Mittag-Leffler-Gould-Hopper polynomials: Symbolic Approach

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Abstract

The paper describes the method of symbolic evaluation that serves as a useful tool to extend the studies of certain special functions including their properties and capabilities. In the paper, we exploit certain symbolic operators to introduce a new family of special polynomials, which is called the Mittag-Leffler-Gould-Hopper polynomials. We obtain the generating function, series definition and symbolic operational rule for these polynomials. This approach give a wide platform to explore the study of classical and hybrid special polynomials. We establish summation formulae and certain identities for these polynomials. Further, we derive the multiplicative and derivative operators to study the quasi-monomiality property of these polynomials. Some concluding remarks are also given.

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Keywords: Mittag-Leffler functions; Gould-Hopper polynomials; Symbolic operator; Operational rules; Summation formulae.

MSC: 33E12; 33C45; 33F10.

1 Introduction

The multi-variable special polynomials provide the solutions of a wide class of partial differential equations often encountered in the field of physical problems. The importance of multi-variable Hermite polynomials has been recognized in dealing with quantum mechanical and optical beam transport problems [8, 14, 15]. It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums involving special functions. Problems of this type arise, for example, in the computation of the higher-order moments of a distribution or to evaluate transition matrix elements in quantum mechanics. It has been shown that the summation formulae of certain special functions, often encountered in applications ranging from electromagnetic processes to combinatorics, can be written in terms of the multi-variable Hermite polynomials. [8].
Throughout this paper, we use the notations: $\mathbb{N} = \{1, 2, 3...\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
Also, as usual $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.

We recall that the Mittag-Leffler function is given by the following series definition [3, 12]:

$$E_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + \beta)}, \ \forall x \in \mathbb{R}, \forall \alpha, \beta \in \mathbb{R}^+, \tag{1.1}$$

which plays an important role in the solution of problems arising in fractional calculus.

The symbolic method provides powerful and efficient means to introduce and to study certain new and known special functions, for instance, the symbolic method is used to obtain certain lacunary generating functions for the Laguerre polynomials by Dattoli [3]. Dattoli and his co-workers [3, 13] introduced a symbolic operator $\hat{c}$, which operates on a vacuum function $\phi_z = \frac{1}{\Gamma(z+1)}$ as [3, 13]:

$$\hat{c}^\alpha \phi_z = \frac{1}{\Gamma(z + \alpha + 1)}, \tag{1.2}$$

which obviously satisfies the properties

$$\hat{c}^\alpha \hat{c}^\beta = \hat{c}^{\alpha+\beta} \quad \text{and} \quad (\hat{c}^\alpha)^r = \hat{c}^{r\alpha}. \tag{1.3}$$

In view of equation (1.2), we have

$$\hat{c}^{\alpha}\phi_0 = \left.\frac{1}{\Gamma(z+\alpha+1)}\right|_{z=0} = \frac{1}{\Gamma(1+\alpha)}. \tag{1.4}$$

In view of equations (1.1) and (1.4), the Mittag-Leffler function can symbolically be defined as [12]:

$$E_{\alpha,\beta}(x) = \hat{c}^{\beta-1}\frac{1}{1-\hat{c}^{\alpha}x}\phi_0. \tag{1.5}$$

Also, Dattoli *et al.* [12] introduced another symbolic operator ${}_{\alpha,\beta}\hat{d}$ $(\alpha,\beta\in\mathbb{R}^+)$, which operates on the vacuum function ψ_0 as:

$${}_{\alpha,\beta}\hat{d}^k\psi_0 = \frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)} \tag{1.6}$$

Evidently, for $k=0$, equation (1.6) gives

$$\psi_0 = \frac{1}{\Gamma(\beta)}. \tag{1.7}$$

In view of equations (1.1) and (1.6), the symbolic definition of Mittag-Leffler function in terms of ${}_{\alpha,\beta}\hat{d}$ can be given as [12]:

$$E_{\alpha,\beta}(x) = e^{x\,{}_{\alpha,\beta}\hat{d}}\psi_0. \tag{1.8}$$

The special polynomials of two variables are important from the point of view of applications. These polynomials allow the derivation of a number of useful identities in a fairly straightforward way and help in introducing new families of special polynomials. For example, Bretti *et al.* [5] introduced general classes of the Appell polynomials of two variables by using properties of an iterated isomorphism related to the Laguerre-type exponentials. To extend this new and significant approach, the hybrid class of the q-Sheffer-Appell polynomials are introduced in [32]. The two variable forms of the Hermite, Laguerre and truncated exponential polynomials as well as their generalizations are studied by several researchers [2, 6, 9, 16, 20, 24, 29, 30].
To solve the problems arising in many branches of mathematics, going from the theory of partial differential equations to abstract group theory, requirement of multi-index and multi-variable special functions are realized. The theory of multi-index and multi-variable Hermite polynomials was initially developed by Hermite [19]. The Hermite polynomials turn up in combinatorics, as an example of an Appell sequence, obeying the umbral calculus, in numerical analysis as Gaussian quadrature, in physics, where they give rise to the eigen states of the quantum harmonic oscillator and also turn up in the solution of the Schr̈odinger equation for the harmonic oscillator [33]. Recently Raza *et al.* studied the properties of Hermite polynomials by using umbral method [27].

The Gould-Hopper polynomials can be realised as a generalization of 2-variable Hermite-Kampé de Fériet polynomials. The Gould-Hopper polynomials (GHP) $H_n^{(m)}(x,y)$ are defined by means of the following generating function and series definition [18, 22]:

$$\sum_{n=0}^{\infty} H_n^{(m)}(x,y)\frac{\xi^n}{n!} = e^{x\xi+y\xi^m} \tag{1.9}$$

and

$$H_n^{(m)}(x,y) = n!\sum_{r=0}^{[\frac{n}{m}]}\frac{x^{n-mr}y^r}{(n-mr)!r!}, \tag{1.10}$$

respectively, where m is positive integer.

The Gould-Hopper polynomials are the solutions of the generalized heat equation [7]:

$$\frac{\partial}{\partial y}f(x,y) = \frac{\partial^m}{\partial x^m}f(x,y), \tag{1.11}$$

with the initial condition $f(x,0)=x^n$.

The Gould- Hopper polynomials are given by the following operational rule [7]:

$$H_n^{(m)}(x,y) = \exp\left(yD_x^m\right)\{x^n\}, \tag{1.12}$$

where

$$D_x := \frac{\partial}{\partial x}. \tag{1.13}$$

Gould and Hopper [18] used the notation $g_n^s(x,y)$ for the Gould-Hopper polynomials, but due to their direct link with the Hermite polynomials, in this paper, we use the notation $H_n^{(m)}(x,y)$ for these polynomials like some other researcher.

In particular, for $m = 2$, we note that

$$H_n^{(2)}(x,y) = H_n(x,y), \tag{1.14}$$

where $H_n(x,y)$ denotes the 2-variable Hermite-Kampé de Fériet polynomials (2VHKdFP), defined by the following generating function [2]:

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2}. \tag{1.15}$$

Also, we note from [1, 2] that

$$H_n(2x,-1) = H_n(x), \tag{1.16}$$

where $H_n(x)$ denotes the ordinary Hermite polynomials.

