Non-fragile containment control of nonlinear multi-agent systems via a disturbance observer-based approach

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Abstract

In this paper, we consider the non-fragile containment control problem of nonlinear multi-agent systems (MASs) with exogenous disturbance where the communication links among agents under consideration is directed. Firstly, based on relative output measurements between the agent and its neighbors, a disturbance observer-based control protocol is proposed to solve the containment control problem of MASs with inherent nonlinear dynamics and exogenous disturbances. Secondly, because of the additional tuning of parameters in the real control systems, uncertainties in the designing of observer and controller gains always occur, and as a result, an output feedback controller with disturbance rejection is conceived and the containment control problem of nonlinear MASs with non-fragility is thoroughly investigated. Then, depending on matrix transformation and inequality technique, sufficient conditions of the designed controller gains exist, which is derived from the asymptotic stability analysis problem of some containment error dynamics of MASs. Finally, two simulation examples are exploited to illustrate the effectiveness of the proposed techniques.

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SUMMARY

In this paper, we consider the non-fragile containment control problem of nonlinear multi-agent systems (MASs) with exogenous disturbance where the communication links among agents under consideration is directed. Firstly, based on relative output measurements between the agent and its neighbors, a disturbance observer-based control protocol is proposed to solve the containment control problem of MASs with inherent nonlinear dynamics and exogenous disturbances. Secondly, because of the additional tuning of parameters in the real control systems, uncertainties in the designing of observer and controller gains always occur, and as a result, an output feedback controller with disturbance rejection is conceived and the containment control problem of nonlinear MASs with non-fragility is thoroughly investigated. Then, depending on matrix transformation and inequality technique, sufficient conditions of the designed controller gains exist, which is derived from the asymptotic stability analysis problem of some containment error dynamics of MASs. Finally, two simulation examples are exploited to illustrate the effectiveness of the proposed techniques.

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KEY WORDS: Multi-agent systems; Non-fragility; Disturbance rejection; Containment control; Output feedback

1. INTRODUCTION

During the last decade, cooperation control of MASs has found substantial success in a variety of applications, such as consensus [1–3], tracking control [4–6], formation control [7, 8], containment

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control [9], cooperative output regulation [10] and optimization [11]. Consensus problem, as a fundamental and non-negligible research topic of MASs, requires that all agents achieve an agreement on a common state, depending on the designed consensus protocols using only relative measurements among neighboring agents. In general, due to the number of leaders in the control system, consensus problem is then called consensus tracking problem (one leader) or containment control problem (multiple leaders). As a matter of fact, it is much more attracting and challenging to investigate the containment control problem in practical application, especially in military areas such as sea hunting.

The fact of containment, generally speaking, is that the states of all followers eventually enter into a given geometric space spanned by those of the leaders. Motivated by the pioneer works on consensus problems of MASs, a number of fruits on containment control problems have already sprung up. By using Z-transformation and Routh Criterion, [12] studies the containment control problem of discrete-time first-order MASs and shows that different step-sizes have a specific effect on the stability of MASs. Utilizing Lasalle's Invariance Principle of hybrid stability theory, [13] researches the containment control problem of continuous-time first-order MASs and the obtained result holds for arbitrary state dimensions. Further, depending on non-smooth analysis and adaptive control method, [14] investigates the robust containment control problem of linear MASs where the time-varying uncertainties exist. Different form the static control schemes in which internal information might be difficult to detect [12–14], the containment control problem of liner MASs is solved in [15–17] via an output feedback approach depending on relative output measurements of the neighboring agents. Note that the aforementioned works [14–17] focus on the linear dynamics, however, the nonlinearity always exists in various engineering applications. Thus, it is of great significance to handle the cooperation control problems of MASs with nonlinear dynamics [18, 19].

As is well known, external disturbances are ubiquitous in practical control systems, especially in large-scale networked systems. Therefore, many fruitful results have been placed on the disturbances of the MASs. For example, by using a state feedback controller, [20] considers the H_∞ consensus problem of general liner MASs with bounded disturbances. Different form the state feedback method proposed in [20], based on a truncated predictor output-feedback strategy, [21] studies the H_∞ scales consensus problem of MASs, in which both Lipschitz nonlinearity and external disturbances are involved. Note that the H_∞ control [20,21] can not directly compensate the influence of disturbances of the system depending on the disturbance estimate. Thus, a disturbance observer-based control scheme is adopted to handle those defects and a great number of achievements have been reported in the literature [22–27]. Specifically, by designing a disturbance observer-based control scheme based on only the relative state information, [22] discusses the consensus disturbance rejection of linear MASs and [23–25] address the consensus tracking problem for nonlinear MASs with disturbance rejection, respectively. For further consideration, by using an observer-based output feedback control scheme, consensus tracking problem of nonlinear dynamics is investigated in [26], in which the disturbances considered are nonlinear. In addition, [27] deals with the containment control problem of MASs with disturbance rejection.

Other than the external disturbances, non-fragility plays an important role in the study of consensus problems of MASs, as well. In particular, by employing a non-fragile state feedback consensus protocol, [28] researches the finite-time consensus of MASs with time-varying input

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delay over switching topologies. In [29], the non-fragile guaranteed-performance H_∞ consensus tracking problem of MASs is discussed, in which Lipschitz nonlinearities and exogenous disturbances are taken into account. Compared to the state feedback control method proposed in [28–30], by adopting an output feedback control method, [31] deals with the non-fragile consensus problem of nonlinear MASs with randomly occurring deception attacks. Further, via a non-fragile output feedback controller, [32] handles the non-fragile cooperative containment control problem of MASs with time delay. Moreover, in order to reduce communication burden, [33] proposed a non-fragile memory sampled-data control scheme to address the consensus problem of MASs. However, to the best of our knowledge, there are few reports concerning the issue of the non-fragile containment control problems of MASs with disturbance rejection. In this paper, via a disturbance observer-based approach, we investigate the non-fragile containment control problem of nonlinear MASs over directed communication topology.

The contributions include the following aspects. Firstly, two kinds of disturbance observer-based control schemes of MASs are developed. The former is used to investigate the containment control problem with disturbance rejection, and the other is used to address that problem with non-fragility. Secondly, by taking advantage of matrix transformation and inequality technique, the feedback controller gain, as well as the corresponding observer gain, is obtained. Finally, only external disturbance or non-fragility is handled in references [17, 22–24, 27, 29–32], however, there exist few works to deal with consensus problems considering both external disturbance and non-fragility. In this paper, to fill the research gaps on consensus problems, we solve the non-fragile containment control problems of MASs with disturbance rejection under directed communication topology, in which all these two factors are involved.

The organizational structure of the paper is as follows. In Section 2, problem formulation is presented. Section 3 respectively discusses the containment control problem and the non-fragile containment control problem of nonlinear MASs with external disturbance. Two simulation examples are given to validate the effectiveness of the developed algorithes in Section 4. Finally, Section 5 summarizes the paper.

