Using decomposition of the nonlinear operator for solving non-differentiable problems

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From decomposition method for operators, we consider Newton-like iterative processes for approximating solutions of nonlinear operators in Banach spaces. These iterative processes maintain the quadratic convergence of Newton's method. Since the operator decomposition method has its highest degree of application in non-differentiable situations, we construct Newton-type methods using symmetric divided differences, which allow us to improve the accessibility of the methods. Experimentally, by studying the basins of attraction of these methods, observe an improvement in the accessibility of derivative-free iterative processes that are normally used in these non-differentiable situations, such as the classic Steffensen's method. In addition, we study both the local and semi-local convergence of the Newton-type methods considered.

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Abstract

From decomposition method for operators, we consider Newton-like iterative processes for approximating solutions of nonlinear operators in Banach spaces. These iterative processes maintain the quadratic convergence of Newton's method. Since the operator decomposition method has its highest degree of application in non-differentiable situations, we construct Newton-type methods using symmetric divided differences, which allow us to improve the accessibility of the methods. Experimentally, by studying the basins of attraction of these methods, observe an improvement in the accessibility of derivative-free iterative processes that are normally used in these non-differentiable situations, such as the classic Steffensen's method. In addition, we study both the local and semi-local convergence of the Newton-type methods considered.

KEYWORDS:
Non-differentiable operator, Kurchatov method, Newton-Kantorovich method, semilocal convergence.

1 | INTRODUCTION

In this paper, we study the problem of approximating a solution of a nonlinear equation

$$H(x) = 0, \tag{1}$$

where $H : \Omega \subseteq X \to X$ is an operator defined on a nonempty convex subset Ω of a Banach space X. If H is a Fréchet differentiable operator, Newton's method[5] is the most used point-to-point iterative process to solve (1), due to its computational efficiency, and it is given by

$$x_{n+1} = x_n - \left[H'(x_n)\right]^{-1} H(x_n), \quad n \geq 0; \quad x_0 \in \Omega \text{ is given.} \tag{2}$$

In addition, this method has good accessibility, so that the domain of starting points of the method is large, but this method needs the existence of $H'(x)$. For this reason, Newton's method cannot be applied when H is non-differentiable. In this situation, point-to-point derivative-free iterative processes are usually considered . Our main goal in this work is to consider an iterative process that has the characteristics of Newton's method, for H differentiable, but it is also applicable in situations where the operator H is not differentiable, maintaining, in the non-differentiable situation, the important properties that the considered iterative process verifies in the differentiable case. To achieve this goal, the first step is to approximate the operator H' when the operator H is non-differentiable. It is common to approximate the derivatives by divided differences using a numerical derivation formula, and as a consequence, iterative processes that use divided differences instead of derivatives are obtained. We shall use, as in[11], the following well known definition for the divided differences of an operator. Let us denote by $\mathcal{L}(X, X)$ the space of bounded linear operators from X to X. An operator $[x, y; D] \in \mathcal{L}(X, X)$ is called a first-order divided difference

for the operator $D : \Omega \subseteq X \to X$, on the points x and y ($x \neq y$), if the following equality holds

$$[x, y; D](x - y) = D(x) - D(y). \tag{3}$$

In [16], Kung and Traub presented a class of point-to-point derivative-free iterative processes which are as simple as Newton's method, having quadratic convergence and the same computational efficiency as Newton's method but, in contrast, this class of iterative processes does not use $H'(x)$. These iterative processes contain Steffensen's method as a special case, where the evaluation of $H'(x)$ in each step of Newton's method is approximated by the first-order divided difference $[x, x + H(x); H]$. Steffensen's method has been widely studied ([1,3,4]) and their algorithm is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n, x_n + H(x_n); H]^{-1} H(x_n), \quad n \geq 0. \end{cases} \tag{4}$$

This method has quadratic convergence and the same computational efficiency as Newton's method.

Other types of approximations to the derivative of the operator have been also considered and Newton-like methods are obtained:

$$x_{n+1} = x_n - \left[A(x_n)\right]^{-1} H(x_n), \quad n \geq 0, \quad x_0 \in \Omega.$$

where $A(x) \in \mathcal{L}(X, X)$ is an approximation of $H'(x)$ for each $x \in X$. The convergence analysis has been given by several authors ([5,7]).

Methods using divided differences in their algorithm have a drawback. As we discuss in Section 2, the accessibility of these methods to the solution of the equation is poor, so that the domains of starting points are reduced. In contrast, this is one of the favorable features of Newton's method (2). So, in this work, we try to improve the accessibility of Steffensen's method. We use a new idea that is to perform a decomposition of the operator H that defines the equation $H(x) = 0$. So, we consider

$$H(x) = F(x) + G(x),$$

where $F, G : \Omega \subseteq X \to X$ are nonlinear operators, F is differentiable ang G is continuous but non-differentiable.

Then, we consider

$$A(x_n) = F'(x_n) + \left[x_n - \varepsilon\, H(x_n), x_n + \varepsilon\, H(x_n); G\right], \text{ with } \varepsilon > 0,$$

and obtain the following Newton-Steffensen-type iterative processes:

$$\begin{cases} x_0 \in \Omega \text{ and } \varepsilon > 0 \text{ are given}, \\ x_{n+1} = x_n - (F'(x_n) + \left[x_n - \varepsilon\, H(x_n), x_n + \varepsilon\, H(x_n); G\right])^{-1} H(x_n), \quad n \geq 0. \end{cases} \tag{5}$$

Notice that, if we consider the case in which the operator H is continuous but non-differentiable, such that $H(x) = F(x) + G(x)$, where F is Fréchet differentiable and G is continuous but non-differentiable, there are two advantages of process (5): first, the differentiable part of the operator is considered in the optimal situation, namely $F'(x_n)$; and second, for the non-differentiable part, the symmetric first-order divided difference $\left[x_n - \varepsilon\, H(x_n), x_n + \varepsilon\, H(x_n); G\right]$ is considered, which improves the results given by $[x_n, x_n + H(x_n); H]$. Thus, a more suitable situation for $A(x_n)$ is considered than the known ones until now. Notice that if H is Fréchet differentiable and $G = 0$, Newton's method (2) is obtained and if $F = 0$, then we obtain a Steffensen-type method with quadratic convergence [2].

In this paper, we start in Section 2 by seeing that the iterative processes considered improve the accessibility of the classical Steffensen's method, one of the most widely used derivative-free methods. After this motivation of our work, we dedicate Sections 3 and 4 to the study of local and semilocal convergence of methods (5) respectively. Finally, a numerical test is presented where the application of methods (5) is justified.

Throughout the paper, we suppose that there exists a divided difference operator $[x, y; G]$ for each pair of distinct points $x, y \in \Omega$.

2 | MOTIVATION

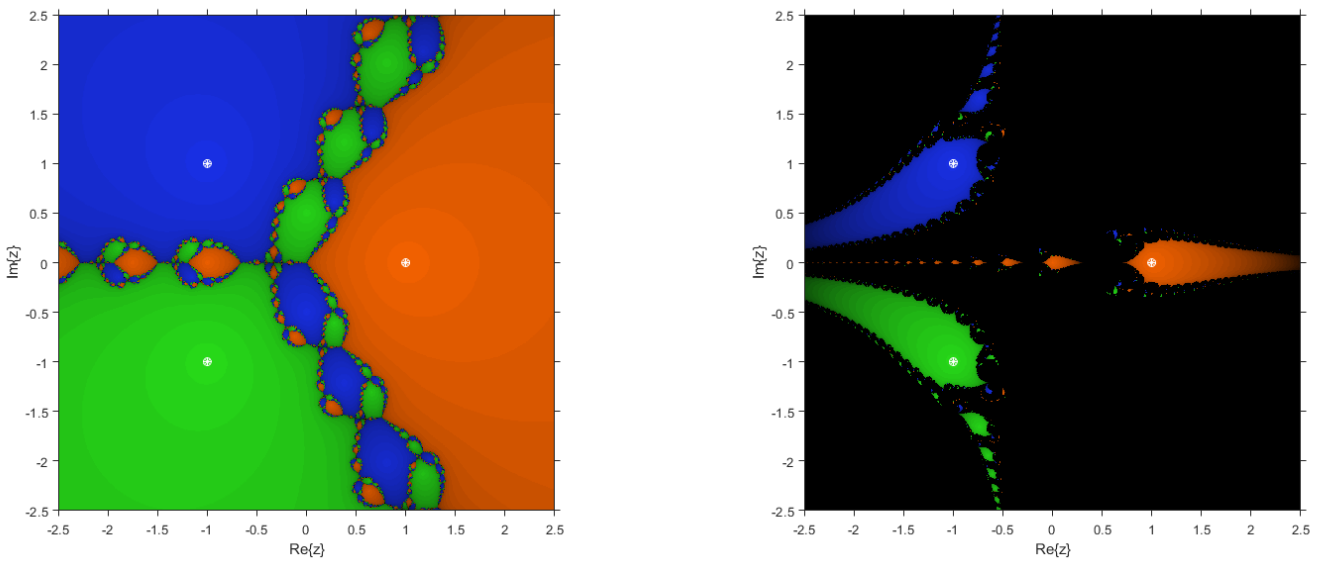
As we have already mentioned, Steffensen's method is a version of Newton's method in which the derivative is replaced by a divided difference operator. This method is as simple as Newton's, it has the same efficiency, the quadratic order of convergence that characterizes Newton's method, and it is also applicable to non-differentiable operators. A priori it seems a more useful method than Newton's. However, it has a major flaw, which is that the region of accessibility of this method is considerably reduced with respect to Newton's. Notice that method (5) is the Newton's method when the operator H is differentiable and we consider $G = 0$.