Many properties of conventional and generalised special polynomials have been shown to be derived, in a straightforward way from [1, 2], within the operational framework which is a consequence of the monomiality principle. The idea of the monomiality is based on the concept of poweroid suggested by Steffensen [31]. It was reformulated and developed by Dattoli [6]. According to the monomiality principle, a polynomial set $\{p_n(x)\}_{n=0}^{\infty}$ is called quasi-monomial, if there exist two operators, multiplicative operator $\hat{M}$ and derivative operator $\hat{P}$, respectively, such that [6]

$$\hat{M}\{p_n(x)\} = p_{n+1}(x) \tag{1.17}$$

and

$$\hat{P}\{p_n(x)\} = np_{n-1}(x). \tag{1.18}$$

Thus, the operators $\hat{M}$ and $\hat{P}$ display a weyl group structure [6]. Several characteristics of polynomial $p_n(x)$ can be obtained by using the operators $\hat{M}$ and $\hat{P}$. If $\hat{M}$ and $\hat{P}$ have differential realizations, then the polynomial $p_n(x)$ satisfies the following differential equation:

$$\hat{M}\hat{P}\{p_n(x)\} = np_n(x). \tag{1.19}$$

Assuming here and in the following $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as:

$$p_n(x) = \hat{M}^n\{1\}. \tag{1.20}$$

In view of equation (1.20), we have

$$\hat{M}_H^n\{1\} = H_n^{(m)}(x,y). \tag{1.21}$$

The multiplicative and derivative operators for the Gould-Hopper polynomials are as follows:

$$\hat{M}_g := x + myD_x^{m-1} \tag{1.22}$$

and

$$\hat{P}_g := D_x. \tag{1.23}$$

There is continuous use of operational methods in research fields like quantum and classical optics.
In this paper, we use symbolic method to introduce and to study the Mittag-Leffler-Gould-Hopper polynomials. In Section 2, the Mittag-Leffler-Gould-Hopper polynomials are introduced and their certain

properties such as generating function, series definition and operational rule are derived. In Section 3, an integral representation and certain summation formulae for Mittag-Leffler-Gould-Hopper polynomials are obtained. In the Section 4, the monomiality property of these polynomials are investigated and their multiplicative and derivative operators are obtained. Further, in Section 5, certain special cases of the results established in this paper, are considered. In Section 6, the graphical interpretation of these polynomials is presented. In the last Section, some concluding remarks are given.

2 Mittag-Leffler-Gould-Hopper Polynomials

In this section, we introduce and study the Mittag-Leffler-Gould-Hopper polynomials. In view of equation (1.8), we define the Mittag-Leffler-Gould-Hopper polynomials (MLGHP) ${}_EH_n^{(m)}(x,y;\alpha,\beta)$ as:

$$ {}_EH_n^{(m)}(x,y;\alpha,\beta) = H_n^{(m)}(x,y_{\alpha,\beta}\hat{d})\psi_0. \tag{2.1}$$

Now, we proceed to obtain the generating function and series definition of the Mittag-Leffler-Gould-Hopper polynomials.

The following result gives the generating function of Mittag-Leffler-Gould-Hopper polynomials:

Theorem 2.1. *The following generating function for the Mittag-Leffler-Gould-Hopper polynomials holds true:*

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = e^{x\xi}E_{\alpha,\beta}(y\xi^m). \tag{2.2}$$

Proof. Using equation (2.1), we have

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} H_n^{(m)}(x,y_{\alpha,\beta}\hat{d})\psi_0, \tag{2.3}$$

which on using (1.9) in the right hand side, it becomes

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}\psi_0. \tag{2.4}$$

Since it is obvious that $[x\xi, y_{\alpha,\beta}\hat{d}\xi^m] = 0$. Therefore, using the Weyl decoupling identity [17]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{\frac{-k}{2}}, \qquad k = [\hat{A},\hat{B}] \qquad (k \in \mathbb{C}), \tag{2.5}$$

in the right hand side of equation (2.4) and then using equation (1.8) in the resultant equation, we get assertion (2.2). □

The following result gives the series definition of Mittag-Leffler-Gould-Hopper polynomials:

Theorem 2.2. *The following series definition of the Mittag-Leffler-Gould-Hopper polynomials holds true:*

$$ {}_EH_n^{(m)}(x,y;\alpha,\beta) = n!\sum_{r=0}^{[\frac{n}{m}]} \frac{x^{n-mr}y^r}{(n-mr)!\Gamma(\alpha r+\beta)} \qquad (\alpha,\beta \in \mathbb{R}^+, m \in \mathbb{N}). \tag{2.6}$$

Proof. In view of equations (1.10) and (2.1), we have

$$ {}_EH_n^{(m)}(x,y;\alpha,\beta) = n!\sum_{r=0}^{[\frac{n}{m}]} \frac{x^{n-mr}y^r\left({}_{\alpha,\beta}\hat{d}\right)^r}{(n-mr)!r!}\psi_0, \tag{2.7}$$

which on using equation (1.6), it gives assertion (2.6). □

Now, we establish the following result for higher order partial derivatives of the Mittag-Leffler-Gould-Hopper polynomials:

Theorem 2.3. *The higher order partial derivatives of the Mittag-Leffler-Gould-Hopper polynomials are as follows:*

$$D_x^s \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-s)!} \ {}_EH_{n-s}^{(m)}(x,y;\alpha,\beta) \qquad (s \in \mathbb{N} \wedge s \leq n) \tag{2.8}$$

and

$$D_y^s \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-sm)!} \ {}_{\alpha,\beta}\hat{d}^s \ {}_EH_{n-sm}^{(m)}(x,y;\alpha,\beta) \qquad (s \in \mathbb{N} \wedge s \leq n). \tag{2.9}$$

Proof. Differentiating equation (2.4) partially with respect to x, we find

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} D_x \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \xi e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m} \psi_0. \tag{2.10}$$

Using equation (2.4) in the right hand side of above equation and then comparing the equal powers of ξ, we have

$$D_x \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = n \ {}_EH_{n-1}^{(m)}(x,y;\alpha,\beta). \tag{2.11}$$

Thus, the result (2.8) holds true for $s = 1$. We assume that this result holds for $s = k$, i.e.

$$D_x^k \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-k)!} \ {}_EH_{n-k}^{(m)}(x,y;\alpha,\beta). \tag{2.12}$$

Differentiating equation (2.12) partially with respect to x, we have

$$D_x^{k+1} \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-k)!} \ D_x \ {}_EH_{n-k}^{(m)}(x,y;\alpha,\beta), \tag{2.13}$$

which on using equation (2.11) in the right hand side, becomes

$$D_x^{k+1} \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-(k+1))!} \ {}_EH_{n-(k+1)}^{(m)}(x,y;\alpha,\beta), \tag{2.14}$$

which proves that the result (2.8) holds true for $s = k+1$. Thus, by the method of mathematical induction, result (2.8) holds for all values of $s \in \mathbb{N}$.

Similarly, differentiating equation (2.4) partially with respect to y, we find

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} D_y \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \xi^m \ {}_{\alpha,\beta}\hat{d} \ e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m} \psi_0. \tag{2.15}$$

Using equation (2.4) in the right hand side of above equation and then comparing the equal powers of ξ, we have

$$D_y \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-m)!} {}_{\alpha,\beta}\hat{d} \ {}_EH_{n-m}^{(m)}(x,y;\alpha,\beta). \tag{2.16}$$

Thus, the result (2.9) holds true for $s = 1$. We assume that the result (2.9) holds for $s = k$, i.e.

$$D_y^k \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-km)!} \ {}_{\alpha,\beta}\hat{d}^k \ {}_EH_{n-km}^{(m)}(x,y;\alpha,\beta). \tag{2.17}$$

Differentiating equation (2.17) partially with respect to y, we have

$$D_y^{k+1} \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-km)!} \ {}_{\alpha,\beta}\hat{d}^k D_y \ {}_EH_{n-km}^{(m)}(x,y;\alpha,\beta), \tag{2.18}$$

which on using equation (2.16) in the right hand side, it becomes

$$D_y^{k+1} \ {}_EH_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-(k+1)m)!} \ {}_{\alpha,\beta}\hat{d}^{k+1} \ {}_EH_{n-(k+1)m}^{(m)}(x,y;\alpha,\beta), \tag{2.19}$$

which proves that the result (2.9) holds true for $s = k+1$. Thus by the method of mathematical induction, result (2.9) holds for all values of $s \in \mathbb{N}$.