Notations: The notations used in this paper are fairly standard. $\otimes$ stands for the Kronecker product, $*$ denotes a symmetric term and I_n represents an identity matrix with dimension n. In addition, an n dimension column vector with all the elements being $1(0)$ is denoted as $1_n(0_n)$ and $diag\{\varepsilon_1,\, \varepsilon_2,\, \cdots,\, \varepsilon_n\}$ is used to represent a block diagonal matrix with diagonal blocks being $\varepsilon_1,\, \varepsilon_2,\, \cdots,\, \varepsilon_n$. Given real symmetric matrix A and B, $A > B(A \geq B)$ denotes that $A - B$ is positive definite (positive semi-definite).

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, the basic knowledge of graph theory is introduced and the considered problem is shown, respectively.

2.1. Preliminaries

Generally, $\mathscr{G} = \{\mathscr{V},\, \mathscr{E}\}$ is used to represent a directed communication topology, where $\mathscr{V} = \{1,\, 2,\, \ldots,\, N\}$ and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ are the set of nodes and that of edges, respectively. $(j,\, i) \in \mathscr{E}$ denotes

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that the ith agent can receive the information from the jth agent in the directed communication topology $\mathscr{G}$, but not vice-versa. $\mathscr{A} = [a_{ij}] \in R^{N\times N}$ is use to stand for the adjacency matrix connected with the directed topology $\mathscr{G}$ by $a_{ii} = 0$, $a_{ij} = 1$ if $(j,\, i) \in \mathscr{E}$, and $a_{ij} = 0$ otherwise. In addition, the Laplacian matrix $L = (l_{ij})_{N\times N}$ is defined as $l_{ij} = -a_{ij}$ with $i \neq j$ and $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$ with $i = j$. The above-mentioned content of graph theory is introduced in [34].

2.2. *Problem Formulation*

We consider a multi-agent system (MAS) consisting of $N+M$ agents with N followers and M leaders, where the dynamics of the ith agent is described by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + Dd_i(t) + Ef(t,\, x_i(t)), \\ y_i(t) &= Cx_i(t), \end{aligned} \tag{1}$$

for $i = 1,\, \cdots,\, N+M$, where $x_i(t) \in R^n$, $u_i(t) \in R^m$ and $y_i(t) \in R^r$ respectively denote the state, control input and output of agent i, and $f:\; R\times R^n \to R^n$ denotes inherent nonlinear dynamics. $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{r\times n}$, $D \in R^{n\times s}$ and $E \in R^{n\times n}$ are constant matrices, and $d_i(t) \in R^s$ is a disturbance that is generated by an exosystem

$$\dot{d}_i(t) = Sd_i(t),\; i = 1,\, \cdots,\, N, \tag{2}$$

with $S \in R^{s\times s}$ being a known constant matrix.

In the paper, the agents indexed by $1,\, 2,\, \ldots,\, N$ are followers and those indexed by $N+1,\, N+2,\, \ldots,\, N+M$ are leaders. We let $\mathscr{F} = \{1,\, 2,\, \ldots,\, N\}$ and $\mathscr{R} = \{N+1,\, N+2,\, \ldots,\, N+M\}$ denote the set of the followers and that of the leaders, respectively. Suppose that the leaders have no parents in the directed communication topology, then the corresponding Laplacian matrix is divided into $L = \begin{bmatrix} L_1 & L_2 \\ 0_{M\times N} & 0_{M\times M} \end{bmatrix}$, where $L_1 \in R^{N\times N}$ and $L_2 \in R^{N\times M}$. Furthermore, the convex hull constructed by the states of the set of multiple leaders $x_{\mathscr{R}}(t) = \{x_{N+1}(t),\, \cdots,\, x_{N+M}(t)\}$ is denoted as $co(x_{\mathscr{R}}(t)) = \left\{\sum_{i=N+1}^{N+M} \alpha_i x_i(t) \middle|\, \sum_{i=N+1}^{N+M} \alpha_i = 1,\; \alpha_i \geq 0\right\}$.

Definition 1 ([15])

The containment control problem of MAS (1) is achieved if each follower asymptotically enters into the convex hull $co(x_{\mathscr{R}}(t))$ formed by the states of the leaders set as time goes to the infinity.

Assumption 1

The directed topology $\mathscr{G}$ is connected. And for each follower, there exists at least one leader that has a directed path to it.

Assumption 2

The control input matrix $B \in R^{n\times m}$ is of full-column rank, namely, $\mathrm{rank}(B) = m$.

Assumption 3

The disturbance is matched, namely, there exists a matrix $V \in R^{m\times s}$ such that $D = BV$.

Assumption 4

Given $\eta_{N+1},\, \eta_{N+2},\, \ldots,\, \eta_{N+M}$ with $\sum_{i=N+1}^{N+M} \eta_i = 1$, and $\eta_i \geq 0$, $i = N+1,\, N+2,\, \ldots,\, N+M$. There

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exists a non-negative constant α such that the nonlinear function f satisfies

$$\left\| f(t,x) - \sum_{i=N+1}^{N+M} \eta_i f(t,y_i) \right\| \leq \alpha \left\| x - \sum_{i=N+1}^{N+M} \eta_i y_i \right\|,$$
$$\forall x,\ y_i \in R^n,\ i = N+1,\ N+2,\ \ldots,\ N+M,\ \forall t \geq 0.$$

Remark 1

Assumption 1 is commonly used in the containment control problems of MASs, see [12–17]. It is worth noting that it is of great importance to study the consensus containment problems of MASs under directed topology than undirected topology in [27]and [29]. The character of B specified in Assumption 2 is in order to make matrix transformation [35] for subsequent containment error analysis. From Assumption 3, the disturbance $d_i(t)$ is the non-vanishing harmonic disturbance and $D = BV$ is the matching condition of MAS (1) with external disturbance, and Assumption 3 is commonly used in the consensus problems [22–24, 26, 27]. All linear and some nonlinear functions such as $c\cos t + v\sin t$ and xe^{-t} satisfy the condition above in Assumption 4. When only one leader exists, the condition converts to the form of Lipschitz condition, namely, $\|f(t,\ x) - f(t,\ y)\| \leq \alpha \|x - y\|$, and the containment control problem is transformed into the consensus tracking problem [18, 19, 23–26, 29].

Lemma 1 ([14])

If Assumption 1 holds. The real parts of all the eigenvalues of L_1 are positive, each entry of $-{L_1}^{-1}L_2$ is nonnegative and the sum of each row of the matrix $-{L_1}^{-1}L_2$ is equal to one.

