{2^{2m}(m+1)^2} \frac{1}{v}$.
(4) $Q(\frac{1}{v^m}\overline{v^n}) = \frac{2^{2m-1}(m-1)\ln 2 - 2^{2m-2}+1}{(m-1)^2} \frac{(m+n-1)^2}{2^{2(m+n)-1}(m+n-1)\ln 2 - 2^{2(m+n)-2}+1} \frac{1}{v^{m+n}}$, $m \geq 2$.
(5) $Q(\frac{1}{v}\overline{v^n}) = 2(\ln 2)^2 \frac{n^2}{2^{2n+1}n\ln 2 - 2^{2n}+1} \frac{1}{v^{n+1}}$, $n \geq 1$.
(6) $Q(\overline{\frac{1}{v^m}} v^n) = \frac{2^{2n+2}-1-(2n+2)\ln 2}{2^{2n}(n+1)^2} \frac{2^{2(m+n)}(m+n+1)^2}{2^{2(m+n)+2}-1-(2(m+n)+2)\ln 2} v^{m+n}$.
(7) $Q(\overline{\frac{1}{v^m}} \frac{1}{v^n}) = 4 \frac{2^{2n-1}(n-1)\ln 2 - 2^{2n-2}+1}{(n-1)^2} \frac{2^{2(m-n)}(m-n+1)^2}{2^{2(m-n)+2}-1-(2(m-n)+2)\ln 2} v^{m-n}$, *if* $m \geq n$, $n \geq 2$.
(8) $Q(\overline{\frac{1}{v^m}} \frac{1}{v}) = 2(\ln 2)^2 \frac{2^{2m} m^2}{2^{2m}-1-2m\ln 2} v^{m-1}$, $m \geq 1$.

(9) $Q(\frac{1}{\overline{v^m}}\frac{1}{v^n}) = \frac{2^{2n-1}(n-1)\ln 2-2^{2n-2}+1}{(n-1)^2}\frac{(n-m-1)^2}{2^{2(n-m)-1}(n-m-1)\ln 2-2^{2(n-m)-2}+1}\frac{1}{v^{n-m}}$, $n-m\geq 2$.

(10) $Q(\frac{1}{\overline{v^m}}\frac{1}{v^{m+1)}}) = \frac{1}{2(\ln 2)^2}\frac{m^2}{2^{2m+1}m\ln 2-2^{2m}+1}\frac{1}{v}$.

3. The results

We will make use of the following computation.

Lemma 3.1. *Let $\psi = \sum_1^\infty d_n\frac{1}{v^n}$ be bounded and analytic on $A_{1/2}$. For k and $l\geq 1$ we have*

$$\langle B_{\overline{\psi}}B_\psi - B_\psi B_{\overline{\psi}}(e_k), e_l\rangle =$$

$$\sum_{\substack{k\geq n\geq 1\\ n+l-k\geq 1}}^\infty \frac{\overline{d_{n+l-k}}d_n 2^k(k+1)2^l(l+1)[2^{2(k-n)+2}-1-(2(k-n)+2)\ln 2]}{\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}2^{2(k-n)}(k-n+1)^2}$$
$$+8(\ln 2)^2\overline{d_{l+1}}d_{k+1}\frac{2^k(k+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}\frac{2^l(l+1)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}$$
$$+4\sum_{n\geq k+2}^\infty \frac{\overline{d_{n+l-k}}d_n 2^k(k+1)2^l(l+1)[2^{2(k-n)-1}(k-n-1)\ln 2-2^{2(k-n)-2}+1]}{\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}(k-n-1)^2}-$$
$$\sum_{n\geq 1}^\infty \frac{\overline{d_{n+l-k}}d_n\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}2^{(n+l)}(n+l+1)^2}{2^k(k+1)2^l(l+1)[2^{2(n+l)+2}-1-(2(n+l)+2)\ln 2]}.$$