□

Next, we establish the following result for partial differential equation satisfied by the Mittag-Leffler-Gould-Hopper:

Theorem 2.4. *The Mittag-Leffler-Gould-Hopper polynomials satisfy the following m^{th}-order partial differential equation:*

$$ {}_{\alpha,\beta}\hat{d}\ D_x^m\ {}_E H_n^{(m)}(x,y;\alpha,\beta) = D_y\ {}_E H_n^{(m)}(x,y;\alpha,\beta) \tag{2.20}$$

with initial condition

$$ {}_E H_n^{(m)}(x,0;\alpha,\beta) = \frac{x^n}{\Gamma(\beta)}. \tag{2.21}$$

Proof. Taking $s = m$ in equation (2.8) and then operating ${}_{\alpha,\beta}\hat{d}$ on both sides of resultant equation, we have

$$ {}_{\alpha,\beta}\hat{d}\ D_x^m\ {}_E H_n^{(m)}(x,y;\alpha,\beta) = \frac{n!}{(n-m)!}{}_{\alpha,\beta}\hat{d}\ {}_E H_{n-m}^{(m)}(x,y;\alpha,\beta). \tag{2.22}$$

Using equation (2.16) in the right hand side of above equation, we get assertion (2.20). Also, taking $y = 0$ in equation (2.6), we get the initial condition (2.21).

□

Further, we obtain the following result for operational definition of the Mittag-Leffler-Gould-Hopper polynomials:

Theorem 2.5. *The Mittag-Leffler-Gould-Hopper polynomials satisfy the following operational rule:*

$$ {}_E H_n^{(m)}(x,y;\alpha,\beta) = e^{y_{\alpha,\beta}\hat{d}D_x^m}\{x^n\psi_0\}. \tag{2.23}$$

Proof. The formal solution of equation (2.20) subject to the initial condition (2.21) is given by

$$ {}_E H_n^{(m)}(x,y;\alpha,\beta) = e^{y_{\alpha,\beta}\hat{d}D_x^m}\left\{\frac{x^n}{\Gamma(\beta)}\right\}, \tag{2.24}$$

which on using equation (1.7), gives assertion (2.23).

□

Now, we establish the following symbolic definition of the Mittag-Leffler-Gould-Hopper polynomials:

Theorem 2.6. *The symbolic definition of the Mittag-Leffler-Gould-Hopper polynomials, is as follows:*

$$ {}_E H_n^{(m)}(x,y;\alpha,\beta) = (x + m y_{\alpha,\beta}\hat{d}D_x^{m-1})^n\psi_0. \tag{2.25}$$

Proof. In view of Crofton identity [17]:

$$ e^{\lambda D_x^m}\{f(x)\} = f(x + m\lambda D_x^{m-1})e^{\lambda D_x^m} \tag{2.26}$$

and equation (2.23), we have

$$ {}_E H_n^{(m)}(x,y;\alpha,\beta) = (x + m y_{\alpha,\beta}\hat{d}D_x^{m-1})^n\psi_0 e^{y_{\alpha,\beta}\hat{d}D_x^m}\{1\}. \tag{2.27}$$

Since $e^{\alpha,\beta\hat{d}D_x^m}\{1\} = 1$, therefore equation (2.27) gives assertion (2.25).

□

In the next section, we obtain an integral representation and certain summation formulae for the Mittag-Leffler-Gould-Hopper polynomials.

3 Integral representation and Summation formulae

In this section, we obtain an integral representation and certain summation formulae for the Mittag-Leffler-Gould-Hopper polynomials.

For suitable substitutions for β and y, we obtain the following integral representation of the Mittag-Leffler-Gould-Hopper polynomials:

Theorem 3.1. *For $\beta = 1$ and $y = t^\alpha$, the Mittag-Leffler-Gould-Hopper polynomials satisfy the following identity*

$$ {}_E H_n^{(m)}(x, t^\alpha; \alpha, 1) = \int_0^\infty n_\alpha(s,t) H_n^{(m)}(x,s) ds, \tag{3.1}$$

where $n_\alpha(s,t)$ denotes the inverse one sided Lévy stable density and is given by

$$n_\alpha(s,t) = \frac{1}{\alpha} \frac{t}{s \sqrt[\alpha]{s}} g_\alpha\left(\frac{t}{\sqrt[\alpha]{s}}\right), \tag{3.2}$$

where $g_\alpha(x)$ is the one sided Lévy stable distribution, whose Laplace transform $x \to u$ is $\hat{g}_\alpha(u) = \exp(-u^\alpha)$, recently obtained for α rational [4, 25].

Proof. For $y = t^\alpha$ and $\beta = 1$ equation (2.2) gives

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} {}_E H_n^{(m)}(x, t^\alpha; \alpha, 1) = e^{x\xi} E_{\alpha,1}(t^\alpha \xi^m), \tag{3.3}$$

which on using equation (1.1) in the right hand side, becomes

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} {}_E H_n^{(m)}(x, t^\alpha; \alpha, 1) = e^{x\xi} \sum_{r=0}^{\infty} \frac{t^{\alpha r}}{\Gamma(\alpha r + 1)} \xi^{mr}, \tag{3.4}$$

If $n_\alpha(s,t)$ is given by equation (3.2), then [4] :

$$\int_0^\infty n_\alpha(s,t) \frac{s^r}{r!} ds = \frac{t^{\alpha r}}{\Gamma(\alpha r + 1)} \qquad (\forall \alpha \in \mathbb{R}^+, \forall t \in \mathbb{R}_0^+). \tag{3.5}$$

Using equation (3.5) in equation (3.4), we get

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} {}_E H_n^{(m)}(x, t^\alpha; \alpha, 1) = e^{x\xi} \sum_{r=0}^{\infty} \int_0^\infty n_\alpha(s,t) \frac{s^r}{r!} \xi^{mr} ds,$$

which can also be written as

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} {}_E H_n^{(m)}(x, t^\alpha; \alpha, 1) = \int_0^\infty n_\alpha(s,t) e^{x\xi + s\xi^m} ds. \tag{3.6}$$

Using equation (1.9) in the right hand side and then comparing the equal powers of ξ, we get assertion (3.1).

□

Now, we establish the following summation formulae for the Mittag-Leffler-Gould-Hopper polynomials:

Theorem 3.2. *The following summation formula for the Mittag-Leffler-Gould-Hopper holds true:*

$${}_E H_n^{(m)}(x + v, y; \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} v^k {}_E H_{n-k}^{(m)}(x, y; \alpha, \beta). \tag{3.7}$$

Proof. Replacing x by $x+v$ in the equation (2.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y;\alpha,\beta)\frac{\xi^n}{n!} =&\exp((x+v)\xi+y_{\alpha,\beta}\hat{d}\xi^m)\psi_0\\ =&e^{x\xi+y_{\alpha,\beta}\hat{d}\xi^m}e^{v\xi}\psi_0.\end{aligned} \tag{3.8}$$

Expanding the second exponential of right hand side of above equation and using equation (2.4), we have

$$\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y;\alpha,\beta)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_EH_n^{(m)}(x,y;\alpha,\beta)\frac{\xi^n}{n!}\frac{v^k\xi^k}{k!}. \tag{3.9}$$

Using series re-arrangement formula, it gives

$$\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y;\alpha,\beta)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}v^k\ {}_EH_{n-k}^{(m)}(x,y;\alpha,\beta)\right)\frac{\xi^n}{n!}, \tag{3.10}$$

comparing the coefficients of like powers of ξ in above equation, we get assertion (3.7).

□

Theorem 3.3. *The following summation formula for the Mittag-Leffler-Gould-Hopper polynomials holds true:*

$${}_EH_{k+l}^{(m)}(w,y;\alpha,\beta) = \sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}\,{}_EH_{k+l-n-r}^{(m)}(x,y;\alpha,\beta), \tag{3.11}$$

where $\sum_{n,r=0}^{k,l} := \sum_{n=0}^{k}\sum_{r=0}^{l}$.