Lemma 2 ([35])

From Assumption 2, it is seen that $\tilde{B} = TBW = \begin{bmatrix} T_1^T & T_2^T \end{bmatrix}^T BW = \begin{bmatrix} Q^T & 0 \end{bmatrix}^T$ with the existence of matrices $T \in R^{n\times n}$ and $W \in R^{m\times m}$, where $T_1 \in R^{m\times n}$ and $T_2 \in R^{(n-m)\times n}$, and $Q = diag\{q_1,\ q_2,\ \ldots,\ q_m\}$ is a diagonal matrix, $q_i\,(i = 1,\ 2,\ \ldots,\ m)$ are nonzero singular value of the matrix B. Assume that the equality $P_1 = T_1^T P_{11} T_1 + T_2^T P_{22} T_2$ holds, there exists a nonsingular matrix $P \in R^{m\times m}$ such that $BP = P_1 B$, where $P_{11} \in R^{m\times m} > 0$ and $P_{22} \in R^{(n-m)\times(n-m)} > 0$.

Remark 2

Finding a solution to Lemma 2's problem of $BP = P_1 B$ for P is intended to help us build the LMI approach to the controller design. Since we can always perform congruence transformation on B, the assumption that B is a full-column rank is purely for presentation convenience and does not lose any generality. If $P_1 = T_1^T P_{11} T_1 + T_2^T P_{22} T_2$ is true, P exists, but it might not be unique unless B is square and nonsingular.

Lemma 3 ([33])

Given matrix $Y = Y^T$, H and E with compatible dimensions, if $Y + HG(t)E + E^T G^T(t) H^T < 0$ holds for all $G(t)$ satisfying $G^T(t)G(t) \leq I$. Then, $Y + \zeta HH^T + \zeta^{-1} E^T E < 0$ holds with any scalar $\zeta > 0$.

Lemma 4 ([32])

For any two real vectors a and b with the same dimension, then $2a^T b \leq a^T U a + b^T U^{-1} b$ holds with $U > 0$.

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3. MAIN RESULTS

In this section, both the distributed cooperation containment control problem and that with non-fragility of nonlinear MASs over directed topology are investigated, respectively.

3.1. Containment Control Problem

In this subsection, the containment control problem is transformed to the stability analysis of a containment error system and the existing conditions on gains of observer and controller of MAS (1) is shown. Motivated by [22,23,27,29,31,32], a distributed disturbance observer-based controller is given as:

$$\begin{aligned} u_i(t) &= 0,\ i\in\mathscr{R}, \\ u_i(t) &= -K\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - V\hat{d}_i(t),\ i\in\mathscr{F}, \end{aligned} \tag{3}$$

with

$$\begin{aligned} \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) - G_1(y_i(t)-\hat{y}_i(t)),\ i\in\mathscr{R}, \\ \hat{d}_i(t) &= 0,\ i\in\mathscr{R}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu_i(t) + D\hat{d}_i(t) - G_1(y_i(t)-\hat{y}_i(t)),\ i\in\mathscr{F}, \\ \dot{\hat{d}}_i(t) &= S\hat{d}_i(t) - G_2\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}((y_i(t)-y_j(t)) - (\hat{y}_i(t)-\hat{y}_j(t))),\ i\in\mathscr{F}, \end{aligned} \tag{5}$$

where $\hat{y}_i(t) = C\hat{x}_i(t)$, $\hat{x}_i(t)$ denotes a state observer, $\hat{d}_i(t)$ denotes the estimate of the disturbance. The feedback gain K, state observer gain G_1 and disturbance observer gain G_2 are constant matrices to be designed. Then, from (3), the system (1) can be rewritten as

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Ef(t,\, x_i(t)),\ i\in\mathscr{R}, \\ \dot{x}_i(t) &= Ax_i(t) - BK\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - BV\hat{d}_i(t) + Dd_i(t) + Ef(t,\, x_i(t)),\ i\in\mathscr{F}, \end{aligned} \tag{6}$$

According to (1)-(6), one gets

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Ef(t,\, x_i(t)),\ i\in\mathscr{R}, \\ \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) - G_1C(x_i(t)-\hat{x}_i(t)),\ i\in\mathscr{R}, \end{aligned}$$

and

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) - BK\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - BV\hat{d}_i(t) + Dd_i(t) + Ef(t,\, x_i(t)),\ i\in\mathscr{F}, \\ \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) - BK\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - G_1C(x_i(t)-\hat{x}_i(t)),\ i\in\mathscr{F}, \end{aligned}$$

which are rewritten in compact forms

$$\begin{aligned} \dot{x}_R(t) &= (I_M\otimes A)x_R(t) + (I_M\otimes E)F(t,\, x_R), \\ y_R(t) &= (I_M\otimes C)x_R(t), \\ \dot{\hat{x}}_R(t) &= (I_M\otimes(A+G_1C))\hat{x}_R(t) - (I_M\otimes G_1C)x_R(t), \end{aligned}$$

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and

$$\begin{aligned}
\dot{x}_F(t) &= (I_N \otimes A)x_F(t) - (L_1 \otimes BK)\hat{x}_F(t) - (L_2 \otimes BK)\hat{x}_R(t) + (I_N \otimes D)e(t) + (I_N \otimes E)F(t,\, x_F),\\
y_F(t) &= (I_N \otimes C)x_F(t),\\
\dot{\hat{x}}_F(t) &= (I_N \otimes (A+G_1C) - L_1 \otimes BK)\hat{x}_F(t) - (I_N \otimes G_1C)x_F(t) - (L_2 \otimes BK)\hat{x}_R(t),
\end{aligned}$$

where $e(t) = [e_1^T(t),\, \cdots,\, e_N^T(t)]^T$ with $e_i(t) = d_i(t) - \hat{d}_i(t)$, $x_F(t) = [x_1^T(t),\, \cdots,\, x_N^T(t)]^T$, $x_R(t) = [x_{N+1}^T(t),\, \cdots,\, x_{N+M}^T(t)]^T$, $y_F(t) = [y_1^T(t),\, \cdots,\, y_N^T(t)]^T$ and $y_R(t) = [y_{N+1}^T(t),\, \cdots,\, y_{N+M}^T(t)]^T$. Moreover, $\bar{F}(t) = (L_1 \otimes E)[F(t,\, x_F) + (L_1{}^{-1}L_2 \otimes I_n)F(t,\, x_R)]$, $F(t,\, x_F) = [f^T(t,\, x_1),\, \ldots,\, f^T(t,\, x_N)]^T$ and $F(t,\, x_R) = [f^T(t,\, x_{N+1}),\, \ldots,\, f^T(t,\, x_{N+M})]^T$. Then, one has

$$\dot{e}(t) = \dot{d}(t) - \dot{\hat{d}}(t) = (I_N \otimes S)e(t) + (L_1 \otimes G_2C)(x_F(t) - \hat{x}_F(t)) + (L_2 \otimes G_2C)(x_R(t) - \hat{x}_R(t)).$$