Proof. We have $\left\langle B_{\overline{\psi}}B_\psi(e_k), e_l\right\rangle =$

$$\sum_{m,n\geq 1}^\infty \frac{(3-2\ln 2)\overline{d_m}d_n 2^k(k+1)2^l(l+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}\frac{\left\langle v^{-n+k}, v^{-m+l}\right\rangle}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}} =$$
$$\sum_{\substack{k\geq n\geq 1\\ n+l-k\geq 1}}^\infty \frac{\overline{d_{n+l-k}}d_n 2^k(k+1)2^l(l+1)[2^{2(k-n)+2}-1-(2(k-n)+2)\ln 2]}{\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}2^{2(k-n)}(k-n+1)^2}$$
$$+8(\ln 2)^2\overline{d_{l+1}}d_{k+1}\frac{2^k(k+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}\frac{2^l(l+1)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}$$
$$+4\sum_{n\geq k+2}^\infty \frac{[\overline{d_{n+l-k}}d_n 2^k(k+1)2^l(l+1)][2^{2(k-n)-1}(k-n-1)\ln 2-2^{2(k-n)-2}+1]}{\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}(k-n-1)^2}-$$
$$\sum_{n\geq 1}^\infty \frac{\overline{d_{n+l-k}}d_n\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}2^{(n+l)}(n+l+1)^2}{2^k(k+1)2^l(l+1)[2^{2(n+l)+2}-1-(2(n+l)+2)\ln 2]}.$$

We also have $\left\langle B_\psi B_{\overline{\psi}}(e_k), e_l \right\rangle =$

$$\sum_{n\geq 1}^{\infty} \frac{(3-2\ln 2)\overline{d_m} d_n 2^k (k+1) 2^l (l+1) \left\langle Q(\frac{1}{v^m} v^k), Q(\frac{1}{v^n} v^l) \right\rangle}{\sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}} =$$

$$\sum_{n\geq 1}^{\infty} \frac{\overline{d_{n+l-k}} d_n \sqrt{[2^{2k+2}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]} 2^{(n+l)} (n+l+1)^2}{2^k (k+1) 2^l (l+1) [2^{2(n+l)+2} - 1 - (2(n+l)+2)\ln 2]}.$$

The result follows. □

Put $\mu_{l,k} = \langle B_{\overline{\psi}} B_\psi - B_\psi B_{\overline{\psi}}(e_k), e_l \rangle$. We have with obvious notations

$$\mu_{l+p,l} = \sum_{1\leq n\leq l} \overline{d_{n+p}} d_n R_{n,p,l} + \overline{d_{l+p+1}} d_{l+1} S_{l,p} + \sum_{n\geq l+2}^{\infty} \overline{d_{n+p}} d_n T_{n,p,l}.$$

If H^2 denotes the Hardy space, denote by T_χ the Toeplitz operator on H^2 with symbol χ. If we denote the matrix of T_χ in the usual basis of H^2 by $(\chi_{i,j})$, it is known that $\chi_{i,i+p} = \chi_{j,j+p}$ for any nonnegative integers i, j, and p. Denote by $\widetilde{\psi}(v) = \sum_{n\geq 1} d_n v^n$ and assume $\widetilde{\psi}' \in H^2$ ie $\sum_{n\geq 1} n^2 |d_n|^2 < \infty$.

Lemma 3.2. *Assume that $\widetilde{\psi}' \in H^2$. Then we have $\lim_{l\to\infty} l^2 \mu_{l+p,l} = \lambda_{s,s+p}$ where $(\lambda_{r,s})_{r,s}$ is the matrix of the operator $T_{|\widetilde{\psi}'|^2}$.*

Proof. A tedious but elementary calculation leads to:

$$\lim_{l\to\infty} l^2 R_{n,p,l} = n(n+p).$$

Set $h_l(n) = l^2 \chi_{\{0,\dots,l\}}(n) \overline{d_{n+p}} d_n R_{n,p,l}$ and let ϵ be the counting measure.

$$\lim_{l\to\infty} h_l(n) = n(n+p)\overline{d_{n+p}} d_n$$

and

$$\int h_l(n) d\epsilon(n) = \sum_{1\leq n\leq l} l^2 \overline{d_{n+p}} d_n R_{n,p,l}.$$

It is not difficult to see that for l large

$$|l^2 \overline{c_{n+p}} c_n R_{n,p,l}| \leq 2n(n+p)|\overline{d_{n+p}} d_n| \leq (n+p)^2 |d_{n+p}|^2 + n^2 |d_n|^2.$$

Since $\widetilde{psi}' \in H^2$ it follows by the dominated convergence theorem that

$$\lim_{l\to\infty} \sum_{1\leq n\leq l} l^2 \overline{d_{n+p}} d_n R_{n,p,l} = \sum_{n\geq 1} n(n+p)\overline{d_{n+p}} d_n.$$