Proof. Replacing ξ by $u+\xi$ in (2.4) and then using the formula [28]

$$\sum_{n=0}^{\infty} f(n)\frac{(x+y)^n}{n!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.12}$$

in the resultant equation, we find the following generating function for the Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x,y;\alpha,\beta)$:

$$\exp\left(x(\xi+u)+y_{\alpha,\beta}\hat{d}(\xi+u)^m\right)\psi_0 = \sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(x,y;\alpha,\beta), \tag{3.13}$$

which can be written as

$$\exp(y_{\alpha,\beta}\hat{d}(\xi+u)^m)\psi_0 = \exp(-x(\xi+u))\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(x,y;\alpha,\beta). \tag{3.14}$$

Multiply both sides of the above equation with $\exp w(\xi+u)$ and then using equation (3.12) in the left hand side of the resultant equation, we find

$$\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(w,y;\alpha,\beta) = \exp\left((w-x)(\xi+u)\right)\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(x,y;\alpha,\beta), \tag{3.15}$$

or equivalently

$$\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(w,y;\alpha,\beta) = \sum_{n=0}^{\infty}\frac{(w-x)^n(\xi+u)^n}{n!}\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(x,y;\alpha,\beta), \tag{3.16}$$

which on using equation (3.12) in the first summation on the right hand side, it gives

$$\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(w,y;\alpha,\beta) = \sum_{n,r=0}^{\infty}\frac{(w-x)^{n+r}\xi^n u^r}{n!r!}\sum_{k,l=0}^{\infty}\frac{\xi^k u^l}{k!l!}\,{}_EH_{k+l}^{(m)}(x,y;\alpha,\beta). \tag{3.17}$$

Now, replacing k by $k-n$, l by $l-r$ and using the following identity [28]:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{3.18}$$

in the right hand side of equation (3.17), we find

$$\sum_{k,l=0}^{\infty} \frac{\xi^k u^l}{k!l!}\, {}_EH_{k+l}^{(m)}(w,y;\alpha,\beta) = \sum_{k,l=0}^{\infty}\sum_{n,r=0}^{k,l} \frac{(w-x)^{n+r}\xi^k u^l}{n!r!(k-n)!(l-r)!}\, {}_EH_{k+l-n-r}^{(m)}(x,y;\alpha,\beta). \tag{3.19}$$

Finally, comparing the coefficients of like powers of ξ and u in equation (3.19), we get assertion (3.11).
□

Remark 3.1. For $l=0$ and $w=x+v$, equation (3.11) reduces to equation (3.7).

Theorem 3.4. *The following symbolic operational identity for the Mittag-Leffler-Gould-Hopper polynomials holds true:*

$${}_EH_n^{(m)}(x+v,y+w;\alpha,\beta) = \sum_{k=0}^{n}\binom{n}{k} H_{n-k}^{(m)}(x,y_{\alpha,\beta}\hat{d})\ {}_EH_k^{(m)}(v,w;\alpha,\beta). \tag{3.20}$$

Proof. Replacing x by $x+v$ and y by $y+w$ in generating function (2.4), we have

$$\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y+w;\alpha,\beta)\frac{\xi^n}{n!} = \exp((x+v)\xi + (y+w)_{\alpha,\beta}\hat{d}\xi^m)\psi_0, \tag{3.21}$$

which on using Weyl decoupling identity (2.5), it gives

$$\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y+w;\alpha,\beta)\frac{\xi^n}{n!} = e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m} e^{v\xi + w_{\alpha,\beta}\hat{d}\xi^m}\psi_0. \tag{3.22}$$

Using equations (1.9) and (2.4) in the right hand side of above equation, we find

$$\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y+w;\alpha,\beta)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_n^{(m)}(x,y_{\alpha,\beta}\hat{d})\ {}_EH_k^{(m)}(v,w;\alpha,\beta)\frac{\xi^n}{n!}\frac{\xi^k}{k!}, \tag{3.23}$$

which on using series re-arrangement gives

$$\sum_{n=0}^{\infty} {}_EH_n^{(m)}(x+v,y+w;\alpha,\beta)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} H_{n-k}^{(m)}(x,y_{\alpha,\beta}\hat{d})\ {}_EH_k^{(m)}(v,w;\alpha,\beta)\frac{\xi^n}{n!}.$$

Comparing equal powers of ξ from both sides of above equation, we get assertion (3.20).
□

In the next section, we discuss the quasi-monomialtiy property of the Mittag-Leffler-Gould-Hopper polynomials.

4 Monomiality property

In order to frame the Mittag-Leffler-Gould-Hopper polynomials within the context of monomiality principle, we obtain the following result:

Theorem 4.1. *The multiplicative and derivative operators for the Mittag-Leffler-Gould-Hopper polynomials are as follows:*

$$\hat{M} = \left(x + my_{\alpha,\beta}\hat{d}D_x^{m-1}\right) \tag{4.1}$$

and

$$\hat{P} = D_x. \tag{4.2}$$

Proof. Differentiating equation (2.4) partially with respect to ξ, we have

$$\sum_{n=0}^{\infty} \frac{n\xi^{n-1}}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = (x + my_{\alpha,\beta}\hat{d}\xi^{m-1})e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}\psi_0. \tag{4.3}$$

Since,

$$D_x(e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}) = \xi(e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}). \tag{4.4}$$

Therefore,

$$D_x^r(e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}) = \xi^r(e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}). \tag{4.5}$$

In view of equations (4.3) and (4.5), we have

$$\sum_{n=1}^{\infty} \frac{\xi^{n-1}}{(n-1)!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = (x + m\ y_{\alpha,\beta}\hat{d}D_x^{m-1})e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}\psi_0,$$

which on using equation (2.4) in the right hand side, gives

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH_{n+1}^{(m)}(x,y;\alpha,\beta) = \left(x + my_{\alpha,\beta}\hat{d}D_x^{m-1}\right)\sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta).$$

Comparing equal powers of ξ from both sides of above equation, we have

$${}_EH_{n+1}^{(m)}(x,y;\alpha,\beta) = \left(x + my_{\alpha,\beta}\hat{d}D_x^{m-1}\right){}_EH_n^{(m)}(x,y;\alpha,\beta). \tag{4.6}$$

In view of equation (1.17) and (4.6), we get assertion (4.1).

Now, differentiating equation (2.4) partially with respect to x, we have

$$D_x \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = \xi e^{x\xi + y_{\alpha,\beta}\hat{d}\xi^m}\psi_0,$$

which on using equation (2.4) in the right hand side, gives

$$\sum_{n=0}^{\infty} D_x\ \frac{\xi^n}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = \sum_{n=0}^{\infty} \frac{\xi^{n+1}}{n!}\, {}_EH_n^{(m)}(x,y;\alpha,\beta). \tag{4.7}$$

Comparing equal powers of ξ from both sides of the equation (4.7), we find

$$D_x\, {}_EH_n^{(m)}(x,y;\alpha,\beta) = n\, {}_EH_{n-1}^{(m)}(x,y;\alpha,\beta), \tag{4.8}$$

which in view of equation (1.18), gives assertion (4.2).

□

Remark 4.1. An alternative proof of assertion (4.1) of Theorem 4.1 is as follows:
Replacing n by $n+1$ in equation (2.25) *and then again using equation* (2.25) *in the right hand side of the resultant equation, we get equation* (4.6), *which in view of equation* (1.17), *gives assertion* (4.1).

Remark 4.2. **(i)** In view of equation (4.6) it can easily be verified that

$$\left(x + my\ {}_{\alpha,\beta}\hat{d}\ D_x^{m-1}\right)^r {}_EH_k^{(m)}(x,y;\alpha,\beta) = {}_EH_{k+r}^{(m)}(x,y;\alpha,\beta). \tag{4.9}$$

(ii) Since from equation (2.6), we have

$$ {}_EH_0^{(m)}(x,y;\alpha,\beta) = \frac{1}{\Gamma(\beta)}, \tag{4.10}$$

therefore, in view of equation (1.7), for $k = 0$ and $r = n$ equation (4.9), it gives equation (2.25).