Let $\eta_i(t) = \sum_{j \in \mathscr{R} \cup \mathscr{F}} a_{ij}(x_i(t) - x_j(t))$ and $\hat{\eta}_i(t) = \sum_{j \in \mathscr{R} \cup \mathscr{F}} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t))$ denote the containment error vectors, $i = 1,\, 2,\, \cdots,\, N$, with $\eta(t) = [\eta_1^T(t),\, \cdots,\, \eta_N^T(t)]^T$ and $\hat{\eta}(t) = [\hat{\eta}_1^T(t),\, \cdots,\, \hat{\eta}_N^T(t)]^T$. Denote $\bar{\eta}_i(t) = \eta_i(t) - \hat{\eta}_i(t)$, $i = 1,\, 2,\, \cdots,\, N$, with $\bar{\eta}(t) = [\bar{\eta}_1^T(t),\, \cdots,\, \bar{\eta}_N^T(t)]^T$. Thus, we have

$$\begin{aligned}
\eta(t) &= (L_1 \otimes I_N)x_F(t) - (L_2 \otimes I_N)x_R(t),\\
\hat{\eta}(t) &= (L_1 \otimes I_N)\hat{x}_F(t) - (L_2 \otimes I_N)\hat{x}_R(t),\\
\bar{\eta}(t) &= \eta(t) - \hat{\eta}(t) = (L_1 \otimes I_N)(x_F(t) - \hat{x}_F(t)) + (L_2 \otimes I_N)(x_R(t) - \hat{x}_R(t)).
\end{aligned}$$

Further, it yields

$$\begin{aligned}
\dot{e}(t) &= (I_N \otimes S)e(t) + (I_N \otimes G_2C)\bar{\eta}(t),\\
\dot{\hat{\eta}}(t) &= (L_1 \otimes I_N)\dot{\hat{x}}_F(t) + (L_2 \otimes I_N)\dot{\hat{x}}_R(t) = (I_N \otimes A - L_1 \otimes BK)\hat{\eta}(t) - (I_N \otimes G_1C)\bar{\eta}(t),\\
\dot{\bar{\eta}}(t) &= \dot{\eta}(t) - \dot{\hat{\eta}}(t) = (L_1 \otimes I_N)(\dot{x}_F(t) - \dot{\hat{x}}_F(t)) + (L_2 \otimes I_N)(\dot{x}_R(t) - \dot{\hat{x}}_R(t))\\
&= (I_N \otimes (A+G_1C))\bar{\eta}(t) + (L_1 \otimes D)e(t) + \bar{F}(t). \qquad (7)
\end{aligned}$$

Based on (7), the error system is expressed in a compact form

$$\dot{\varepsilon}(t) = \bar{A}\varepsilon(t) + \tilde{F}(t), \qquad (8)$$

where

$$\varepsilon(t) = \begin{bmatrix} e(t) \\ \hat{\eta}(t) \\ \bar{\eta}(t) \end{bmatrix},\ \bar{A} = \begin{bmatrix} I_N \otimes S & 0 & I_N \otimes G_2C \\ 0 & I_N \otimes A - L_1 \otimes BK & -I_N \otimes G_1C \\ L_1 \otimes D & 0 & I_N \otimes (A+G_1C) \end{bmatrix},\ \tilde{F}(t) = \begin{bmatrix} 0 \\ 0 \\ \bar{F}(t) \end{bmatrix}.$$

Now, we are in the position to present our result as follows.

Theorem 1

Suppose that the directed topology satisfies Assumption 1, the feedback gain, state observer gain and disturbance observer gain are designed by $K = WQ^{-1}P_{11}{}^{-1}QW^TX$, $G_1 = P_1^{-1}Y_1$ and $G_2 = P_1^{-1}Y_2$ such that MAS (1) with external disturbance system (2) under the control protocol (3) solves the containment disturbance rejection problem if there exist matrices $P_{11} > 0$, $P_{22} > 0$ and matrices X,

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Y_1 and Y_2 such that the following LMI holds:

$$\begin{bmatrix} \theta_{11} & 0 & \theta_{13} & I_N \otimes P_1 & 0 & 0 & 0 & 0 \\ * & \theta_{22} & I_N \otimes Y_1 C & 0 & I_N \otimes P_1 & 0 & \alpha(I_N \otimes E^T) & 0 \\ * & * & \theta_{33} & 0 & 0 & I_N \otimes P_1 & 0 & \alpha(I_N \otimes E^T) \\ * & * & * & -I_{nN} & 0 & 0 & 0 & 0 \\ * & * & * & * & -I_{nN} & 0 & 0 & 0 \\ * & * & * & * & * & -I_{nN} & 0 & 0 \\ * & * & * & * & * & * & -I_{nN} & 0 \\ * & * & * & * & * & * & * & -I_{nN} \end{bmatrix} < 0, \tag{9}$$

where

$$\theta_{11} = I_N \otimes (P_1 S + S^T P_1), \ \theta_{22} = I_N \otimes (P_1 A + A^T P_1) - L_1 \otimes BX - (L_1 \otimes BX)^T,$$
$$\theta_{13} = (L_1 \otimes P_1 D)^T + I_N \otimes Y_2 C, \ \theta_{33} = I_N \otimes (P_1 A + A^T P_1) + I_N \otimes Y_1 C + (I_N \otimes Y_1 C)^T.$$

Proof
Select the following Lyapunov functional for system (8)

$$V_1(t) = \varepsilon^T(t)\bar{P}\varepsilon(t), \tag{10}$$

where

$$\bar{P} = \begin{bmatrix} I_N \otimes P_1 & 0 & 0 \\ 0 & I_N \otimes P_1 & 0 \\ 0 & 0 & I_N \otimes P_1 \end{bmatrix},$$

and P_1 is positive definite and symmetric. Calculating the time derivative of $V_1(t)$ along the trajectory of (8) yields

$$\dot{V}_1(t) = \varepsilon^T(t)(\bar{A}^T\bar{P} + \bar{P}\bar{A})\varepsilon(t) + 2\varepsilon^T(t)\bar{P}\tilde{F}(t).$$

By Lemma 4 and Assumption 4, it has $2\varepsilon^T(t)\bar{P}\tilde{F}(t) \le \varepsilon^T(t)\bar{P}\bar{P}^T\varepsilon(t) + \tilde{F}^T(t)\tilde{F}(t) \le \varepsilon^T(t)\bar{P}\bar{P}^T\varepsilon(t) + \varepsilon^T(t)\Omega\varepsilon(t)$, where

$$\Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha^2(I_N \otimes E^T E) & 0 \\ 0 & 0 & \alpha^2(I_N \otimes E^T E) \end{bmatrix},$$