In practice, we estimate the accessibility of a method by depicting its basins of attraction (the set of points in space such that the initial conditions chosen in the dynamic set evolve to a particular attractor[15,18]) when it is applied to solve a complex equation $H(z) = 0$, where $H : \mathbb{C} \to \mathbb{C}$ and $z \in \mathbb{C}$.

In this article we are going to show the accesibility of the Newton and Steffensen method that are generated by solving the complex equation $H(z) = z^3 + z^2 - 2$. We know that the solutions of this equation are $z_1^* = 1$, $z_2^* = -1 + -i$ and $z_3^* = -1 + i$.

We are interested in showing the basins of attraction of the methods for comparison. We will choose the rectangle $D = [-2.5, 2.5] \times [-2.5, 2.5]$ that contains the three solutions. We will take a grid of 1000×1000 points in D that will be the starting points and we will begin to iterate in both methods. The initial points that are in the rectangle can converge to one of the solutions or diverge according to the iterative method we use.

For both methods we choose the stopping criterion 10^{-3} and a maximum of 50 iterations. If the tolerance is not less than 10^{-3} in 50 iterations, we will say that the method does not converge starting from that initial point and this point will be colored black. In case it converges, we have assigned a different color depending on which solution converges to. Specifically, we have chosen the orange color for the basins of attraction of the real root $z_1^* = 1$ and the colors green and blue for the basin of attraction of the complex roots z_2^* and z_3^* , respectively. Note that for all the starting points $z_0 \in D$ the real part is located on the abscissa axis and the imaginary part on the ordinate axis.



(a) Newton's method.

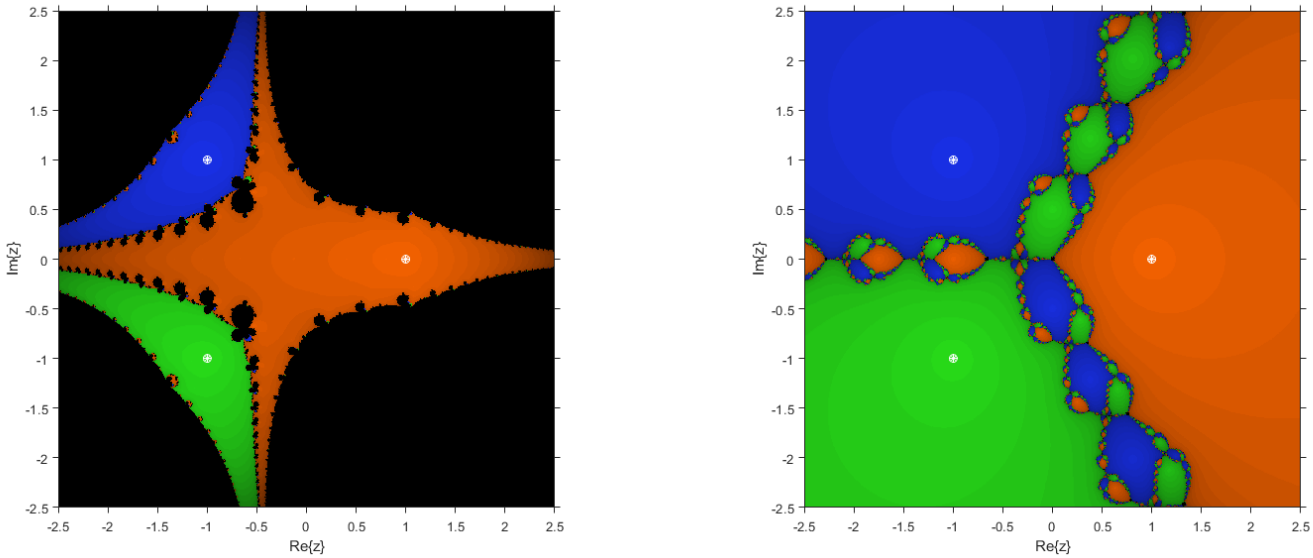
(b) Steffensen's method.

FIGURE 1 Basins of attraction for the polynomial $H(z) = z^3 + z^2 - 2$.

We observe in the Figure 1 that Steffensen's accessibility is clearly worse than Newton's, and therefore of method (5). We can see this in the black areas that are the points of the rectangle that, after 50 iterations, do not match any of the three solutions. However, it is sometimes necessary to use this method without derivatives despite its low accessibility due to the non-applicability

of Newton when H is non-differentiable.

If we now apply the method (5) to solve the complex equation $H(z) = z^3 + z^2 - 2$ assuming $F(z) = 0$, $G(z) = z^3 + z^2 - 2$ and $\varepsilon = 1$, we see in Figure 1 that the accessibility improves with respect to Steffensen's method since we use centered divided differences. If we also choose $\varepsilon = 0.001$, the basins of attraction shown in Figure 2 are more similar to those of Newton's method than to those of Steffensen's method even though we consider $F(z) = 0$ and do not use derivatives.



(a) $\varepsilon = 1$. **(b)** $\varepsilon = 0.001$.

FIGURE 2 $H(z) = z^3 + z^2 - 2$ for the iterative method (5).

In addition to the dynamic planes, we would like to see numerically that the accessibility of the method (5) is quite close to that of Newton's method. In Table 1 we study the percentage of initial points that converge to some solution. Also in this table we can observe the average number of iterations required for each method to achieve convergence, taking into account that we work with a maximum of 50 iterations.

TABLE 1 Comparison of the percentage and the mean of the methods.

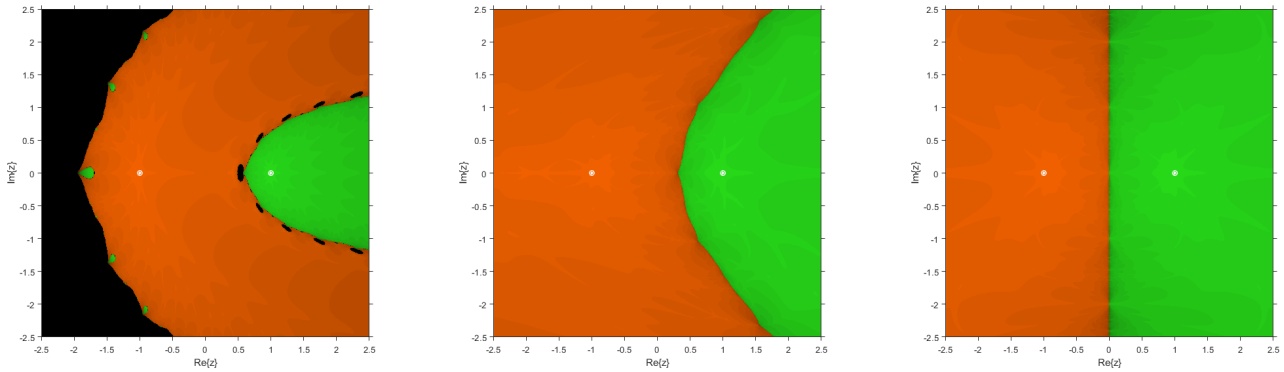
Method	ε	percentage	mean
Newton		99.9998%	5.7795
Steffensen		10.72645%	45.6607
Method (5)	1	30.2849%	36.7588
Method (5)	0.001	99.9378%	5.7988

As we see both in the dynamical planes and in the table, the method (5) greatly improves the accessibility of the Steffensen method and, in turn, is applicable to non-differentiable operators. For this reason, our goal in this article is to decompose the nonlinear operator into the sum of a differentiable and a non-differentiable part, if possible, trying to preserve the good accessibility of Newton's method for the differentiable part of H.

Next, see Figure 3 , we are going to see how the accessibility of Steffensen's method improves using Newton-Steffensen's

method with advanced divided differences of the article[13] and the method (5) for the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$. We know that the solutions of this equation are $z_1^* = 1$ and $z_2^* = -1$. In the case of Newton-Steffensen-type methods, we decompose $H(z)$ into $F(z) = z^2 - 2$, the differentiable part of the equation, and $G(z) = |z|$, the non-differentiable part.

We see how the both methods, Newton-Steffensen method and (5), improve the accessibility of the Steffensen method and, in turn, are applicable to non-differentiable operators.