Next, we establish the following recurrence relations for the Mittag-Leffler-Gould-Hopper polynomials.

Theorem 4.2. *The Mittag-Leffler-Gould-Hopper polynomials satisfy the following symbolic and differential recurrence relations:*

$$ {}_EH_{n+1}^{(m)}(x,y;\alpha,\beta) = x\,{}_EH_n^{(m)}(x,y;\alpha,\beta) + my\frac{n!}{(n-m+1)!}\ {}_{\alpha,\beta}\hat{d}\ {}_EH_{n-m+1}^{(m)}(x,y;\alpha,\beta) \tag{4.11}$$

and

$$(n+1)\,{}_EH_{n+1}^{(m)}(x,y;\alpha,\beta) = (n+1)x\,{}_EH_n^{(m)}(x,y;\alpha,\beta) + myD_y\,{}_EH_{n+1}^{(m)}(x,y;\alpha,\beta), \tag{4.12}$$

respectively.

Proof. Using equation (2.8) for $s = m-1$ in the right hand side of equation (4.6), we get assertion (4.11).

Also, replacing n by $n+1$ in equation (2.16) and then using the resultant equation in the right hand side of equation (4.11), we get assertion (4.12).

□

Since, the multiplicative and derivative operators of the Mittag-Leffler-Gould-Hopper polynomials have symbolic-differential realization, therefore we obtain the following result for the symbolic differential equation of the Mittag-Leffler-Gould-Hopper polynomials:

Theorem 4.3. *The Mittag-Leffler-Gould-Hopper polynomials are the solutions of following m^{th}- order symbolic-differential equation:*

$$\left(my_{\alpha,\beta}\hat{d}D_x^m + xD_x - n\right) {}_EH_n^{(m)}(x,y;\alpha,\beta) = 0. \tag{4.13}$$

Proof. In view of equation (1.19), equations (4.1) and (4.2) gives the assertion (4.13). □

In the next section, we give some applications of the results obtained in this paper.

5 Special Cases

In this section, we consider certain special cases of the Mittag-Leffler-Gould-Hopper polynomials and obtain some of their properties by substituting appropriate values for parameters m, α, β and variables x, y in the results established in this paper.

I. Since, in view of equation (1.14) for $m = 2$, the Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x,y;\alpha,\beta)$ reduce to Mittag-Leffler-Hermite polynomials (MLHP) ${}_EH_n(x,y;\alpha,\beta)$, i.e.

$$ {}_EH_n^{(2)}(x,y;\alpha,\beta) = {}_EH_n(x,y;\alpha,\beta). \tag{5.1}$$

Therefore, in view of equation (2.1), ${}_EH_n(x,y;\alpha,\beta)$ are defined as:

$$ {}_EH_n(x,y;\alpha,\beta) = H_n(x,y_{\alpha,\beta}\hat{d})\psi_0. \tag{5.2}$$

Thus, taking $m=2$ in equations (2.2), (2.6), (2.23), (2.25), (3.1), (3.7), (3.11), (3.20), (4.1), (4.2), (4.11), (4.12) and (4.13), we get the respective properties of the Mittag-Leffler-Hermite polynomials, which are listed in Table 5.1.

Table 5.1 Properties of the Mittag-Leffler-Hermite polynomials $_EH_n(x,y;\alpha,\beta)$:

S. No.	Name of the properties	Results
I.	Generating function	$e^{x\xi}E_{\alpha,\beta}(y\xi^2)$
II.	Series definition	$n!\sum_{r=0}^{[\frac{n}{2}]}\frac{x^{n-2r}y^r}{(n-2r)!\Gamma(\alpha r+\beta)}$
III.	Symbolic operational identity	$e^{y_{\alpha,\beta}\hat{d}D_x^2}\{x^n\psi_0\}$
IV.	Symbolic definition	$(x+2y_{\alpha,\beta}\hat{d}D_x)^n\psi_0$
V.	Integral representation	$\int_0^\infty n_\alpha(s,t)H_n(x,s)ds$
VI.	Summation formulae	$_EH_n(x+v,y;\alpha,\beta)=\sum_{k=0}^n\binom{n}{k}v^k\,{}_EH_{n-k}(x,y;\alpha,\beta)$ $_EH_{k+l}(w,y;\alpha,\beta)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}\,{}_EH_{k+l-n-r}(x,y;\alpha,\beta)$
VII	Symbolic operational identity	$_EH_n(x+v,y+w;\alpha,\beta)=\sum_{k=0}^n\binom{n}{k}H_{n-k}(x,y_{\alpha,\beta}\hat{d})\ {}_EH_k(v,w;\alpha,\beta)$
VIII.	Multiplicative and derivative operator	$\hat{M}_1=(x+2\,y_{\alpha,\beta}\hat{d}D_x)$ and $\hat{P}_1=D_x$.
IX.	Symbolic and differential recurrence relation	$_EH_{n+1}(x,y;\alpha,\beta)=x\,{}_EH_n(x,y;\alpha,\beta)+2ny\ {}_{\alpha,\beta}\hat{d}\ {}_EH_{n-1}(x,y;\alpha,\beta)$ $(n+1)\,{}_EH_{n+1}(x,y;\alpha,\beta)=(n+1)x\,{}_EH_n(x,y;\alpha,\beta)+2yD_y\,{}_EH_{n+1}(x,y;\alpha,\beta)$
X.	Symbolic differential equation	$\left(2y_{\alpha,\beta}\hat{d}\ D_x^2+xD_x-n\right){}_EH_n(x,y;\alpha,\beta)=0$

II. Since from equation (1.1), it is clear that $E_{1,1}(x)=e^x$, therefore taking $\alpha=1$ and $\beta=1$ in equation (2.2), we get

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\ {}_EH_n^{(m)}(x,y;1,1)=e^{x\xi+y\xi^m}, \tag{5.3}$$

which in view of equation (1.9), gives

$$_EH_n^{(m)}(x,y;1,1)=H_n^{(m)}(x,y). \tag{5.4}$$

Thus, keeping in view that $_{1,1}\hat{d}^k\psi_0=1$ $(k\in\mathbb{N})$ and taking $\alpha=1$, $\beta=1$ in equations (2.2), (2.6), (2.23), (2.25), (3.7), (3.11), (3.20), (4.1), (4.2), (4.11), (4.12), and (4.13), we get the respective properties of the Gould-Hopper polynomials which are listed in Table 5.2.

Table 5.2 Properties of the Gould-Hopper polynomials $H_n^{(m)}(x,y)$:

S. No.	Name of the properties	Results
I.	Generating function	$e^{x\xi+y\xi^m}$ [18]
II.	Series definition	$n!\sum_{r=0}^{[\frac{n}{m}]}\frac{x^{n-mr}y^r}{(n-mr)!r!}$ [18]
III.	Operational identity	$e^{yD_x^m}\{x^n\}$ [7]
IV.	Operational definition	$(x+myD_x^{m-1})^n\{1\}$ [7]
V.	Summation formula	$H_n^{(m)}(x+v,y)=\sum_{k=0}^n\binom{n}{k}v^kH_{n-k}^{(m)}(x,y)$ [22] $H_{k+l}^{(m)}(w,y)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}H_{k+l-n-r}^{(m)}(x,y)$ [22] $H_n^{(m)}(x+v,y+w)=\sum_{k=0}^n\binom{n}{k}H_{n-k}{}^{(m)}(x,y)\ H_k^{(m)}(v,w)$ [22]
VI.	Multiplicative and derivative operator	$\hat{M}_g=(x+myD_x^{m-1})$ and $\hat{P}_g=D_x$ [7]

VII.	Pure and differential recurrence relation	$H_{n+1}^{(m)}(x,y) = xH_n^{(m)}(x,y) + my\frac{n!}{(n-m+1)!}H_{n-m+1}^{(m)}(x,y)$ [11] $(n+1)H_{n+1}^{(m)}(x,y) = (n+1)xH_n^{(m)}(x,y) + myD_y H_{n+1}^{(m)}(x,y)$ [11]
VIII.	Differential equation	$(myD_x^m + xD_x - n)H_n^{(m)}(x,y) = 0$ [7]

III. In view of equation (1.14), taking $m = 2$ in equation (5.4), we get

$$ {}_EH_n^{(2)}(x,y;1,1) = H_n(x,y). \tag{5.5}$$

Thus, keeping in view that ${}_{1,1}\hat{d}^k\psi_0 = 1$ ($k \in \mathbb{N}$) and taking $m = 2$, $\alpha = 1$, $\beta = 1$ in equations (2.2), (2.6), (2.23), (2.25), (3.7), (3.11), (3.20), (4.1), (4.2), (4.11), (4.12) and (4.13), we get the respective properties of the 2VHKdFP $H_n(x,y)$, which are listed in Table 5.3.