Obviously, it gives

$$\dot{V}_1(t) \le \varepsilon^T(t)\Pi\varepsilon(t), \tag{11}$$

where

$$\Pi = \begin{bmatrix} \phi_{11} & 0 & \phi_{13} & I_N \otimes P_1 & 0 & 0 & 0 & 0 \\ * & \phi_{22} & I_N \otimes P_1 G_1 C & 0 & I_N \otimes P_1 & 0 & \alpha(I_N \otimes E^T) & 0 \\ * & * & \phi_{33} & 0 & 0 & I_N \otimes P_1 & 0 & \alpha(I_N \otimes E^T) \\ * & * & * & -I_{nN} & 0 & 0 & 0 & 0 \\ * & * & * & * & -I_{nN} & 0 & 0 & 0 \\ * & * & * & * & * & -I_{nN} & 0 & 0 \\ * & * & * & * & * & * & -I_{nN} & 0 \\ * & * & * & * & * & * & * & -I_{nN} \end{bmatrix},$$

$$\phi_{11} = I_N \otimes (P_1 S + S^T P_1),\ \phi_{22} = I_N \otimes (P_1 A + A^T P_1) - L_1 \otimes P_1 BK - (L_1 \otimes P_1 BK)^T,$$
$$\phi_{13} = (L_1 \otimes P_1 D)^T + I_N \otimes P_1 G_2 C,\ \phi_{33} = I_N \otimes (P_1 A + A^T P_1) + I_N \otimes P_1 G_1 C + (I_N \otimes P_1 G_1 C)^T.$$

If there exist matrices P_{11} and P_{22} satisfying Lemma 2, by using $P_1 T^T [Q^T\ 0]^T W^T = T^T [Q^T\ 0]^T W^T P$ in Lemma 2, then a nonsingular matrix $P = (W^T)^{-1} Q^{-1} P_{11} Q W^T$ is obtained which satisfies $BP = P_1 B$. Furthermore, by $K = WQ^{-1} P_{11}{}^{-1} Q W^T X$, $G_1 = P_1^{-1} Y_1$ and $G_2 = P_1^{-1} Y_2$ in Theorem 1, it is obtained that $X = PK$, $Y_1 = P_1 G_1$ and $Y_2 = P_1 G_2$.

Therefore, it is obvious to see that $\Pi < 0$ in (11) is equivalent to (9), implying that $\dot{V}_1(t) < 0$ holds. Further, it follows from the closed-loop systems (8) that $\bar{\eta}(t) \to 0$, $\hat{\eta}(t) \to 0$ and $e(t) \to 0$ as $t \to \infty$, then it has $\eta(t) \to 0$ as $t \to \infty$, which means $(L_1 \otimes I_n) x_F(t) + (L_2 \otimes I_n) x_R(t) \to 0$. Hence, by Lemma 1, it can be derived that $x_F(t) \to -(L_1{}^{-1} L_2 \otimes I_n) x_R(t)$ as $t \to \infty$. That is to say, all followers asymptotically converge to the convex spanned by all leaders as $t \to \infty$ which complies with the character in Definition 1, meaning that the distributed containment control problem for MASs with inherent nonlinear dynamics and disturbance rejection has been solved. The proof is complete. □

Remark 3

In [27], an output feedback observer is presented to solve the containment control problem with disturbance rejection, and the topology considered here is undirected. And in [17], the containment control problem is addressed under directed topology, in which no exogenous disturbance is considered. Compared with the linear dynamics of [27] and [17], however, the nonlinear dynamics considered in [23–25] are more practical and challenging. In view of these facts, the protocol (3) is conceived based on the output feedback strategy to handle the containment control problem where exogenous disturbances and inherent nonlinear dynamics are involved simultaneously, and the communication topology considered here is directed. Moreover, it is easy to find that no matter how many agents are involved, processing the LMI (9) yields the gain matrices K, G_1 and G_2 in Theorem 1. Even though the form of followers N becomes more complex and the calculation of LMI (9) increases, the Theorem 1's scalability and reasonableness for many followers is still guaranteed.

3.2. Non-fragile Containment Control Problem

In the last subsection, the distributed containment control problem for MASs is discussed, in which no controller or observer gain variations are considered. In order to handle the problem with gains variations in the controller and observer designs, a novel non-fragile disturbance observer-based

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control protocol is proposed as follows:

$$\begin{aligned} &u_i(t) = 0,\ i \in \mathscr{R}, \\ &u_i(t) = -(K+\Delta K)\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - V\hat{d}_i(t),\ i\in\mathscr{F}, \end{aligned} \tag{12}$$

with

$$\begin{aligned} &\dot{\hat{x}}_i(t) = A\hat{x}_i(t) - (G_1+\Delta G_1)(y_i(t)-\hat{y}_i(t)),\ i\in\mathscr{R}, \\ &\hat{d}_i(t) = 0,\ i\in\mathscr{R}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + D\hat{d}_i(t) - (G_1+\Delta G_1)(y_i(t)-\hat{y}_i(t)),\ i\in\mathscr{F}, \\ &\dot{\hat{d}}_i(t) = S\hat{d}_i(t) - (G_2+\Delta G_2)\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}((y_i(t)-y_j(t)) - (\hat{y}_i(t)-\hat{y}_j(t))),\ i\in\mathscr{F}, \end{aligned} \tag{14}$$

where $\hat{x}_i(t)$, $\hat{y}_i(t)$, $\hat{d}_i(t)$, K, G_1 and G_2 are the same as those defined in the last subsection. Note that the uncertainties matrices ΔK, ΔG_1 and ΔG_2 denote the possible controller and observer gain variation. Furthermore, the gain perturbations ΔK, ΔG_1 and ΔG_2 are represented in the following form:

$$\Delta G_1 = E_1H_1(t)F_1,\ \Delta G_2 = E_2H_2(t)F_2,\ \Delta K = E_3H_3(t)F_3, \tag{15}$$

where E_1, F_1, E_2, F_2, E_3 and F_3 are known matrices with appropriate dimensions, and the unknown matrices $H_1(t)$, $H_2(t)$ and $H_3(t)$ are described by $H_1(t)^TH_1(t)\le I$, $H_2(t)^TH_2(t)\le I$ and $H_3(t)^TH_3(t)\le I$.

Then, based on (12), the system (1) is rewritten as

$$\begin{aligned} &\dot{x}_i(t) = Ax_i(t) + Ef(t,\,x_i(t)),\ i\in\mathscr{R}, \\ &\dot{x}_i(t) = Ax_i(t) - B(K+\Delta K)\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - BV\hat{d}_i(t) + Dd_i(t) + Ef(t,\,x_i(t)),\ i\in\mathscr{F}, \end{aligned} \tag{16}$$