Steffensen's method. Newton-Steffensen's method. Method (5) for $\epsilon = 0.001$.

FIGURE 3 Basins of attraction for the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$.

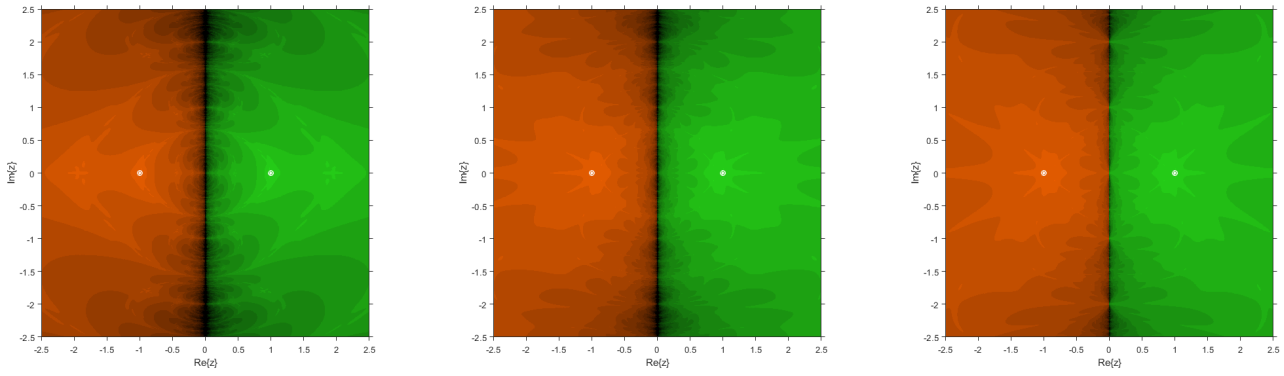
Another motivation for this work is to study how the centered divided difference operator used in (5) improves the Newton-Steffensen method, which uses leading divided differences. In the case of these two methods, which of the two has better accessibility is not so reflected, since no non-convergence zones are seen. The only thing that should be noted is that the basins of attraction of the solutions for the method (5) are symmetric while for the Newton-Steffensen method more initial points converge to the solution z_2^* than a z_1^*.

To better see the differences between three methods in the non-differentiable case, we study in Table 2 the percentage of initial points that converge to any of the solutions and the mean of iterations necessary to do so, taking into account that the maximum number iterations is 50.

TABLE 2 Comparison of the percentage and the mean of methods.

Method	percentage	mean
Steffensen	77.2919%	15.8885
Newton-Steffensen	100%	4.5630
Method (5)	100%	4.5091

Finally, see Figure 4 , using the same non-differentiable operator, we are going to study the accessibility of method (5) for different values of the ϵ parameter. To better see the different dynamic planes generated by the different values of ϵ, we are going to study the accessibility of the previous equation for a maximum of 25 iterations. That is, all those points that have not satisfied the 10^{-3} tolerance in less than 25 iterations, will belong to the black zone in each of the dynamic planes. Observing figures 4 , we can conclude that, for smaller values of ϵ, the convergence speed improves considerably since in the cases $\epsilon = 1$ and $\epsilon = 10$ we can observe more extended black areas, that is, initial points that do not converge to any of the two solutions in 25 iterations.



$\epsilon = 10.$ $\epsilon = 1.$ $\epsilon = 0.001.$

FIGURE 4 Basins of attraction for (5), for different values of ϵ, for the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$.

3 | LOCAL ERROR AND LOCAL CONVERGENCE

In this section, firstly we study the local order of convergence for the method given by (5), assuming that the nonlinear operator H is differentiable and we can obtain Taylor's expansion in a neighborhood of the solution x^*, thus, we can obtain an expression of the error. Secondly, we pay our attention to the analysis of the local convergence. The study of the local convergence of an iterative process is based on demanding conditions in the solution x^*, from certain conditions on the operator H, and provide the so-called ball of convergence of the iterative process, that shows the accessibility to the solution x^* from the initial approximations belonging to the ball

3.1 | Local order of convergence

First of all, as usual[8,9,10], we consider the case in which the operator $H : \Omega \subseteq X \to X$ is sufficiently differentiable and any decomposition is performed $H(x) = F(x) + G(x)$. This will allow us to obtain the error equation for the decomposition method (5) and thus obtain its order of convergence.

From the Genochi-Hermite formula (see[16]) for the divided difference operator

$$[x + h, x - h; G] = \frac{1}{2} \int_{-1}^{1} G'(x + th)dt \tag{6}$$

and by the Taylor series expansion of $G'(x + th)$ at the point x and integrating, we obtain the following development

$$[x + h, x - h; G] = G'(x) + \frac{1}{6} G'''(x)h^2 + O(h^3), \tag{7}$$

which we will use in the proof of the following Theorem when we obtain the convergence order of the method.

Theorem 1. Let $H : \Omega \subseteq X \to X$ such that $H(x) = F(x) + G(x)$, where F and G are sufficiently differentiable operators, and $x^* \in \Omega$ such that $H(x^*) = 0$. If the initial approximation x_0 is chosen sufficiently close to x^*, then the sequence $\{x_n\}$, given by (5), converges to x^* and the local order of convergence of the method (5) is at least 2. Moreover, the error equation is:

$$e_{n+1} = (\Gamma A_2 + \Upsilon B_2)(\Gamma + \Upsilon)^{-1} e_n^2 + O(e_n^3), \tag{8}$$

where $\Gamma = F'(x^*)$, $\Upsilon = G'(x^*)$ and $A_i = \frac{1}{i!} F'(x^*)^{-1} F^{(i)}(x^*)$ and $B_i = \frac{1}{i!} G'(x^*)^{-1} G^{(i)}(x^*)$ such that $A_i, B_i \in \mathcal{L}_i(X, X)$, $i = 1, 2, 3$, where $\mathcal{L}_i(X, X)$ is the space of bounded i-linear symmetric operators.

Proof. We calculate the Taylor series of $F(x_n)$ and $G(x_n)$ around x^*, which is the solution, and obtain

$$F(x_n) = \Gamma\left(A_0 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\right),$$
$$G(x_n) = \Upsilon\left(B_0 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + O(e_n^4)\right),$$

where $e_n = x_n - x^*$, $\Gamma = F'(x^*)$, $\Upsilon = G'(x^*)$ and $A_i = \frac{1}{i!}F'(x^*)^{-1}F^{(i)}(x^*)$, $B_i = \frac{1}{i!}G'(x^*)^{-1}G^{(i)}(x^*) \in \mathcal{L}_i(X, X)$. By the way to define A_i and B_i, we have that $A_1 = B_1 = I$ in the expansions of $F(x_n)$ and $G(x_n)$.

We calculate the expression of $H(x_n) = F(x_n) + G(x_n)$, using the previous developments. To do this, we are going to take into account that $\Gamma A_0 + \Upsilon B_0 = F(x^*) + G(x^*) = H(x^*) = 0$. We obtain

$$H(x_n) = (\Gamma + \Upsilon)e_n + (\Gamma A_2 + \Upsilon B_2)e_n^2 + (\Gamma A_3 + \Upsilon B_3)e_n^3 + O(e_n^4). \tag{9}$$

Now, deriving the Taylor series of $F(x_n)$ with respect to x_n we obtain that $F'(x_n)$ is of the form:

$$F'(x_n) = \Gamma\left(I + 2A_2 e_n + 3A_3 e_n^2 + 4A_4 e_n^3 + O(e_n^4)\right).$$

Using the divided difference operator seen in (7), we have:

$$[x_n + h, x_n - h; G] = G'(x_n) + \frac{1}{6}G'''(x_n)h^2 + O(h^3) \tag{10}$$

where $h = \varepsilon\, H(x_n)$.