Table 5.3 Properties of the 2VHKdFP $H_n(x,y)$:

S. No.	Name of the properties	Results
I.	Generating function	$e^{x\xi+y\xi^2}$ [2]
II.	Series definition	$n!\sum_{r=0}^{[\frac{n}{2}]}\frac{x^{n-2r}y^r}{(n-2r)!r!}$ [2]
III.	Operational identity	$e^{yD_x^2}\{x^n\}$ [6]
IV.	Operational definition	$(x+2yD_x)^n\{1\}$ [6]
V.	Summation formula	$H_k(w,y) = \sum_{n=0}^{k}\binom{k}{n}(w-x)^n H_{k-n}(x,y)$ [22] $H_{k+l}(w,y) = \sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}H_{k+l-n-r}(x,y)$ [22] $H_n(x+v,y+w) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}(x,y)\ H_k(v,w)$ [22]
VI.	Multiplicative and derivative operator	$\hat{M}_H = (x+2yD_x)$ and $\hat{P}_H = D_x$ [6]
VII.	Pure and differential recurrence relation	$H_{n+1}(x,y) = xH_n(x,y) + 2nyH_{n-1}(x,y)$ [11] $(n+1)H_{n+1}(x,y) = (n+1)xH_n(x,y) + 2yD_yH_{n+1}(x,y)$ [11]
VIII.	Differential equation	$(2yD_x^2 + xD_x - n)H_n(x,y) = 0$ [6]

IV. In view of equation (1.16), replacing x with $2x$ and y with -1 in equation (5.5), we get

$$ {}_EH_n^{(2)}(2x,-1;1,1) = H_n(x). \tag{5.6}$$

Thus, keeping in view that ${}_{1,1}\hat{d}^k\psi_0 = 1$ ($k \in \mathbb{N}$) and taking $m = 2$, $\alpha = 1$, $\beta = 1$, and replacing x with $2x$ and y with $= -1$ in equations (2.2), (2.6), (2.25), (3.7), (3.11), (3.20), (4.1), (4.2), (4.11) and (4.13), we get the respective properties of the Hermite polynomials $H_n(x)$, which are listed in Table 5.4.

Table 5.4 Properties of the Hermite polynomials $H_n(x)$:

S. No.	Name of the properties	Results
I.	Generating function	$e^{2x\xi-\xi^2}$ [1, 28]
II.	Series definition	$n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(-1)^r(2x)^{n-2r}}{(n-2r)!r!}$ [1, 28]
III.	Operational definition	$(2x-D_x)^n\{1\}$ [28]
IV.	Summation formula	$H_k(w) = \sum_{n=0}^{k}\binom{k}{n}(2(w-x))^n H_{k-n}(x)$ [21] $H_{k+l}(w) = \sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(2(w-x))^{n+r}H_{k+l-n-r}(x)$ [21] $H_n(x+v) = \sum_{k=0}^{n}\binom{n}{k}2^k x^k H_{n-k}(v)$ [21]

V.	Multiplicative and derivative operator	$\hat{M}_H = (2x - D_x)$ and $\hat{P}_H = \frac{1}{2}D_x$ [10]
VI.	Pure recurrence relation	$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ [1]
VII.	Differential equation	$(D_x^2 - 2xD_x + 2n)H_n(x) = 0$ [1]

V. In view of equations (1.16) and (5.2), the 1-variable Mittag-Leffler-Hermite polynomials (1VMLHP) ${}_EH_n(x;\alpha,\beta)$ are defined as:

$$ {}_EH_n(x;\alpha,\beta) = H_n(2x, -{}_{\alpha,\beta}\hat{d})\ \psi_0. \tag{5.7}$$

Thus, taking $m = 2$ and replacing x with $2x$ and y with -1 in equations (2.2), (2.6), (2.25), (3.7), (3.11), (4.1), (4.2), (4.11) and (4.13), we get the respective properties of the 1-variable Mittag-Leffler-Hermite polynomials ${}_EH_n(x;\alpha,\beta)$, which are listed in Table 5.5.

Table 5.5 Properties of the 1-variable Mittag-Leffler-Hermite polynomials ${}_EH_n(x;\alpha,\beta)$:

S. No.	Name of the properties	Results
I.	Generating function	$e^{2x\xi}E_{\alpha,\beta}(-\xi^2)$
II.	Series definition	$n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(-1)^r(2x)^{n-2r}}{(n-2r)!\Gamma(\alpha r+\beta)}$
III.	symbolic definition	$(2x - {}_{\alpha,\beta}\hat{d}\ D_x)^n\psi_0$
IV.	Summation formulae	${}_EH_n(x+v;\alpha,\beta) = \sum_{k=0}^{n}\binom{n}{k}(2v)^k{}_EH_{n-k}(x;\alpha,\beta)$ ${}_EH_{k+l}(w;\alpha,\beta) = \sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(2(w-x))^{n+r}{}_EH_{k+l-n-r}(x;\alpha,\beta)$
V.	Multiplicative and derivative operator	$\hat{M}_1 = (2x - {}_{\alpha,\beta}\hat{d}D_x)$ and $\hat{P}_1 = \frac{1}{2}D_x$
VI.	Symbolic recurrence relation	${}_EH_{n+1}(x;\alpha,\beta) = 2x{}_EH_n(x;\alpha,\beta) - 2n{}_{\alpha,\beta}\hat{d}{}_EH_{n-1}(x;\alpha,\beta)$
VII.	Symbolic differential equation	$\left({}_{\alpha,\beta}\hat{d}\ D_x^2 - 2xD_x + 2n\right){}_EH_n(x;\alpha,\beta) = 0$

VI. In view of equations (5.1) and (5.7), the 1-variable Mittag-Leffler-Gould-Hopper polynomials (1VMLGHP) ${}_EH_n^{(m)}(x;\alpha,\beta)$ can be introduced as

$$ {}_EH_n^{(m)}(x;\alpha,\beta) = H_n^{(m)}(2x, -{}_{\alpha,\beta}\hat{d})\ \psi_0. \tag{5.8}$$

Since, in view of equation (2.1) and (5.8), it is clear that

$$ {}_EH_n^{(m)}(2x,-1;\alpha,\beta) = {}_E\ H_n^{(m)}(x;\alpha,\beta). \tag{5.9}$$

Therefore, the respective properties of the 1-variable Mittag-Leffler-Gould-Hopper polynomials (1VMLGHP) ${}_EH_n^{(m)}(x;\alpha,\beta)$ can be obtained by replacing x with $2x$ and y with -1 in equations (2.2), (2.6), (2.25), (3.7), (3.11), (3.20), (4.1), (4.2), (4.11) and (4.13), which are listed in Table 5.6.