According to (1) and (12)-(16), one gets

$$\begin{aligned} &\dot{x}_i(t) = Ax_i(t) + Ef(t,\,x_i(t)),\ i\in\mathscr{R}, \\ &\dot{\hat{x}}_i(t) = A\hat{x}_i(t) - (G_1+\Delta G_1)C(x_i(t)-\hat{x}_i(t)),\ i\in\mathscr{R}, \end{aligned}$$

and

$$\begin{aligned} &\dot{x}_i(t) = Ax_i(t) - B(K+\Delta K)\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - BF\hat{d}_i(t) + Dd_i(t) + Ef(t,\,x_i(t)),\ i\in\mathscr{F}, \\ &\dot{\hat{x}}_i(t) = A\hat{x}_i(t) - B(K+\Delta K)\sum_{j\in\mathscr{R}\cup\mathscr{F}} a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) - (G_1+\Delta G_1)C(x_i(t)-\hat{x}_i(t)),\ i\in\mathscr{F}, \end{aligned}$$

which are rewritten in compact forms

$$\begin{aligned} &\dot{x}_R(t) = (I_M\otimes A)x_R(t) + (I_M\otimes E)F(t,\,x_R), \\ &y_R(t) = (I_M\otimes C)x_R(t), \\ &\dot{\hat{x}}_R(t) = (I_M\otimes(A+(G_1+\Delta G_1)C))\hat{x}_R(t) - (I_M\otimes(G_1+\Delta G_1)C)x_R(t), \end{aligned}$$

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and

$$\begin{aligned}
\dot{x}_F(t) =&(I_N \otimes A)x_F(t) - (L_1 \otimes B(K+\Delta K))\hat{x}_F(t) - (L_2 \otimes B(K+\Delta K))\hat{x}_R(t)\\
&+ (I_N \otimes D)e(t) + (I_N \otimes E)F(t,\, x_F),\\
y_F(t) =&(I_N \otimes C)x_F(t),\\
\dot{\hat{x}}_F(t) =&(I_N \otimes (A+(G_1+\Delta G_1)C) - L_1 \otimes B(K+\Delta K))\hat{x}_F(t) - (I_N \otimes (G_1+\Delta G_1)C)x_F(t)\\
&- (L_2 \otimes B(K+\Delta K))\hat{x}_R(t),
\end{aligned}$$

Then, one has

$$\dot{e}(t) = (I_N \otimes S)e(t) + (L_1 \otimes (G_2+\Delta G_2)C)(x_F(t) - \hat{x}_F(t)) + (L_2 \otimes (G_2+\Delta G_2)C)(x_R(t) - \hat{x}_R(t)).$$

Further, according to the same procedures in the last subsection, it yields

$$\begin{aligned}
\dot{e}(t) &= (I_N \otimes S)e(t) + (I_N \otimes (G_2+\Delta G_2)C)\bar{\eta}(t),\\
\dot{\hat{\eta}}(t) &= (I_N \otimes A - L_1 \otimes B(K+\Delta K))\hat{\eta}(t) - (I_N \otimes (G_1+\Delta G_1)C)\bar{\eta}(t),\\
\dot{\bar{\eta}}(t) &= (I_N \otimes (A+(G_1+\Delta G_1)C))\bar{\eta}(t) + (L_1 \otimes D)e(t) + \bar{F}(t).
\end{aligned} \tag{17}$$

Based on (17), the error system is expressed in a compact form

$$\dot{\varepsilon}(t) = \tilde{A}\varepsilon(t) + \tilde{F}(t), \tag{18}$$

where

$$\tilde{A} = \begin{bmatrix} I_N \otimes S & 0 & I_N \otimes (G_2+\Delta G_2)C \\ 0 & I_N \otimes A - L_1 \otimes B(K+\Delta K) & -I_N \otimes (G_1+\Delta G_1)C \\ L_1 \otimes D & 0 & I_N \otimes (A+(G_1+\Delta G_1)C) \end{bmatrix}.$$

Thus, the responding results about non-fragile containment control with external disturbance is presented as follows.

Theorem 2

Suppose that the directed topology satisfies Assumption 1, the feedback gain, state observer gain and disturbance observer gain are designed by $K = WQ^{-1}P_{11}{}^{-1}QW^TX$, $G_1 = P_1^{-1}Y_1$ and $G_2 = P_1^{-1}Y_2$ such that MAS (1) with external disturbance system (2) under the control protocol (12) solves the distributed non-fragile containment control problem with external disturbance if there exist matrices $P_{11} > 0$, $P_{22} > 0$ and matrices X, Y_1 and Y_2 such that the following LMI holds:

$$\begin{bmatrix} \tilde{\Pi} & \tilde{M} \\ * & -\tilde{N} \end{bmatrix} < 0, \tag{19}$$

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where

$$\tilde{\Pi}=\begin{bmatrix}\theta_{11} & 0 & \theta_{13} & I_N\otimes P_1 & 0 & 0 & 0 & 0\\ * & \theta_{22} & I_N\otimes Y_1C & 0 & I_N\otimes P_1 & 0 & \alpha(I_N\otimes E^T) & 0\\ * & * & \theta_{33} & 0 & 0 & I_N\otimes P_1 & 0 & \alpha(I_N\otimes E^T)\\ * & * & * & -I_{nN} & 0 & 0 & 0 & 0\\ * & * & * & * & -I_{nN} & 0 & 0 & 0\\ * & * & * & * & * & -I_{nN} & 0 & 0\\ * & * & * & * & * & * & -I_{nN} & 0\\ * & * & * & * & * & * & * & -I_{nN}\end{bmatrix},$$

$$\tilde{M}=[M_1,\, M_2,\, M_3,\, M_4,\, k_1N_1^T,\, k_2N_2^T,\, k_3N_3^T,\, k_4N_4^T],\ \tilde{N}=diag\{k_1,\, k_2,\, k_3,\, k_4,\, k_1,\, k_2,\, k_3,\, k_4\},$$
$$M_1^T=[(I_N\otimes P_1E_2)^T,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0],\ M_2^T=[0,\, -(L_1\otimes P_1BE_3)^T,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0],$$
$$M_3^T=[0,\, (I_N\otimes P_1E_1)^T,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0],\ M_4^T=[0,\, 0,\, (I_N\otimes P_1E_1)^T,\, 0,\, 0,\, 0,\, 0,\, 0],$$
$$N_1=[0,\, 0,\, I_N\otimes F_2C,\, 0,\, 0,\, 0,\, 0,\, 0],\ N_2=[0,\, I_N\otimes F_3,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0],$$
$$N_3=[0,\, 0,\, I_N\otimes F_1C,\, 0,\, 0,\, 0,\, 0,\, 0],\ N_4=[0,\, 0,\, I_N\otimes F_1C,\, 0,\, 0,\, 0,\, 0,\, 0].$$

and θ_{11}, θ_{22}, θ_{13} and θ_{33} are defined in Theorem 1.