Also we can readily see that for l large

$$|l^2 \overline{d_{l+p+1}} d_{l+1} S_{l,p}| \leq C[(l+p+1)^2 |\overline{d_{l+p+1}}|^2 + (l+1)^2 |d_{l+1}|^2]$$

for some constant C, so

$$\lim_{l\to\infty} l^2 \overline{d_{l+p+1}} d_{l+1} S_{l,p} = 0.$$

Finally we have

$$|l^2 \overline{c_{n+p}} c_n T_{n,p,l}| \leq n(n+p)|d_n||d_{n+p}| \leq \frac{1}{2}(n^2 |d_n|^2 + (n+p)^2 |d_{n+p}|^2)$$

for l large and $n \geq l+2$. It follows from the dominated convergence theorem that

$$\lim_{l\to\infty} \sum_{n\geq l+2}^{\infty} l^2 \overline{d_{n+p}} d_n T_{n,p,l} = 0.$$

We obtain that

$$\lim_{l\to\infty} l^2 \mu_{l+p,l} = \sum_{n\geq 1} n(n+p)\overline{d_{n+p}} d_n = \lambda_{s,s+p}.$$

□

We deduce one of our main results

Theorem 3.3. *Let* $\psi = \sum_1^\infty d_n \frac{1}{v^n}$ *and* $\phi = \sum_1^\infty t_n \frac{1}{v^n}$ *be bounded and analytic on* $A_{1/2}$ *and assume that* $\widetilde{\psi}' \in H^2$. *If* $B_{\psi+\overline{\phi}}$ *is hyponormal then* $\widetilde{\phi}' \in H^2$ *and* $|\psi'| \geq |\phi'|$ *a.e on the unit circle.*

Proof. Denote by $(\Psi_{i,j})$ the matrix of $(B_{\overline{\psi}} B_\psi - B_\psi B_{\overline{\psi}}) - (B_{\overline{\phi}} B_\phi - B_\phi B_{\overline{\phi}})$, by $(\mu_{i,j})$ the matrix of $B_{\overline{\psi}} B_\psi - B_\psi B_{\overline{\psi}}$ and $(\theta_{i,j})$ the matrix of $B_{\overline{\phi}} B_\phi - B_\phi B_{\overline{\phi}}$. Hyponormality of $B_{\psi+\overline{\phi}}$ gives $\theta_{l,l} \leq \mu_{l,l}$. We deduce that

$$\sum_{1\leq n\leq l} l^2 |d_n|^2 R_{n,0,l} + l^2 |d_{l+1}|^2 S_{l,0} + \sum_{n\geq l+2}^{\infty} l^2 |d_n|^2 T_{n,0,l} \geq$$
$$\sum_{1\leq n\leq l} l^2 |t_n|^2 R_{n,0,l} + l^2 |t_{l+1}|^2 S_{l,0} + \sum_{n\geq l+2}^{\infty} l^2 |t_n|^2 T_{n,0,l}.$$

We have

$$\lim_{l\to\infty} l^2 |d_{l+1}|^2 S_{l,0} = 0 = \lim_{l\to\infty} l^2 |t_{l+1}|^2 S_{l,0}.$$

Using the fact that $\sum_{l\geq 1} l^2 |d_l|^2 < \infty$ and the dominated convergence theorem we see that

$$\lim_{l\to\infty} \sum_{n\geq l+2}^{\infty} l^2 |d_n|^2 T_{n,0,l} = 0.$$

Also since $\sum_1^\infty |t_n|^2 < \infty$, by the dominated convergence theorem we get

$$\lim_{l\to\infty} \sum_{n\geq l+2}^{\infty} l^2 |t_n|^2 T_{n,0,l} = 0.$$

Writing $\sum_{1\leq n\leq l} l^2 |t_n|^2 S_{n,0,l}$ as an integral with respect to the counting measure, and using Fatou's lemma in the last inequality we get:

$$\sum_{1\leq n} n^2 |d_n|^2 \geq \sum_{1\leq n} n^2 |t_n|^2.$$

Thus $\widetilde{\phi}' \in H^2$. From the previous lemma 3.2 we obtain that

$$\lim_{l\to\infty} l^2 (\sigma_{l+p,l} - \theta_{l+p,l}) = \Lambda_{s,s+p}.$$

We recognize $(\Lambda_{s,s+p})$ as the matrix of the operator $T_{|\widetilde{\psi'}|^2-|\widetilde{\phi'}|^2}$. Hyponormality and a property of Toeplitz forms [5] lead to $|\widetilde{\psi'}| \geq |\widetilde{\phi'}|$ a.e on the unit circle. This is equivalent to $|\psi'| \geq |\phi'|$ a.e on the unit circle. $\square$

The following computation is needed.