Table 5.6 Properties of the 1-variable Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x;\alpha,\beta)$:

S. No.	Name of the properties	Results
I.	Generating function	$e^{2x\xi}E_{\alpha,\beta}(-\xi^m)$
II.	Series definition	$n!\sum_{r=0}^{[\frac{n}{m}]}\frac{(-1)^r(2x)^{n-mr}}{(n-mr)!\Gamma(\alpha r+\beta)}$
III.	symbolic definition	$(2x - m{}_{\alpha,\beta}\hat{d}\ D_x^{m-1})^n\psi_0$
IV.	Summation formulae	${}_EH_n^{(m)}(x+v;\alpha,\beta) = \sum_{k=0}^{n}\binom{n}{k}(2v)^k{}_EH_{n-k}^{(m)}(x;\alpha,\beta)$ ${}_EH_{k+l}^{(m)}(w;\alpha,\beta) = \sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(2(w-x))^{n+r}{}_EH_{k+l-n-r}^{(m)}(x;\alpha,\beta)$
V.	Multiplicative and derivative operator	$\hat{M}_1 = (2x - m\ {}_{\alpha,\beta}\hat{d}D_x^{m-1})$ and $\hat{P}_1 = \frac{1}{2}D_x$

VI.	Symbolic recurrence relation	${}_EH_{n+1}^{(m)}(x;\alpha,\beta) = 2x\,{}_EH_n^{(m)}(x;\alpha,\beta) - m_{\alpha,\beta}\hat{d}\,\frac{n!}{(n-m+1)!}\,{}_EH_{n-m+1}^{(m)}(x;\alpha,\beta)$
VII.	Symbolic differential equation	$\left(m_{\alpha,\beta}\hat{d}\,D_x^m - 2xD_x + 2n\right){}_EH_n^{(m)}(x;\alpha,\beta) = 0$

In the next section, we give graphical representations of the polynomials discussed in this paper.

6 Graphical representations

In this section, we obtain the graphical representations of the Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x,y;\alpha,\beta)$, Mittag-Leffler-Hermite polynomials ${}_EH_n(x,y;\alpha,\beta)$, 1-variable Mittag-Leffler-Hermite polynomials ${}_EH_n(x;\alpha,\beta)$ and 1-variable Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x;\alpha,\beta)$ by using their series expansions for suitable choices of parameters in the software MATLAB.

We assign the appropriate values to n and other parameters in the respective series expansions of these polynomials given by equation (2.6), Tables 5.1 (II), 5.5 (II) and 5.6(II) to obtain the expressions, which are required to plot their graphs by MATLAB.

Figures 1 and 2; Figures 3 and 4; Figure 5; Figures 6,7 and 8 show the following graphical representations of the Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x,y;\alpha,\beta)$, Mittag-Leffler-Hermite polynomials ${}_EH_n(x,y;\alpha,\beta)$, 1-variable Mittag-Leffler-Hermite polynomials ${}_EH_n(x;\alpha,\beta)$ and 1-variable Mittag-Leffler-Gould-Hopper polynomials ${}_EH_n^{(m)}(x;\alpha,\beta)$, respectively.

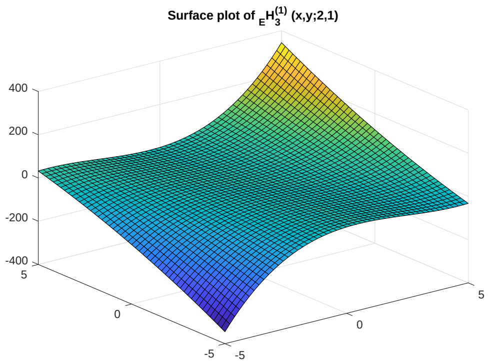

Figure 1: MLGHP ${}_EH_3^{(1)}(x,y;2,1)$

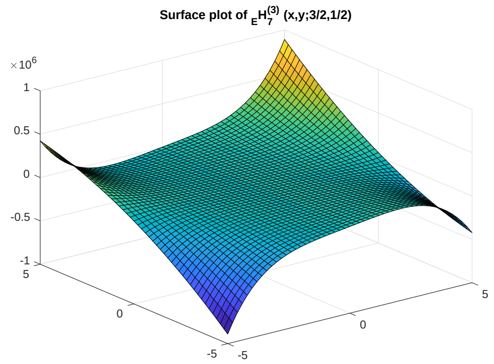

Figure 2: MLGHP ${}_EH_7^{(3)}(x,y;3/2,1/2)$

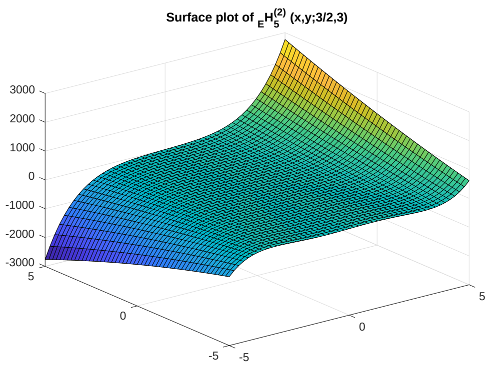

Figure 3: MLHP ${}_EH_5(x,y;3/2,3)$

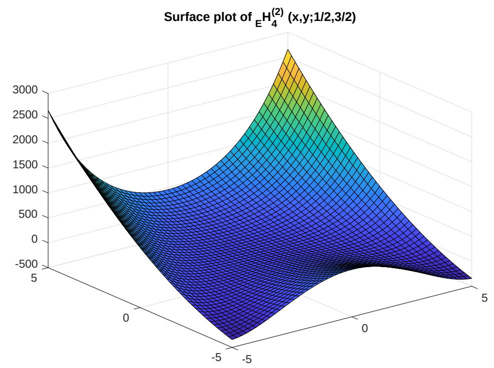

Figure 4: MLHP ${}_EH_4(x,y;1/2,3/2)$

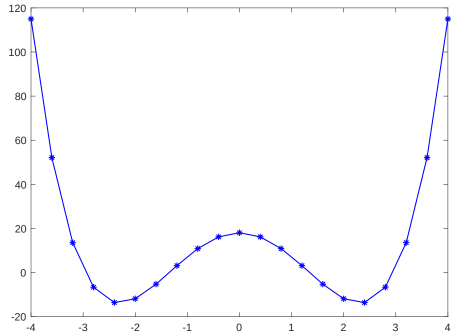

Figure 5: 1VMLHP ${}_EH_4(x;1/2,3/2)$

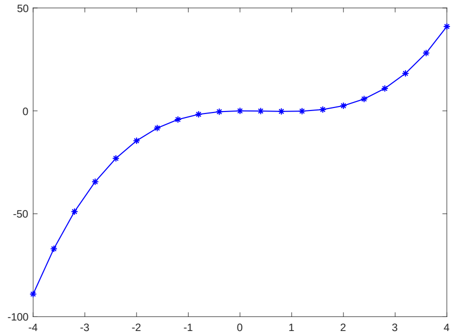

Figure 6: 1VMLGHP ${}_EH_3^{(1)}(x;2,1)$

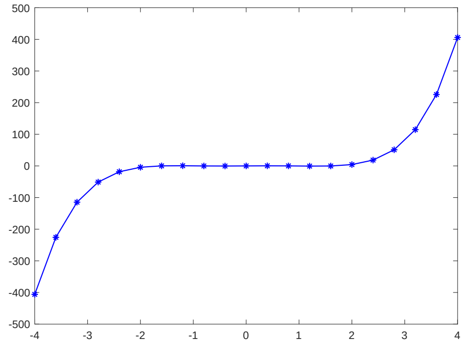

Figure 7: 1VMLGHP ${}_EH_5^{(1)}(x;3/2,3)$

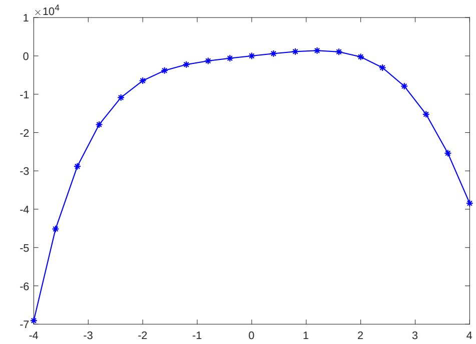

Figure 8: 1VMLGHP ${}_EH_7^{(3)}(x;3/2,1/2)$

7 Concluding remarks

In view of Weyl decoupling identity (2.5), we have

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}(x+2y_{\alpha,\beta}\hat{d}D_x)^n = e^{x\xi}e^{y_{\alpha,\beta}\hat{d}\xi^2}e^{2\xi\ y_{\alpha,\beta}\hat{d}D_x}, \tag{7.1}$$

which on using equation (1.15), becomes

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}(x+2y_{\alpha,\beta}\hat{d}D_x)^n = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{\xi^n}{n!s!}H_n(x,y_{\alpha,\beta}\hat{d})(2\xi y_{\alpha,\beta}\hat{d}D_x)^s. \tag{7.2}$$