Proof

Construct the following Lyapunov functional for system (18)

$$V_2(t)=\varepsilon^T(t)\bar{P}\varepsilon(t), \tag{20}$$

According to the same method of calculating the time derivative of $V_1(t)$, one has

$$\dot{V}_2(t)\leq\varepsilon^T(t)(\Pi+\Delta\Pi)\varepsilon(t), \tag{21}$$

where Π is given in the proof of Theorem 1, and

$$\Delta\Pi=\begin{bmatrix}0 & 0 & I_N\otimes P_1E_2H_2(t)F_2C\\ * & \Delta\Pi_{22} & I_N\otimes P_1E_1H_1(t)F_1C\\ * & * & \Delta\Pi_{33}\end{bmatrix},$$
$$\Delta\Pi_{22}=-L_1\otimes P_1BE_3H_3(t)F_3-(L_1\otimes P_1BE_3H_3(t)F_3)^T,$$
$$\Delta\Pi_{33}=I_N\otimes P_1E_1H_1(t)F_1C+(I_N\otimes P_1E_1H_1(t)F_1C)^T.$$

By using matrix transformation, it yields that $\Pi+\Delta\Pi<0$ is equivalent to

$$\begin{aligned}\Pi+N_1^TH_2^T(t)M_1^T+&M_1H_2(t)N_1+N_2^TH_3^T(t)M_2^T+M_2H_3(t)N_2\\ &+N_3^TH_1^T(t)M_3^T+M_3H_1(t)N_3+N_4^TH_1^T(t)M_4^T+M_4H_1(t)N_4<0,\end{aligned} \tag{22}$$

applying Lemma 3 to (22), it is derived that

$$\begin{aligned}\Pi+k_1N_1^TN_1+k_1^{-1}M_1M_1^T+k_2N_2^TN_2+&k_2^{-1}M_2M_2^T\\ &+k_3N_3^TN_3+k_3^{-1}M_3M_3^T+k_4N_4^TN_4+k_4^{-1}M_4M_4^T<0,\end{aligned} \tag{23}$$

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according to Schur complement, then (23) is equivalent to

$$\begin{bmatrix} \Pi & \tilde{M} \\ * & -\tilde{N} \end{bmatrix} < 0. \tag{24}$$

Thus, it is obvious to see that the inequality (24) is equivalent to the inequality (19), implying that $\dot{V}_2(t) < 0$ holds. Then, by using the same method made in the proof of Theorem 1, it follows from the closed-loop systems (18) that $\bar{\eta}(t) \to 0$, $\hat{\eta}(t) \to 0$ and $e(t) \to 0$ as $t \to \infty$, then it has $\eta(t) \to 0$ as $t \to \infty$, which means $(L_1 \otimes I_n)x_F(t) + (L_2 \otimes I_n)x_R(t) \to 0$. Hence, by Lemma 1, it can be derived that $x_F(t) \to -({L_1}^{-1}L_2 \otimes I_n)x_R(t)$ as $t \to \infty$. Namely, all followers asymptotically converge to the convex spanned by the leaders as $t \to \infty$, meaning that the non-fragile containment control problem for MASs with inherent nonlinear dynamics and external disturbance has been solved. The proof is complete. □

Remark 4

Compared with [27] where a distributed disturbance observer-type protocol with output feedback control strategy is proposed for undirected communication topology, a non-fragile observer-type containment protocol (12) is conceived to solve the containment disturbance rejection problem, in which both the controller gain variations and directed communication topology are considered. Furthermore, in the works of [17, 22–24, 27, 29–32] where only part factors of non-fragility, exogenous disturbances and nonlinearity are considered. In light of these, the containment control problem involved with all these three factors for nonlinear MASs in this paper is addressed.

Remark 5

Note that the relevant system parameters should be chose reasonably during the simulation to guarantee the solvability of the LMIs (9) and (19). Furthermore, it is shown that the control input matrix B is full-column rank. Besides, the nonlinear function $f(t,\ x_i(t))$ considered here is locally Lipschitz continuous and satisfies Assumption 4.

4. SIMULATION EXAMPLE

In this section, two examples are given to demonstrate the theoretical results. The communication topology considered here is a directed communication topology in Fig. 1, where the agents indexed by $7-9$ are leaders and the others are followers. The initial states of x_{i1} and x_{i2} are randomly chosen within $[-15\ \ 15] \times [-15\ \ 15]$.

Example 1: The parameters of (1) and (2) are selected as follows:

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, S = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}, V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T,$$

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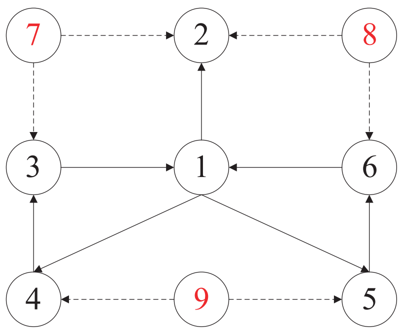
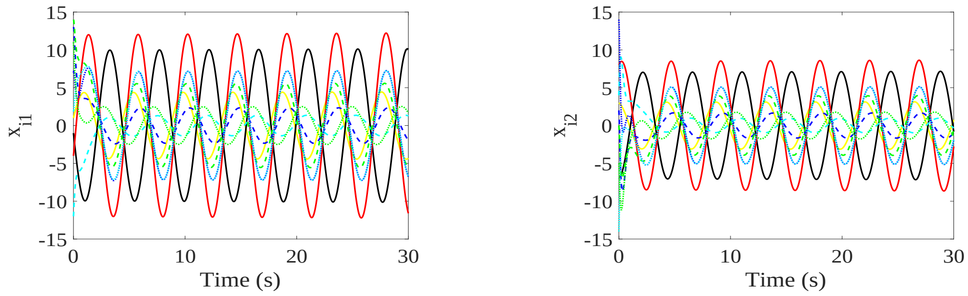


Figure 1. The communication topology.



(a) $x_{i1}(t)$ (b) $x_{i2}(t)$

Figure 2. State trajectories of the leaders and followers of Theorem 1.

and suppose $\alpha = 0.5$ and $f(t,\, x_i(t)) = 0.5 \times \sin(t) \times x_i(t)$. By solving the LMI (9) in Theorem 1, feasible results are derived that:

$$P_{11} = 11.61,\ P_{22} = 19.08,\ X = [57.67\ \ 96.57],$$
$$P_1 = \begin{bmatrix} 17.58 & 2.99 \\ 2.99 & 13.10 \end{bmatrix},\ Y_1 = \begin{bmatrix} -62.87 \\ -94.78 \end{bmatrix},\ Y_2 = \begin{bmatrix} -0.03 \\ -37.03 \end{bmatrix},$$

and the controller and observer gain matrices are obtained as:

$$K = [4.97\ \ 8.32],\ G_1 = \begin{bmatrix} -2.44 \\ -6.68 \end{bmatrix},\ G_2 = \begin{bmatrix} 0.50 \\ -2.94 \end{bmatrix}.$$

Fig. 2 depicts that the state trajectories of the followers stay inside the region spanned by those of the leaders whose curves are denoted by solid lines. Furthermore, Fig. 3 and Fig. 4 show the disturbance observer errors and containment errors of the close-loop system asymptotically converge to zero as $t \to \infty$. Then, the containment control problem of MASs with external disturbance is solved.