To calculate the above equation, first we will see what the expressions for $G'(x_n)$ and $G'''(x_n)$ look like. From the Taylor series of $G(x_n)$ around x^* we obtain

$$G'(x_n) = \Upsilon\left(I + 2B_2 e_n + 3B_3 e_n^2 + 4B_4 e_n^3 + O(e_n^4)\right),$$
$$G'''(x_n) = \Upsilon\left(6B_3 + 24B_4 e_n + 60B_5 e_n^2 + O(e_n^2)\right).$$

Substituting these developments in (10) and taking into account that $h = \varepsilon\, H(x_n)$ with $H(x_n)$ developed in (9), we calculate

$$\begin{aligned}[x_n - h, x_n + h; G] = \Upsilon(I + 2B_2 e_n + (3I + \varepsilon^2(\Gamma + \Upsilon)^2)B_3 e_n^2 \\ + 2(\varepsilon^2(\Gamma + \Upsilon)(\Gamma A_2 + \Upsilon B_2)B_3 + 2B_4 + 2\varepsilon^2(\Gamma + \Upsilon)^2 B_4)e_n^3 + O(e_n^4)).\end{aligned}$$

We denote $A(x_n) = F'(x_n) + [x_n - \varepsilon\, H(x_n), x_n + \varepsilon\, H(x_n); G]$. Using the Taylor series expansion of $F'(x_n)$ around x^* and the last expression obtained, $A(x_n)$ is expressed as follows:

$$A(x_n) = (\Gamma + \Upsilon) + (2\Gamma A_2 + 2\Upsilon B_2)e_n + (3\Gamma A_3 + \Upsilon(3I + \varepsilon^2(\Gamma + \Upsilon)^2)B_3)e_n^2 + O(e_n^3).$$

Finally, calculating $A(x_n)^{-1} = (F'(x_n) + [x_n - \varepsilon\, H(x_n), x_n + \varepsilon\, H(x_n); G])^{-1}$ and substituting this development in the method (5) together with the expansion of $H(x_n)$ from (9), the following error equation is obtained:

$$e_{n+1} = x_{n+1} - x^* = x_n - x^* - [A(x_n)]^{-1}H(x_n) = (\Gamma A_2 + \Upsilon B_2)(\Gamma + \Upsilon)^{-1}e_n^2 + O(e_n^3).$$

□

3.2 | Local convergence and uniquiness of solutions

Second, we are going to obtain a local convergence result applicable to the operator H both when it is differentiable and non-differentiable.

We consider $H(x) = F(x) + G(x)$ where $F, G : \Omega \subseteq X \to X$, F is a differentiable Fréchet operator and G is continuous but not differentiable. Note that if $H(x)$ is differentiable, we consider $H(x) = F(x)$ and we would be studying local convergence for Newton's method. In the opposite case, that is, if $H(x)$ is a non-differentiable operator such that $H(x) = G(x)$, we would be studying the local convergence of the Steffensen method.

To ensure the local convergence of the method (5), we assume conditions on the operators F and G and also on the solution x^* of the equation $H(x) = 0$. Recall that local convergence provides what we call the convergence ball $B(x^*, r)$ where x^* is the center and r is its radius. Depending on how large the radius of convergence is, we can say that the method has more or less accessibility because the larger the ball, the more starting points within it will converge to the solution.

First, we consider the following conditions:

(A) H is μ_0-Lipschitz continuous operator on x^* such that

$$\|H(x) - H(x^*)\| \leq \mu_0(\|x - x^*\|), \quad x \in \Omega, \tag{11}$$

where $\mu_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function, with $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

(B) F' is μ_1-Lipschitz continuous operator such that

$$\|F'(x) - F'(y)\| \leq \mu_1(\|x - y\|), \quad x, y \in \Omega, \tag{12}$$

where $\mu_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function. We assume that there exists a function $h : [0, 1] \to \mathbb{R}^+$, which is continuous and non-decreasing, such that $\mu_1(tz) \leq h(t)\mu_1(z)$, with $t \in [0, 1]$ and $z \in [0, +\infty)$. In addition, we define $M = \int_0^1 h(t)dt$.

(C) We suppose that there exists the divided difference operator of the form $[u, v; G]$ for each pair $u, v \in \Omega$, $u \neq v$. Therefore, $[-, -; G]$ is a μ_2-Lipschitz continuous operator such that

$$\|[x, y; G] - [u, v; G]\| \leq \mu_2(\|x - u\|, \|y - v\|), \quad x, y, u, v \in \Omega, \tag{13}$$

where $\mu_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function respect to both arguments.

Note that since G is a non-differentiable operator, $\mu_2(0, 0) > 0$. If we consider that $\mu_2(0, 0) = 0$, this would imply that G is differentiable as proved in [14] and, therefore, we would not be doing a study of local convergence of the method (5) for any operator $H(x)$, be it differentiable or non-differentiable.

Now, we are going to introduce a lemma to make sure that the sequence of iterations $\{x_n\}$ given by the method (5) is well defined.

Lemma 1. Under conditions **(A)**, **(B)** and **(C)**, we assume that x_{n-1}, $x_{n-1} - \varepsilon H(x_{n-1})$, $x_{n-1} + \varepsilon H(x_{n-1}) \in \Omega$ for $\varepsilon > 0$, $n \geq 1$, and the following conditions are verified:

(D) Let x^* be the solution of $H(x) = 0$ and considering $\tilde{x} \in \Omega$, with $\|\tilde{x} - x^*\| \leq \delta$ and $\delta > 0$, so that $L^{-1} = (F'(x^*) + [x^*, \tilde{x}; G])^{-1}$ exists and $\|L^{-1}\| \leq \gamma$.

(E)

$$\alpha_{n-1} = \gamma(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \delta + \|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|))) < 1.$$

Then, x_n is well defined and it holds that:

$$\|x_n - x^*\| \leq \Gamma_{n-1}\|x_{n-1} - x^*\| \quad \text{where} \quad \Gamma_{n-1} = \frac{\tilde{\alpha}_{n-1}}{1 - \alpha_{n-1}} \quad \text{and}$$

$$\tilde{\alpha}_{n-1} = \gamma\left(M\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \varepsilon\mu_0(\|x_{n-1} - x^*\|))\right).$$

Proof. Note that $x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}) \in \Omega$ for $\varepsilon > 0$ and we assume that $x_{n-1} - \varepsilon H(x_{n-1}) \neq x_{n-1} + \varepsilon H(x_{n-1})$. We assume the above because otherwise we would have to $2\varepsilon H(x_{n-1}) = 0$ and, since $\varepsilon > 0$, we have that $H(x_{n-1}) = 0$ and so $x_m = x_{n-1} = x^*$, $\forall m \geq n - 1$ so that the result would be obtained in a simple way. So if $x_{n-1} - \varepsilon H(x_{n-1}) \neq x_{n-1} + \varepsilon H(x_{n-1})$, the operator exists $[x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G]$.

To simplify the proof notation, we define $A(x) = F'(x) + [x - \varepsilon H(x), x + \varepsilon H(x); G]$. Now, taking into account the conditions **(A)** - **(B)**,

$$\begin{aligned}
\|I - L^{-1}A(x_{n-1})\| &= \|I - L^{-1}(F'(x_{n-1}) + [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G])\| \\
&\leq \|L^{-1}\|\|L - F'(x_{n-1}) - [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G]\| \\
&\leq \|L^{-1}\|(\|F'(x^*) - F'(x_{n-1})\| + \|[x^*, \tilde{x}; G] - [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G]\|) \\
&\leq \gamma\left(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x^* - x_{n-1} + \varepsilon H(x_{n-1})\|, \|\tilde{x} - x_{n-1} - \varepsilon H(x_{n-1})\|)\right) \\
&\leq \gamma(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\|H(x_{n-1}) - H(x^*)\|, \|\tilde{x} - x^*\| + \|x_{n-1} - \varepsilon H(x_{n-1}) - x^*\|)) \\
&\leq \gamma(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \delta + \|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|))) \\
&\leq \alpha_{n-1} < 1.
\end{aligned}$$

By Banach's Lemma for invertible operators (see[6]), there exists $A(x_{n-1})^{-1}$ and

$$\|A(x_{n-1})^{-1}\| \leq \frac{\|L^{-1}\|}{1-\|L^{-1}A(x_{n-1})\|} \leq \frac{\gamma}{1-\alpha_{n-1}}.$$

Since method (5) says that

$$x_n = x_{n-1} + (F'(x_{n-1}) + [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G])^{-1} H(x_{n-1})$$

and we have seen that $A(x_{n-1})^{-1}$ exists, so x_n is well defined.

From the method (5) we deduce:

$$\begin{aligned} x_{n-1} - x^* &= x_{n-1} - A(x_{n-1})^{-1}H(x_{n-1}) - x^* = A(x_{n-1})^{-1}(A(x_{n-1})(x_{n-1}-x^*) - H(x_{n-1})) \\ &= A(x_{n-1})^{-1}((F'(x_{n-1}) + [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G])(x_{n-1}-x^*) - F(x_{n-1}) - G(x_{n-1})). \end{aligned}$$

Since $\int_{x_{n-1}}^{x^*}(F'(z) - F'(x_{n-1}))dz = F(x^*) - F(x_{n-1}) - F'(x_{n-1})(x^* - x_{n-1})$, substituting in the previous equation:

$$\begin{aligned} x_{n-1} - x^* &= A(x_{n-1})^{-1} \int_{x_{n-1}}^{x^*} (F'(z) - F'(x_{n-1}))dz \\ &\quad + A(x_{n-1})^{-1}([x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G](x_{n-1} - x^*) + G(x^*) - G(x_{n-1})) \\ &= A(x_{n-1})^{-1}\left(\int_0^1 (F'(x_{n-1} + t(x^* - x_{n-1})) - F'(x_{n-1}))(x^* - x_{n-1})dt \right) \\ &\quad + A(x_{n-1})^{-1}\left([x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G] - [x^*, x_{n-1}; G](x_{n-1} - x^*) \right). \end{aligned}$$

Taking norms in the previous expression and taking into account the conditions **(A)**-**(C)**, we obtain:

$$\begin{aligned} \|x_n - x^*\| &\leq \|A(x_{n-1})^{-1}\| \Big(\int_0^1 \mu_1(\|t(x^* - x_{n-1})\|)dt + \mu_2(\|x_{n-1} - \varepsilon H(x_{n-1}) - x^*\|, \|\varepsilon H(x_{n-1})\|) \Big) \|x_{n-1} - x^*\| \\ &\leq \frac{\gamma}{1-\alpha_{n-1}} (M\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \varepsilon\mu_0(\|x_{n-1} - x^*\|)))\|x_{n-1} - x^*\| \\ &= \frac{\tilde{\alpha}_{n-1}}{1-\alpha_{n-1}} \|x_{n-1} - x^*\| = \Gamma_{n-1}\|x_{n-1} - x^*\|. \end{aligned}$$

□

To prove that $\{x_n\}$ given by the method (5) converges to x^*, we are interested in the conditions under which $\{\|x_n - x^*\|\}_{n=0}^{\infty}$ is a strictly decreasing sequence of positive real numbers. By the *Lemma 1*, $\|x_n - x^*\| < \|x_{n-1} - x^*\|$ if $\Gamma_{n-1} < 1$ and this is true if only if $\tilde{\alpha}_{n-1} + \alpha_{n-1} < 1$.