Comparing the equal powers of ξ from both sides of the above equation, we have

$$(x+2y_{\alpha,\beta}\hat{d}D_x)^n = n!\sum_{s=0}^{n}\frac{H_{n-s}(x,y_{\alpha,\beta}\hat{d})(2y_{\alpha,\beta}\hat{d}D_x)^s}{s!(n-s)!}. \tag{7.3}$$

For any function $f(x,y;\psi_0)$, we get the following result:

$$(x+2y_{\alpha,\beta}\hat{d}D_x)^n f(x,y;\psi_0) = n!\sum_{s=0}^{n}\frac{H_{n-s}(x,y_{\alpha,\beta}\hat{d})(2y_{\alpha,\beta}\hat{d})^s}{s!(n-s)!}f_x^s(x,y;\psi_0), \tag{7.4}$$

where $f_x^s(x,y;\psi_0) := D_x^s f(x,y;\psi_0)$.

For $f(x,y;\psi_0) = {}_EH_k^{(m)}(x,y;\alpha,\beta)$ equation (7.4), gives

$$(x+2y_{\alpha,\beta}\hat{d}D_x)^n\,{}_EH_k^{(m)}(x,y;\alpha,\beta) = n!\sum_{s=0}^{n}\frac{H_{n-s}(x,y_{\alpha,\beta}\hat{d})(2y_{\alpha,\beta}\hat{d}D_x)^s}{s!(n-s)!}\,{}_EH_k^{(m)}(x,y;\alpha,\beta), \tag{7.5}$$

which on using equation (4.9) in the left hand side and equation (2.8) in the right hand side, gives the following identity:

$$ {}_EH^{(m)}_{k+n}(x,y;\alpha,\beta) = \sum_{s=0}^{n} \binom{n}{s} \frac{k! H_{n-s}(x, y_{\alpha,\beta}\hat{d})(2y_{\alpha,\beta}\hat{d})^s\, {}_EH^{(m)}_{k-s}(x,y;\alpha,\beta)}{(k-s)!}, \tag{7.6} $$

If we take $\alpha = 1$ and $\beta = 1$ in equation (7.3), we get the Burchnall identity [17]:

$$ (x + 2yD_x)^n = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(x,y)(2y)^s D_x^s. \tag{7.7} $$

Again, since, $[2\xi y_{\alpha,\beta}\hat{d}D_x, y_{\alpha,\beta}\hat{d}\xi^2] = 0$, therefore, equation (7.1) can be written as

$$ \sum_{n=0}^{\infty} \frac{\xi^n}{n!}(x + 2y_{\alpha,\beta}\hat{d}D_x)^n = e^{x\xi} e^{2\xi\ y_{\alpha,\beta}\hat{d}D_x} e^{y_{\alpha,\beta}\hat{d}\xi^2}, \tag{7.8} $$

which on using equation (1.15), gives

$$ \sum_{n=0}^{\infty} \frac{\xi^n}{n!}(x + 2y_{\alpha,\beta}\hat{d}D_x)^n = e^{x\xi} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} H_n(2y_{\alpha,\beta}\hat{d}D_x, y_{\alpha,\beta}\hat{d}). \tag{7.9} $$

Expanding the exponential in the right hand side of equation (7.9), we find

$$ \sum_{n=0}^{\infty} \frac{\xi^n}{n!}(x + 2y_{\alpha,\beta}\hat{d}D_x)^n = \sum_{n=0}^{\infty} \left(\sum_{s=0}^{\infty} \frac{x^s \xi^s}{s!} \right) \frac{\xi^n}{n!} H_n(2y_{\alpha,\beta}\hat{d}D_x, y_{\alpha,\beta}\hat{d}), \tag{7.10} $$

which on using equation (3.18), in the right hand side gives

$$ \sum_{n=0}^{\infty} \frac{\xi^n}{n!}(x + 2y_{\alpha,\beta}\hat{d}D_x)^n = \sum_{n=0}^{\infty} \left(\sum_{s=0}^{n} \frac{x^s}{(n-s)!s!} \right) H_{n-s}(2y_{\alpha,\beta}\hat{d}D_x, y_{\alpha,\beta}\hat{d})\xi^n. \tag{7.11} $$

Comparing the equal powers of ξ from both sides of the above equation, we get the following result:

$$ (x + 2y_{\alpha,\beta}\hat{d}D_x)^n = n! \sum_{s=0}^{n} \frac{x^s}{(n-s)!s!} H_{n-s}(2y_{\alpha,\beta}\hat{d}D_x, y_{\alpha,\beta}\hat{d}). \tag{7.12} $$

From equation (7.12), we have

$$ (x + 2y_{\alpha,\beta}\hat{d}D_x)^n\, {}_EH_k(x,y;\alpha,\beta) = n! \sum_{s=0}^{n} \frac{x^s}{(n-s)!s!} H_{n-s}(2y_{\alpha,\beta}\hat{d}D_x, y_{\alpha,\beta}\hat{d})\, {}_EH_k(x,y;\alpha,\beta), \tag{7.13} $$

which on using equation (4.9) for $m = 2$ in the left hand side gives the following identity:

$$ {}_EH_{n+k}(x,y;\alpha,\beta) = n! \sum_{s=0}^{n} \frac{x^s}{(n-s)!s!} H_{n-s}(2y_{\alpha,\beta}\hat{d}\ D_x, y_{\alpha,\beta}\hat{d})\, {}_EH_k(x,y;\alpha,\beta). \tag{7.14} $$

Further, since in view of equation (1.7), for $\beta = 1$, we have $\psi_0 = 1$ and $[x\xi,_{\alpha,1}\ \hat{d}\ y\xi^m] = 0$. Therefore, we have

$$ e^{-y_{\alpha,1}\hat{d}\xi^m}(e^{x\xi + y_{\alpha,1}\hat{d}\xi^m})\{1\} = e^{x\xi}, \tag{7.15} $$

which in view of equation (2.4), gives

$$ e^{-y_{\alpha,1}\hat{d}\xi^m} \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, {}_EH^{(m)}_n(x,y;\alpha,1) = \exp(x\xi), \tag{7.16} $$

or equivalently,

$$ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (-y\ _{\alpha,1}\hat{d})^k\, {}_EH^{(m)}_n(x,y;\alpha,1) \frac{\xi^{n+mk}}{n!k!} = \sum_{n=0}^{\infty} \frac{x^n \xi^n}{n!}. \tag{7.17} $$

Using identity [28]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n+mk,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} A(n-mk,k) \tag{7.18}$$

in the left hand side of equation (7.17) and then comparing equal powers of ξ, we get

$$n!\sum_{k=0}^{[\frac{n}{m}]} \frac{\left(-y_{\alpha,1}\hat{d}\right)^k}{k!(n-mk)!} {}_E H_{n-mk}^{(m)}(x,y;\alpha,1) = x^n. \tag{7.19}$$

The Mittag-Leffler function [12] plays a central role in the theory of fractional derivatives [26]. It has been thoroughly investigated [26] but the increasing interest for fractional derivatives in applications demands for further studies [23], eventually leading to further properties or to more efficient methods of computation and analysis.

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