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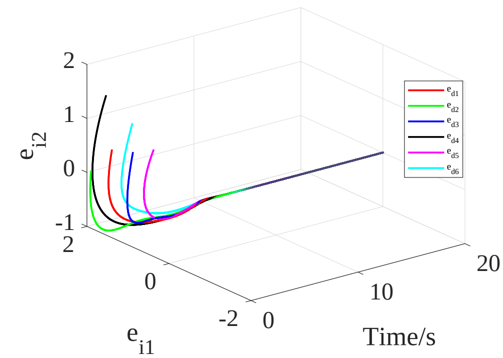

Figure 3. Disturbance observer errors of each states of followers of Theorem 1.

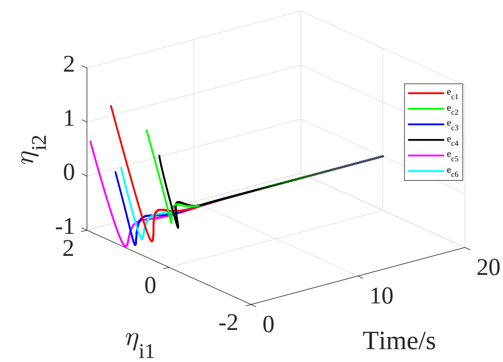

Figure 4. Containment errors of each states of followers of Theorem 1.

Example 2: In this example, the effectiveness of Theorem 2 is illustrated. The parameters of system (1) with (2) and control protocol (12) with (13) are selected as follows:

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, A = \begin{bmatrix} 0 & 2.5 \\ -1.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$S = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T, E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$E_3 = [0.5 \;\; 0], F_1 = 0.5, F_2 = 0.5, F_3 = [0.5 \;\; 0],$$

and suppose $\alpha = 0.5$ and $f(t, x_i(t)) = 0.5 \times \sin(t) \times x_i(t)$. By solving the LMI (19) in Theorem 2, feasible results are derived that:

$$P_{11} = 3.48, P_{22} = 5.98, k_1 = 18.23, k_2 = 13.58, k_3 = 18.10, k_4 = 17.65,$$
$$X = [30.06 \;\; 51.04], P_1 = \begin{bmatrix} 5.48 & 1.00 \\ 1.00 & 3.98 \end{bmatrix}, Y_1 = \begin{bmatrix} -24.16 \\ -38.02 \end{bmatrix}, Y_2 = \begin{bmatrix} 0.01 \\ -9.73 \end{bmatrix},$$

and the controller and observer gain matrices are obtained as:

$$K = [8.65 \;\; 14.67], G_1 = \begin{bmatrix} -2.79 \\ -8.85 \end{bmatrix}, G_2 = \begin{bmatrix} 0.47 \\ -2.56 \end{bmatrix}.$$

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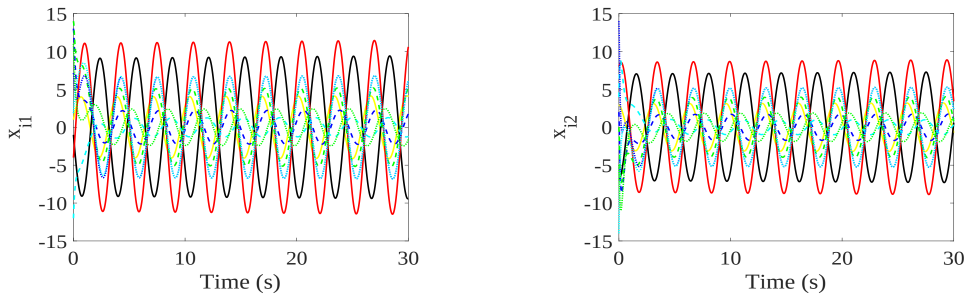
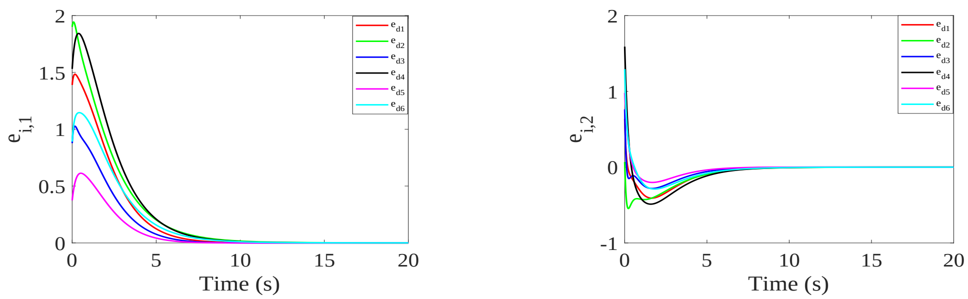


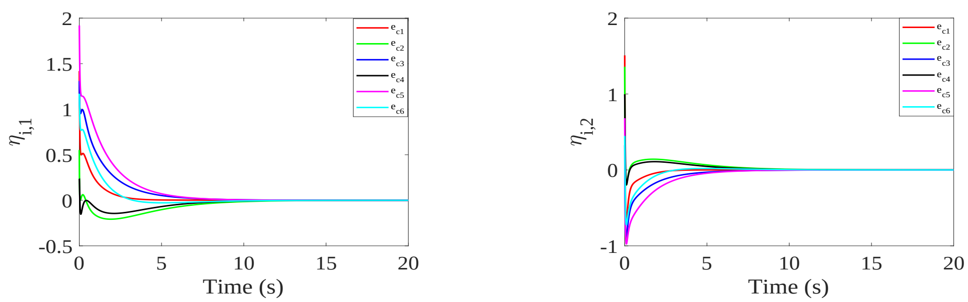
(a) $x_{i1}(t)$ (b) $x_{i2}(t)$

Figure 5. State trajectories of the leaders and followers of Theorem 2.



(a) $x_{i1}(t)$ (b) $x_{i2}(t)$

Figure 6. Disturbance observer errors of each states of followers of Theorem 2.



(a) $x_{i1}(t)$ (b) $x_{i2}(t)$

Figure 7. Containment errors of each states of followers of Theorem 2.

Fig. 5 depicts that the state trajectories of the followers stay inside the region spanned by those of the leaders whose curves are denoted by solid lines. Furthermore, Fig. 6 and Fig. 7 show the disturbance observer errors and containment errors of the close-loop system asymptotically converge to zero as $t \to \infty$, which illustrates that the theoretical results are effective.

5. CONCLUSION

In the paper, the non-fragile containment control problems of MASs over directed communication network with external disturbances have been studied. The disturbances generated by an exogenous system, as well as the uncertainties in the designing of the observer and controller gains, are allowed to take place in the practical control systems. A class of distributed disturbance observer-based controller depending on output feedback strategy is developed. Furthermore, by transforming such problem into the asymptotic stability analysis problem of some containment error dynamics of MASs, the corresponding designed observer and controller gains are obtained if the derived LMI is solvable. Thus, the containment disturbance rejection problem involved with non-fragility is solved by using the disturbance observer-based method. Finally, simulation results are presented to verify the effectiveness of the proposed control schemes. One of the future research topics would be the non-fragile formation-containment control problems of heterogeneous MASs with external disturbances.

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