Thus, if there exists at least a positive real root for the real equation:

$$\gamma\left(\mu_1(t) + \mu_2(t + \varepsilon\mu_0(t), \delta + t + \varepsilon\mu_0(t))\right) + \gamma\left(M\mu_1(t) + \mu_2(t + \varepsilon\mu_0(t), \varepsilon\mu_0(t))\right) - 1 = 0,$$

we denote by r the smallest positive real root.

So, for $x_{n-1} \in B(x^*, r)$ and $x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}) \in \Omega$ such that

$$\begin{aligned} \|x_{n-1} + \varepsilon H(x_{n-1}) - x^*\| &\leq \|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|) < r + \varepsilon\mu_0(r), \\ \|x_{n-1} - \varepsilon H(x_{n-1}) - x^*\| &\leq \|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|) < r + \varepsilon\mu_0(r), \end{aligned}$$

we get that

$$\alpha_{n-1} < m = \gamma(\mu_1(r) + \mu_2(r + \varepsilon\mu_0(r), \delta + r + \varepsilon\mu_0(r))) < 1$$

since μ_1 and μ_2 are non-decresing functions in $\mathbb{R}^+ \times \mathbb{R}^+$. Also,

$$\tilde{\alpha}_{n-1} < \tilde{m} = \gamma(M\mu_1(r) + \mu_2(r + \varepsilon\mu_0(r), \varepsilon\mu_0(r))).$$

Thus, since $m + \tilde{m} = 1$, we have:

$$\Gamma_{n-1} < \frac{\tilde{m}}{1-m} = 1.$$

where $\{\Gamma_{n-1}\}$ is a strictly decreasing sequence.

Bearing in mind these new results, we modify the condition **(E)** from the previous study and consider the following condition

(E') We assume that the equation

$$\gamma\left[(1+M)\mu_1(t)+\mu_2(t+\varepsilon\mu_0(t),\delta+t+\varepsilon\mu_0(t))+\mu_2(t+\varepsilon\mu_0(t),\varepsilon\mu_0(t))\right]-1=0 \tag{14}$$

has at least one positive real root and $B(x^*, r+\varepsilon\mu_0(r)) \subset \Omega$, where r is the smallest positive real root of (14).

The following result shows the local convergence results.

Theorem 2. Under the conditions **(A)**-**(D)** and **(E')**, if we choose an initial approximation $x_0 \in B(x^*, r)$, then the sequence $\{x_n\}$ given by the method (5) is well defined, belongs to the ball of convergence $B(x^*, r)$ and converges to the solution x^* from the equation $H(x) = 0$.

Proof. We have $x_0 + \varepsilon H(x_0) \neq x_0 - \varepsilon H(x_0)$ since otherwise $H(x_0) = 0$ and therefore $x_0 = x^*$ and $x_n = x^*$, $\forall n \leq 1$ and the result would be proved.

From *Lemma 1*,

$$\begin{aligned}\alpha_0 &= \gamma(\mu_1(\|x_0-x^*\|)+\mu_2(\|x_0-x^*\|+\varepsilon\mu_0(\|x_0-x^*\|),\delta+\|x_0-x^*\|+\mu_0(x_0-x^*)))\\ &< \gamma(\mu_1(r)+\mu_2(r+\varepsilon\mu_0(r),\delta+r+\varepsilon\mu_0(r))) = m < 1,\end{aligned}$$

we have that there exists $A(x_0)^{-1}$ and $\|A(x_0)^{-1}\| \leq \frac{\gamma}{1-\alpha_0}$. Hence, x_1 is well defined and

$$\|x_1 - x^*\| \leq \Gamma_0\|x_0 - x^*\| < \|x_0 - x^*\| < r.$$

Therefore $x_1 \in B(x^*, r)$. On the other hand,

$$\begin{aligned}\|x_1+\varepsilon H(x_1)-x^*\| &\leq \|x_1-x^*\|+\varepsilon\|H(x_1)\| \leq \|x_1-x^*\|+\varepsilon\mu_0(\|x_1-x^*\|)\\ &< \|x_0-x^*\|+\varepsilon\mu_0(\|x_0-x^*\|) < r+\varepsilon\mu_0(r),\\ \|x_1-\varepsilon H(x_1)-x^*\| &\leq \|x_1-x^*\|+\varepsilon\mu_0(\|x_1-x^*\|) < r+\varepsilon\mu_0(r).\end{aligned}$$

So, $x_1 + \varepsilon H(x_1), x_1 - \varepsilon H(x_1) \in \Omega$.

By induction on $n \geq 2$, we prove that, if $x_{n-1} \in B(x^*, r)$ and $x_{n-1} + \varepsilon H(x_{n-1})$, $x_{n-1} - \varepsilon H(x_{n-1}) \in \Omega$, with $x_{n-1} + \varepsilon H(x_{n-1}) \neq x_{n-1} - \varepsilon H(x_{n-1})$, then x_n is well defined,

$$\|x_n - x^*\| < \|x_{n-1} - x^*\|, \quad \|x_n + \varepsilon H(x_n) - x^*\| < r + \varepsilon\mu_0(r) \quad \text{and} \quad \|x_n - \varepsilon H(x_n) - x^*\| < r + \varepsilon\mu_0(r).$$

Assuming that the hypotheses are true for $n = 2, \ldots, k$, let's see that it is true for $n = k + 1$.

Since $x_k \in B(x^*, r)$ and $x_k + \varepsilon H(x_k), x_k - \varepsilon H(x_k) \in \Omega$, by Lemma 1 there exists $A(x_k)^{-1}$. This implies that x_{k+1} is well defined and so

$$\|x_{k+1} - x^*\| \leq \Gamma_k\|x_k - x^*\| < \|x_k - x^*\| < \ldots < \|x_0 - x^*\| < r.$$

Moreover,

$$\begin{aligned}\|x_{k+1}+\varepsilon H(x_{k+1})-x^*\| &\leq \|x_{k+1}-x^*\|+\varepsilon\mu_0(\|x_{k+1}-x^*\|) < r+\varepsilon\mu_0(r),\\ \|x_{k+1}-\varepsilon H(x_{k+1})-x^*\| &\leq \|x_{k+1}-x^*\|+\varepsilon\mu_0(\|x_{k+1}-x^*\|) < r+\varepsilon\mu_0(r).\end{aligned}$$

Then, $\{x_n\} \subset B(x^*, r)$ and $\{\|x_n - x^*\|\}$ is a strictly decreasing sequence of positive real numbers. Since $\{Gamma_{n-1}\}$ is also strictly decreasing, we have that

$$\|x_n - x^*\| < \Gamma_{n-1}\|x_{n-1} - x^*\| < \ldots < \Gamma_0^n\|x_0 - x^*\|.$$

Therefore, when $n \to \infty$, $\{x_n - x^*\} \to 0$ is satisfied, and therefore $\{x_n\}$ converges to x^*. □

Now we are going to present a result that ensures uniqueness of solution under certain conditions and allows us to define the uniqueness ball.

Theorem 3. Under the conditions **(A)**-**(D)** and **(E')**, we assume that the equation

$$E_1(t) = \gamma(M\mu_1(t) + \mu_2(0, t+\delta)) - 1 = 0 \tag{15}$$

has at least one positive real root, where R is the smallest positive real root of (15). Then the solution x^* is the only solution of the equation $H(x) = 0$ in $\overline{B(x^*, R)} \cap \Omega$.