Lemma 3.4. *Let $\psi = \sum_1^{\infty} d_n \frac{1}{v^n}$ be bounded and analytic on $A_{1/2}$. For k and l greater than or equal to 3 we have* $\langle B_{\overline{\psi}} B_\psi - B_\psi B_{\overline{\psi}}(e_{-k}), e_{-l}\rangle =$

$$= \sum_{\substack{n\geq 1\\ n+k-l\geq 1}}^{\infty} \frac{\overline{d_{n+k-l}} d_n (k-1)(l-1)[2^{2(n+k)}(n+k-1)\ln 2 - 2^{2(n+k)-2}+1]}{\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}(n+k-1)^2}$$
$$-4\sum_{n\geq k}^{\infty} \frac{\overline{d_n} d_{n+l-k}\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}2^{2(n-k)}(n-k+1)^2}{(k-1)(l-1)2^{2(n-k)+2}-1-(2(n-k)+2)\ln 2}$$
$$-\frac{1}{2(\ln 2)^2}\frac{\overline{d_{k-1}} d_{l-1}\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}}{(k-1)(l-1)}$$
$$-\sum_{n\leq k-2}\frac{\overline{d_n} d_{n+l-k}\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}(k-n-1)^2}{(k-1)(l-1)2^{2(k-n)-1}(k-n-1)\ln 2 - 2^{2(k-n)-2}+1}$$

Proof. We have $\left\langle B_{\overline{\psi}} B_\psi(e_{-k}), e_{-l}\right\rangle =$

$$\sum_{m,n\geq 1}^{\infty} \frac{\overline{d_m} d_n (k-1)(l-1)\left\langle \frac{1}{v^{n+k}}, \frac{1}{v^{m+l}}\right\rangle}{\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}}$$
$$= \sum_{\substack{n\geq 1\\ n+k-l\geq 1}}^{\infty} \frac{\overline{d_{n+k-l}} d_n (k-1)(l-1)(2^{2(n+k)-1}(n+k-1)\ln 2 - 2^{2(n+k)-2}+1)}{\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}(n+k-1)^2}$$

We also have $\left\langle B_\psi B_{\overline{\psi}}(e_{-k}), e_{-l}\right\rangle =$

$$\sum_{m,n\geq 1}^{\infty} \frac{(3-2\ln 2)\overline{d_n} d_m (k-1)(l-1)\left\langle Q(\frac{1}{\overline{v^n}}\frac{1}{v^k}), Q(\frac{1}{\overline{v^m}}\frac{1}{v^l})\right\rangle}{4\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}}$$
$$= \sum_{n\geq k}^{\infty} \frac{4\overline{d_n} d_{n+l-k}\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}2^{2(n-k)}(n-k+1)^2}{(k-1)(l-1)2^{2(n-k)+2}-1-(2(n-k)+2)\ln 2}$$
$$-\frac{1}{2(\ln 2)^2}\frac{\overline{d_{k-1}} d_{l-1}\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}}{(k-1)(l-1)}$$
$$-\sum_{n\leq k-2}\frac{\overline{d_n} d_{n+l-k}\sqrt{[2^{2k-1}(k-1)\ln 2 - 2^{2k-2}+1][2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1]}(k-n-1)^2}{(k-1)(l-1)2^{2(k-n)-1}(k-n-1)\ln 2 - 2^{2(k-n)-2}+1}$$

$\square$

As before we denote the matrix of $B_{\overline{\psi}}B_\psi - B_\psi B_{\overline{\psi}}$ by $(\mu_{i,j})$. Then $\mu_{-l-p,-l} =$

$$\sum_{n\geq 1}^{\infty} \frac{\overline{d_n}d_{n+p}(l-1)(l+p-1)(2^{2(n+p+l)-1}(n+p+l-1)\ln 2 - 2^{2(n+p+l)-2}+1)}{\sqrt{[2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1][2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2}+1]}(n+p+l-1)^2}$$
$$-\sum_{n\geq l}^{\infty} \frac{4\overline{d_n}d_{n+p}\sqrt{[2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1][2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2}+1]}2^{2(n-l)}(n-l+1)^2}{(l-1)(l+p-1)(2^{2(n-l)+2}-1-(2(n-l)+2)\ln 2)}$$
$$-\frac{1}{2(\ln 2)^2}\frac{\overline{d_{l-1}}d_{l+p-1}\sqrt{[2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1][2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2}+1]}}{(l-1)(l+p-1)}$$
$$-\sum_{1\leq n\leq l-2} \frac{\overline{d_n}d_{n+p}\sqrt{[2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1][2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2}+1]}(l-n-1)^2}{(l-1)(l+p-1)(2^{2(l-n)-1}(l-n-1)\ln 2 - 2^{2(l-n)-2}+1)}$$

We can thus write without ambiguity

$$\mu_{-l-p,-l} = \sum_{1\leq n\leq l-2} \overline{d_n}d_{n+p}U_{n,p,l} + \overline{d_{l-1}}d_{l-1+p}V_{n,p,l} + \sum_{n\geq l}^{\infty}\overline{d_n}d_{n+p}W_{n,p,l}$$

where for example $U_{n,p,l} =$

$$\frac{(l-1)(l+p-1)(2^{2(n+p+l)-1}(n+p+l-1)\ln 2 - 2^{2(n+p+l)-2}+1)}{\sqrt{[2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1][2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2}+1]}(n+p+l-1)^2}$$
$$-\frac{\sqrt{[2^{2l-1}(l-1)\ln 2 - 2^{2l-2}+1][2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2}+1]}(l-n-1)^2}{(l-1)(l+p-1)[2^{2(l-n)-1}(l-n-1)\ln 2 - 2^{2(l-n)-2}+1]}.$$

Set $\psi_2(v) = \sum_{n\geq 1} 2^n d_n v^n$. Then obviously

$$\psi_2' \in H^2 \Leftrightarrow \sum_{n\geq 1} n^2 2^{2n}|d_n|^2 < \infty.$$

Denote the matrix of the Hardy space Toeplitz operator $T_{|\psi_2'|^2}$ by $(\xi_{i,j})$.

Lemma 3.5. *Assume* $\sum_{n\geq 1} n^2 2^{2n}|d_n|^2 < \infty$. *We have*

$$\lim_{l\to\infty} l^2\mu_{-l-p,-l}{=}\xi_{s+p,s}.$$

Proof. We proceed as in the proof Lemma 3.2. A computation shows that

$$\lim_{l\to\infty} l^2 U_{n,p,l} = n(n+p)2^{2n+p}.$$

We can also show that for l large

$$|l^2\overline{d_n}d_{n+p}U_{n,l,p}| \leq Cn(n+p)2^{2n+p}|d_n||d_{n+p}| \leq \frac{C}{2}(n^2 2^{2n}|d_n|^2 + (n+p)^2 2^{2n+2p}|d_{n+p}|^2)$$

for some constant C. Expressing $\sum_{1\leq n\leq l-2} l^2\overline{d_n}d_{n+p}U_{n,p,l}$ as an integral with respect to the counting measure and applying the dominated convergence theorem we obtain

$$\lim_{l\to\infty}\sum_{1\leq n\leq l-2} l^2\overline{d_n}d_{n+p}U_{n,p,l} = \sum_{n\geq 1} n(n+p)2^{2n+p}\overline{d_n}d_{n+p}.$$

We can show that $\lim_{l\to\infty} l^2\overline{d_{l-1}}d_{l+p-1}V_{n,p,l} = 0$. Also using the dominated convergence theorem and the hypothesis $\sum_{n\geq 1} n^2 2^{2n}|d_n|^2 < \infty$, it is possible to show that

$$\lim_{l\to\infty}\sum_{n\geq l} l^2\overline{d_n}d_{n+p}W_{n,p,l} = 0.$$

Thus we deduce that $\lim_{l\to\infty} l^2\mu_{-l-p,-l} = \xi_{s+p,s}$. $\square$

If $\psi(v) = \sum_{n\geq 1} d_n\frac{1}{v^n}$ and $\phi = \sum_{n\geq 1} t_n\frac{1}{v^n}$ and as before $\psi_2(v) = \sum_{n\geq 1} 2^n d_n v^n$ and $\phi_2(v) = \sum_{n\geq 1} 2^n t_n v^n$, then $|\phi_2'| \leq |\psi_2'|$ a.e on the circle is equivalent to $|\phi'| \leq |\psi'|$ a.e on $\{z, |z| = 1/2\}$. Using the previous lemma 3.5 and proceeding as in the proof of Theorem 3.3, we deduce the following result.