Proof. Let $y^* \in \overline{B(x^*, R)} \cap \Omega$ and $H(y^*) = 0$. We define the following operator:

$$P = \int_0^1 F'(x^* + t(y^* - x^*))dt + [x^*, y^*; G].$$

Using **(A)** and **(B)**, we get:

$$\begin{aligned}
\|L^{-1}P - I\| &\leq \|L^{-1}\|\|P - L\| \\
&\leq \|L^{-1}\|\left[\int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x^*)\|dt + \|[x^*, y^*; G] - [x^*, \tilde{x}; G]\|\right] \\
&\leq \gamma\left[\int_0^1 \mu_1(\|t(y^* - x^*)\|)dt + \mu_2(0, \|y^* - \tilde{x}\|)\right] \\
&< \gamma(M\mu_1(R) + \mu_2(0, R+\delta)) = 1.
\end{aligned}$$

In the last inequality we have used the hypothesis that there is at least one positive real root of $E_1(t)$ since this implies that at least one of the functions $\mu_1(t)$ or $\mu_2(t)$ is strictly increasing.

Therefore, there exists $P^{-1} \in \mathcal{L}(X, Y)$ and, for the identity

$$\begin{aligned}
0 = H(x^*) - H(y^*) &= F(x^*) - F(y^*) + G(x^*) - G(y^*) \\
&= \left(\int_0^1 F'(x^* + t(y^* - x^*))dt + [x^*, y^*; G]\right)(x^* - y^*) \\
&= P(x^* - y^*),
\end{aligned}$$

we deduce that $x^* = y^*$. □

4 | SEMILOCAL CONVERGENCE

In this section we are going to analyze the semilocal convergence of iterative method (5), that is, we are going to impose the necessary conditions so that, given a starting point x_0, the iterative method (5) converges to a solution. Moreover, we determine the existence and uniqueness convergence domains.

For this, we consider $x_0 \in \Omega$ and we assume:

(I) Exists $A_0^{-1} = A(x_0)^{-1} = (F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0)])^{-1}$ such that $||A_0^{-1}|| \leq \beta$. Furthermore, $||H(x_0)|| \leq \eta_0$ such that $||A_0^{-1}H(x_0)|| \leq \beta\eta_0 = \omega$.

(II) $||H(x) - H(y)|| \leq \mu_0(||x - y||)$, $x, y \in \Omega$ where $\mu_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function.

(III) $||F'(x) - F'(y)|| \leq \mu_1(||x - y||)$, $x, y \in \Omega$ where $\mu_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function. We assume that exists a continuous non-decreasing function $h : [0, 1] \to \mathbb{R}^+$ such that $\mu_1(tz) \leq h(t)\mu_1(z)$ with $z \in [0, 1]$ and $t \in [0, \infty)$. We denote $M = \int_0^1 h(t)dt$.

Note that h always exists, by taking $h(t) = 1$, as a consequence of μ_1 is a non-decreasing function.

(IV) We suppose that exists $[x, y; G]$ for each pair $x, y \in \Omega$, $x \neq y$ such that the divided differences operator satisfies:

$$||[x, y; G] - [u, v; G]|| \leq \mu_2(||x - u||, ||y - v||); \text{for all } x, y, u, v \in \Omega,$$

where $\mu_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function with respect to both arguments.

Note that since G is non-differentiable, $\mu_2(0,0) > 0$. Then, we will give a generalized result of semilocal convergence. We will do it by fixing the radius r of the domain of existence and we will try to calculate it so that the sequence $\{x_n\}$ is contained in the ball of convergence $B(x_0, r)$ whose center is the initial iteration and whose radius is our fixed r.

Theorem 4. Under the conditions **(I) − (IV)**, we consider the following parameter $s = \mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega)$, and the functions

$$m(t) = \beta(\mu_1(t) + \mu_2(t + \varepsilon\mu_0(t), t + \varepsilon\mu_0(t))) \text{ and } \lambda(t) = \frac{\beta s}{1 - m(t)},$$

with $t \in \mathbb{R}^+$. Let us assume that the following equation has at least one positive real root,

$$\frac{1}{1 - \lambda(t)}\omega - t = 0, \tag{16}$$

by denoting r the smallest one and if it is verified that $s\beta < 1$, $m(r) < 1$ and $\lambda(r) < 1$. Then, the sequence $\{x_n\}$ given by the iterative method (5) is well defined, remains in $B(x_0, r)$ and converges to the solution x^* of the equation $H(x) = 0$.

Proof. To simplify the notation, we denote

$$A_n = A(x_n) = F'(x_n) + [x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G].$$

First, we prove by induction that the sequence given by the method (5) is well defined, that is, in each step $n \geq 1$, the operator A_n is invertible and the iteration x_{n+1} can be obtained.

We start from the fact that x_1 is well defined since by the condition **(I)**, we have:

$$||x_1 - x_0|| = ||A_0^{-1}H(x_0)|| \leq \omega < r.$$

Considering the conditions **(I) − (IV)** and the auxiliary function implied, we have:

$$\begin{aligned}
||I - A_0^{-1}A_1|| &\leq ||A_0^{-1}||\ ||A_0 - A_1|| \\
&\leq ||A_0^{-1}||\ ||F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G] - F'(x_1) - [x_1 + \varepsilon H(x_1), x_1 - \varepsilon H(x_1)]|| \\
&\leq \beta\left(\mu_1(||x_1 - x_0||) + \mu_2(||x_1 - x_0|| + \varepsilon\mu_0(||x_1 - x_0||), ||x_1 - x_0|| + \varepsilon\mu_0(||x_1 - x_0||))\right) \\
&\leq \beta\left(\mu_1(r) + \mu_2(r + \varepsilon\mu_0(r), r + \varepsilon\mu_0(r))\right) = m(r) < 1.
\end{aligned}$$

So, by applying Banach's lemma, there exists A_1^{-1} and

$$||A_1^{-1}|| \leq \frac{\beta}{1 - m(r)}.$$

Since F is Fréchet differentiable, one has

$$\begin{aligned}
F(x_1) &= F(x_0) + \int_0^1 F'(x_0 + t(x_1 - x_0))(x_1 - x_0)dt \\
&= F(x_0) + F'(x_0)(x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt.
\end{aligned}$$

On the other hand, for the operator G we use the definition of divided difference:

$$G(x_1) = G(x_0) - [x_0, x_1; G](x_0 - x_1).$$

Therefore, since $H(x) = F(x) + G(x)$, we have:

$$H(x_1) = H(x_0) + F'(x_0)(x_1 - x_0) + [x_0, x_1; G](x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt.$$

So, by substituting $H(x_0)$ by its obtained expression with the function iteration of ((5)), we obtain:

$$\begin{aligned}
H(x_1) &= -[F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G]](x_1 - x_0) \\
&\quad + F'(x_0)(x_1 - x_0) + [x_0, x_1; G](x_1 - x_0) \\
&\quad + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt \\
&= ([x_0, x_1; G] - [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G])(x_1 - x_0) \\
&\quad + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt.
\end{aligned}$$

Applying norms to the above expression one gets

$$\begin{aligned}
||H(x_1)|| &\leq (\mu_2(\varepsilon\eta_0, ||x_1 - x_0|| + \varepsilon\eta_0) + \mu_1(||x_1 - x_0||) \int_0^1 h(t)dt)||x_1 - x_0|| \\
&\leq (\mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega))||x_1 - x_0|| \leq s||x_1 - x_0|| < s\beta\eta_0 < \eta_0.
\end{aligned}$$

Therefore, we can say that x_2 is well defined thanks to the existence of A_1^{-1} and verifies

$$\begin{aligned}
||x_2 - x_1|| &\leq ||A_1^{-1}H(x_1)|| \\
&\leq \frac{\beta}{1 - m} s||x_1 - x_0|| \\
&= \frac{\beta s}{1 - m(r)}||x_1 - x_0|| = \lambda(r)||x_1 - x_0|| < \omega.
\end{aligned}$$

So,

$$||x_2 - x_0|| \leq ||x_2 - x_1|| + ||x_1 - x_0|| \leq (\lambda(r) + 1)||x_1 - x_0|| \leq (\lambda(r) + 1)\omega < r$$

and $x_2 \in B(x_0, r)$.

By mathematical induction, we assume that for $k = 2, \ldots, n - 1$, the following are verified:

(1) There exists $A_{k-1}^{-1} = \left(F'(x_k) + [x_k + \varepsilon H(x_k), x_k - \varepsilon H(x_k); G]\right)^{-1}$ such that $||A_{k-1}^{-1}|| \leq \frac{\beta}{1 - m(r)}$.

(2) $||x_k - x_{k-1}|| \leq \lambda(r)||x_{k-1} - x_{k-2}|| \leq \lambda(r)^{k-1}||x_1 - x_0|| < \omega$.

(3) $||H(x_k)|| \leq s||x_k - x_{k-1}|| < \eta_0$.

(4) $||x_k - x_0|| \leq \frac{1 - \lambda(r)^k}{1 - \lambda(r)}\omega < r$ and $x_k \in B(x_0, r)$.

So, we are going to see that (1) − (4) are true for $k = n$.

First, let us see that there exists A_{n-1}^{-1}.

$$\begin{aligned}
||I - A_0^{-1}A_{n-1}|| &\leq \beta||F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0)] - F'(x_{n-1}) - [x_{n-1} + \varepsilon H(x_{n-1}), x_{n-1} - \varepsilon H(x_{n-1}); G]|| \\
&\leq \beta\left(\mu_1(r) + \mu_2(r + \varepsilon\mu_0(r), r + \varepsilon\mu_0(r))\right) = m(r) < 1.
\end{aligned}$$

So, by Banach's lemma, exists A_{n-1}^{-1} and

$$||A_{n-1}^{-1}|| \leq \frac{\beta}{1 - m(r)}.$$

Then, we have that x_n is well defined and, furthermore, **(3)** holds:

$$||x_n - x_{n-1}|| \leq ||A_{n-1}^{-1}H(x_{n-1})|| \leq \frac{\beta}{1 - m(r)} s||x_{n-1} - x_{n-2}|| \leq \lambda(r)^{n-1}\omega \leq \omega < r.$$

Also, thanks to this bound, we can ensure that $x_n \in B(x_0, r)$ since:

$$\begin{aligned}||x_n - x_0|| &\leq ||x_n - x_{n-1}|| + ||x_{n-1} - x_0|| \\ &\leq \lambda(r)^{n-1}\omega + \frac{1-\lambda(r)^{n-1}}{1-\lambda(r)}\omega \leq \frac{1-\lambda(r)^n}{1-\lambda(r)}\omega < \frac{1}{1-\lambda(r)}\omega = r.\end{aligned}$$

On the other hand,

$$\begin{aligned}H(x_n) &= -[F'(x_{n-1}) + [x_{n-1} + \varepsilon H(x_{n-1}), x_{n-1} - \varepsilon H(x_{n-1}); G]](x_n - x_{n-1}) \\ &\quad + F'(x_{n-1})(x_n - x_{n-1}) + [x_{n-1}, x_n; G](x_n - x_{n-1}) \\ &\quad + \int_0^1 (F'(x_{n-1} + t(x_n - x_{n-1})) - F'(x_{n-1}))(x_n - x_{n-1})dt \\ &= ([x_{n-1}, x_n; G] - [x_{n-1} + \varepsilon H(x_{n-1}), x_{n-1} - \varepsilon H(x_{n-1}); G])(x_n - x_{n-1}) \\ &\quad + \int_0^1 (F'(x_{n-1} + t(x_n - x_{n-1})) - F'(x_{n-1}))(x_n - x_{n-1})dt.\end{aligned}$$

Applying norms to the above expression one gets

$$\begin{aligned}||H(x_n)|| &\leq (\mu_2(\varepsilon\eta_0, ||x_n - x_{n-1}|| + \varepsilon\eta_0) + \mu_1(||x_n - x_{n-1}||)\int_0^1 h(t)dt)||x_n - x_{n-1}|| \\ &\leq (\mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega))||x_n - x_{n-1}|| \leq s||x_n - x_{n-1}|| < s\lambda(r)^{n-1}||x_1 - x_0|| \\ &\leq s\lambda(r)^{n-1}\beta\eta_0 \leq \eta_0.\end{aligned}$$

Therefore, since **(1)-(5)** holds for $k = n$, we have shown that these hypotheses are true $\forall n \in \mathbb{N}$.

Now, using these hypotheses, we are going to prove that the sequence of iterations $\{x_n\}$ is a Cauchy sequence. For $k \geq 1$,

$$\begin{aligned}||x_{n+k} - x_{n-1}|| &\leq ||x_{n+k} - x_{n+k-1}|| + \ldots + ||x_n - x_{n-1}|| \\ &\leq \left(\lambda(r)^k + \lambda(r)^{k-1} + \ldots + 1\right)||x_n - x_{n-1}|| \\ &\leq \frac{1-\lambda(r)^{k+1}}{1-\lambda(r)}||x_n - x_{n-1}|| \leq \frac{1-\lambda(r)^k}{1-\lambda(r)}\lambda(r)^{n-1}||x_1 - x_0|| \\ &< \frac{1}{1-\lambda(r)}\lambda(r)^{n-1}||x_1 - x_0||.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in B(x_0, r)$ where

$$r = \frac{1}{1-\lambda(r)}\omega.$$

Finally, we see that x^* is a zero of $H(x)$. Since

$$||H(x_n)|| \leq (\mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega))||x_n - x_{n-1}||.$$

Moreover, by taking limits when $n \to \infty$, as $||x_n - x_{n-1}|| \to 0$, thus, by the continuity of the operator H, $H(x^*) = 0$.

□

Theorem 5. Under the conditions **(I) − (IV)**, we suppose that the equation

$$E_2(t) = \beta(\mu_1(r + t) + \mu_2(r + \varepsilon\mu_0(t), r + \varepsilon\mu_0(t))) - 1 = 0 \tag{17}$$

has at least one positive real root, we denote R the smallest one. Then, the solution x^* is the unique solution of the equation $H(x) = 0$ in $\overline{B(x_0, R)} \cap \Omega$.

Proof. To see uniqueness, we assume that there is another solution $y^* \in \overline{B(x_0, R)} \cap \Omega$ and consider the operator

$$P = \int_0^1 F'(x^* + t(y^* - x^*))dt + [x^*, y^*; G].$$

Since $P(y^* - x^*) = H(y^*) - H(x^*) = 0$, if P is an invertible operator then $x^* = y^*$. So, in order to apply Banach lemma we have:

$$\begin{aligned}
||A_0^{-1}P - I|| &\leq \beta\Big(\int_0^1 ||F'(x^* + t(y^* - x^*)) - F'(x_0)||dt + |||[x^*, y^*; G] - [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G]||| \Big) \\
&\leq \beta\Big(\int_0^1 \mu_1(||x^* - x_0 + ty^* - tx^*)||)dt + \mu_2(||x^* - x_0|| + \varepsilon\|H(x_0)\|, ||y^* - x_0|| + \varepsilon||H(x0)||) \Big) \\
&\leq \beta\Big(\int_0^1 \mu_1(||(1-t)(x^* - x_0) + t(y^* - x_0)||)dt + \mu_2(||x^* - x_0|| + \varepsilon\mu_0(||x^* - x_0||), ||y^* - x_0|| + \varepsilon\mu_0(||y^* - x_0||)) \Big) \\
&< \beta(\mu_1(r + R) + \mu_2(r + \varepsilon\mu_0(R), r + \varepsilon\mu_0(R))) = 1.
\end{aligned}$$

In the last inequality we have used the hypothesis that there is at least one positive real root of $E_2(t)$ since this implies that at least one of the functions $\mu_1(t)$ or $\mu_2(t)$ is strictly increasing.

Therefore, P^{-1} exists, and the result holds. □

5 | NUMERICAL EXPERIENCE

In this section we deal with the application of the previously obtained local convergence results. We consider the nonlinear elliptic boundary value problem solved in [12], that describes properties in the gas dynamic theory. But, now we add a nondifferentiable part in order to illustrate the theory developed in our study.

$$\begin{aligned}
&u_{ss} + u_{tt} = u^3 + |u| \quad \text{with S=}\{(s,t) \in \mathbb{R}^2 : s,t \in [0,1]\} \\
&u(s,0) = 2s^2 - s + 1, \quad u(s,1) = 2, \quad 0 \leq s \leq 1 \\
&u(0,t) = 2t^2 - t + 1, \quad u(1,t) = 2, \quad 0 \leq t \leq 1
\end{aligned}$$

We create a mesh for discretising the problem being: $h = \frac{1}{n+1}$ and $\tau = \frac{1}{m+1}$, we have the mesh points (s_i, t_j) with $s_i = ih,\ i = 0, ..., n+1$ and $t_j = j\tau, j = 0, ..., m+1$, such that:

$$u_{ss}(s_i, t_j) + u_{tt}(s_i, t_j) = u(s_i, t_j)^3 + |u((s_i, t_j)|.$$

So, by approximating the partial derivatives by central divided differences we have the following equation:

$$\frac{u(s_{i+1}, t_j) - 2u(s_i, t_j) + u(s_{i-1}, t_j)}{h^2} + \frac{u(s_i, t_{j-1}) - 2u(s_i, t_j) + u(s_i, t_{j+1})}{\tau^2} = u(s_i, t_j)^3 + |u(s_i, t_j)|$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

The boundary conditions are:

$$\begin{aligned}
&u(s_i, t_0) = 2s_i^2 - s_i + 1, \quad u(s_i, t_{q+1}) = 2, \quad i = 1, \ldots, n \\
&u(s_0, t_j) = 2t_j^2 - t_j + 1, \quad u(s_{p+1}, t_j) = 2, \quad j = 0, \ldots, m+1.
\end{aligned}$$