Proposition 3.6. *Let $\psi = \sum_1^\infty d_n\frac{1}{v^n}$ and $\phi = \sum_1^\infty t_n\frac{1}{v^n}$ be bounded and analytic on $A_{1/2}$. If $B_{\psi+\overline{\phi}}$ is hyponormal and if $\sum_{n\geq 1} n^2 2^{2n}|d_n|^2 < \infty$ then $\sum_{n\geq 1} n^2 2^{2n}|t_n|^2 < \infty$ and $|\phi'| \leq |\psi'|$ a.e on $\{z, |z| = 1/2\}$.*

We consequently get the following theorem.

Theorem 3.7. *Let $\psi = \sum_1^\infty d_n\frac{1}{v^n}$ and $\phi = \sum_1^\infty t_n\frac{1}{v^n}$ be bounded and analytic on $A_{1/2}$ and assume $\sum_{n\geq 1} n^2 2^{2n}|d_n|^2 < \infty$. If $B_{\psi+\overline{\phi}}$ is hyponormal then $\sum_{n\geq 1} n^2 2^{2n}|t_n|^2 < \infty$ and $|\phi'| \leq |\psi'|$ a.e on $\{z, |z| = 1\} \cup \{z, |z| = 1/2\}$.*

An application of the maximum modulus principle leads to:

Corollary 3.8. *Let $\psi = \sum_1^\infty d_n\frac{1}{v^n}$ and $\phi = \sum_1^\infty t_n\frac{1}{v^n}$ be bounded and analytic and univalent on an open set containing $A_{1/2}$. If $B_{\psi+\overline{\phi}}$ is normal, then $\phi = a\psi + b$ for some constants a and b with $|a| = 1$.*

Using similar techniques and notations we show the following lemma where we assume $\psi = \sum_1^\infty d_n v^n$ with $\psi' \in H^2$ and $(\lambda_{i,j})_{i,j\geq 0}$ denotes the matrix of $T_{|\psi'|^2}$.

Lemma 3.9. *Let $\psi = \sum_1^\infty d_n v^n$ and assume $\psi' \in H^2$. Then*

$$\lim_{l\to\infty} l^2\mu_{l,l+p} = \lambda_{i,i+p}.$$

We get the following theorems

Theorem 3.10. *Let $\psi = \sum_1^\infty d_n v^n$ and $\phi = \sum_1^\infty t_n\frac{1}{v^n}$ assume $\psi' \in H^2$. If $B_{\psi+\overline{\phi}}$ is hyponormal then $\widetilde{\phi}' \in H^2$ and $|\phi'(e^{-i\theta}| \leq |\psi'(e^{i\theta})|$ a.e on the unit circle.*

Theorem 3.11. *Let $\psi = \sum_1^\infty d_n v^n$ and $\phi = \sum_1^\infty t_n\frac{1}{v^n}$. If $B_{\psi+\overline{\phi}}$ is hyponormal then $\sum_{n\geq 1} n^2 2^{2n}|t_n|^2 < \infty$ and $|\phi'(\frac{1}{2}e^{-i\theta})| \leq |\psi'(\frac{1}{2}e^{i\theta})|$ a.e on $\{z, |z| = 1/2\}$.*

If $\widetilde{\phi}(v) = \sum\limits_{1}^{\infty} t^n v^n$ we get the following corollary.

Corollary 3.12. *Let* $\psi = \sum\limits_{1}^{\infty} d_n v^n$ *and* $\phi = \sum\limits_{1}^{\infty} t_n \frac{1}{v^n}$. *Assume both are analytic and univalent on an open set containing* $A_{1/2}$. *If* $B_{\psi+\overline{\phi}}$ *is normal then* $\widetilde{\phi} = a\psi + b$ *for some constants* a *and* b *with* $|a| = 1$.

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Department of mathematics, College of Sciences, King Saud University, P. O Box 2455 Riyadh 11451, Saudi Arabia.

Email address: `sadrawi@ksu.edu.sa, halouani@ksu.edu.sa`