Now, we denote $u(s_i, t_j) = u_{i,j}$ simplifying the notation we obtain:

$$2(1+\left(\frac{h}{\tau}\right)^2)u_{i,j}-(u_{i-1}+u_{i+1,j})-\left(\frac{h}{\tau}\right)^2(u_{i,j-1}+u_{i,j+1})=-h^2(u_{ij}+|u_{ij}|) \tag{18}$$

for $i=1,\dots,n$ and $j=1,\dots,m$ with

$$u_{i,0}=2s_i^2-s_i+1,\quad u_{i,q+1}=2,\quad i=1,\dots,n$$
$$u_{0,j}=2t_j^2-t_j+1,\quad u_{p+1,j}=2,\quad j=0,\dots,m+1.$$

The equation (18) together with the boundary conditions form a nonlinear system of size $(nm)\times(nm)$ given by:

$$Tu+h^2v(u)=\omega$$

where

$$T=\begin{pmatrix} A & B & 0 & \cdots & 0\\ B & A & B & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & B\\ 0 & \dots & 0 & B & A\end{pmatrix},\qquad A=\begin{pmatrix} 2(1+(\frac{h}{\tau})^2) & -1 & 0 & \cdots & 0\\ -1 & 2(1+(\frac{h}{\tau})^2) & -1 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -1\\ 0 & \dots & 0 & -1 & 2(1+(\frac{h}{\tau})^2)\end{pmatrix},$$

$B=-(\frac{h}{\tau})^2I_{n\times n}$, $u=(u_1,\dots,u_d)^T$, $v(u)=(u_1^3,\dots,u_d^3)^T+(|u_1|,\dots,|u_d|)^T$ being $d=nm$, and W is a vector formed by the boundary conditions. Then, the nonlinear system can be formulated as follows:

$$H(u)=Tu+h^2v(u)-W=0, \tag{19}$$

being its differenctiable part $F(u)=T(u_1,\dots,u_d)+h^2(u_1^3,\dots,u_d^3)^T-W$, and the non-differentiable part $G(u)=h^2(|u_1|,\dots,|u_d|)^T$, such that $F(u)+G(u)=0$.

We calculate the linear operator $F'(u)=T+3h^2\mathrm{diag}[u_1^2,\dots,u_d^2]$. Moreover we characterize the divided differences considering: $[u,v;G]=([u,v;G]_{ij})_{i,j=1}^d\in\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)$ where

$$[u,v;G]_{ij}=\frac{1}{u_j-v_j}(G_i(u_1,\dots,u_j,v_{j+1},\dots,v_d)-G_i(u_1,\dots,u_{j-1},v_j,\dots,v_d))$$

and $u=(u_1,\dots,u_d)^T$, $v=(v_1,\dots,v_d)^T$, then

$$[u,v;G]=h^2\cdot diag\left(\frac{|u_1|-|v_1|}{u_1-v_1},\dots,\frac{|u_d|-|v_d|}{u_d-v_d}\right).$$

To solve the problem (19), we approximate the exact solution u^* with starting point $u^{(0)}=(1,1,...,1)$, denoting the iteration i$-th$ by the $u^{(i)}=(u_1^{(i)},...,u_d^{(i)}$ when running the iterative scheme (5) in Matlab2020 by using variable precision arithmetic with 100 digits, using as stopping criteria $\|u^{(n+1)}-u^{(n)}\|<10^{-30}$ and with $n=m=4$ and $\varepsilon=1$. The solution obtained after 6 iterations verifies $||u^{(6)}-u^{(5)}||<5.2930e-37$ and the norm of nonlinear operator H at this point verifies $||H(u^{(6)})||\leq 3.3391e-75$, reshaping the approximated solution in $\mathbb{R}^{16}$ to the initial size 4×4 for $i,j=1,...,4$ is

$$u^{(6)}(s_i,t_j)=\begin{pmatrix} 0.92416 & 0.98713 & 1.11328 & 1.39532\\ 1.01807 & 1.10905 & 1.25039 & 1.51249\\ 1.20200 & 1.27954 & 1.39495 & 1.60315\\ 1.50792 & 1.54714 & 1.61111 & 1.73408\end{pmatrix}$$

Finally, we change the starting point to $u^{(0)}=(1.5,1.5,...,1.5)$ noting that the iterative method converges to the same solution, now we have $||u^{(6)}-u^{(5)}||<1.4351e-41$ and the norm of nonlinear operator H at this point verifies $||H(u^{(6)})||\leq 2.1086e-84$.

So, we use this approximated solution to establish that if there exists the exact solution u^* then $u^*\in B(\bar{1},l)$, and so we take $\Omega=B(\bar{1},l)$ with $l\geq1$ and $\bar{1}=(1,1,...,1)$. Therefore, we can obtain the bounds for the conditions assumed in the semilocal

convergence study and establishing the auxiliary functions μ_0, μ_1 and μ_2 involved in conditions $(I) - (IV)$, we observe that:

$$\begin{aligned} ||H(u) - H(v)|| &= ||F(u) + G(u) - F(v) - G(v)|| \\ &\leq ||F(u) - F(v)|| + ||G(u) - G(v)||. \end{aligned}$$

But taking into account that F is differentiable, we have, by applying the mean value theorem, that:

$$||F(u) - F(v)|| \leq F'(\tilde{u})||u - v|| \text{ with } \tilde{u} \in B(\bar{1}, l),$$

so,

$$\begin{aligned} ||F(u) - F(v)|| &\leq (||T|| + 3h^2||\text{diag}(\tilde{u}_1^2, \ldots, \tilde{u}_d^2))||||u - v|| \\ &\leq (||T|| + 3h^2(1 + l)^2)||u - v||. \end{aligned}$$

On the other hand, by using the definition of the divided differences we have:

$$||G(u) - G(v)|| \leq |||[u, v, G]||||u - v||.$$

Therefore, we define $\mu_0(t) = (||T|| + 3h^2(1+l)^2 + h^2)t$. From the expression of $F'(u)$ deduced before we get $\mu_1(t) = 6h^2(1+l)t$, and using the characterization of divided differences we define $\mu_2(s,t) = 2h^2$.

By using these auxiliary functions we focus on applying the semilocal convergence results obtained previously, by taking the initial approximation

$$u^{(0)} = (1, 1, 1.2, 1.6, 1.1, 1.2, 1.4, 1.6, 1.2, 1.4, 1.5, 1.7, 1.4, 1.6, 1.6, 1.8)^T,$$

we obtain

$$\beta = 0.96767, \qquad \eta_0 = 0.34784, \qquad \omega = 0.33659$$

and solving equations (16) and (17) we obtain the existence and uniqueness radius, r and R involved in our semilocal convergence study. We consider different values for l, while $\varepsilon = 0.1$ in scheme (5). In Table 3 we can see the radii obtained r and R while in the last two columns we check the conditions $m(r) < 1$ and $\lambda(r) < 1$. We obtain better results when l is smaller because in this case the bounds used are more accurate.

TABLE 3 Numerical results with different values of l.

l	r	R	$m(r)$	$\lambda(r)$
1.5	0.45918	1.12985	0.34401	0.26696
2.0	0.51367	0.81052	0.43529	0.34473
2.25	0.58617	0.63616	0.51984	0.42577

Finally, once has been proved the existence of an exact solution, u^*, we focus on the local convergence study also in $\Omega = B(\bar{1}, l)$ with $l \geq 1$, so we set in condition (D) that $||\tilde{u} - u^*|| \leq \delta < l$ obtaining parameter γ, the convergence and uniqueness radius, r and R of section 3.2. We take different values for l and $u_0 = \tilde{u}$ while we consider u^* as the approximate solution given by the scheme (5) and $\varepsilon = 0.1$. In Table 4 we can observe the results, it can be checked that the accessibility improves when we work in a smaller domain, that is due to the fact that we can find more precise bounds for our local convergence restrictions.

6 | CONCLUSIONS

In this work we use a decomposition technique that consists on considering the non-differentiable and nonlinear operator $H(x)$ as $H(x) = F(x) + G(x)$, where F is Fréchet differentiable and G is continuous but non-differentiable. So we define a new iterative method to solve $H(x) = 0$ that allows us to maintain the quadratic order of convergence due to Newton's method for

TABLE 4 Numerical results with different values of u_0 and l.

$u^{(0)} = \tilde{u}$	l	$\delta = \|u^* - \tilde{u}\|$	γ	r	R
$(1, 1, 1, \dots, 1)^T$	1.5	0.73408	1.27893	0.69101	2.33968
$(1, 1, 1, \dots, 1)^T$	2	0.73408	1.27893	0.575838	1.94974
$(1, 1, 1, \dots, 1)^T$	3	0.73408	1.27893	0.43188	1.46231
$(-0.25, \dots, -0.25)^T$	1.5	1.98408	1.29837	0.67799	2.30065
$(-0.25, \dots, -0.25)^T$	2	1.98408	1.29837	0.56499	1.91721
$(-0.25, \dots, -0.25)^T$	3	1.98408	1.29837	0.42375	1.43791
$(-1, -1, \dots, -1)^T$	2	2.73408	1.33098	0.54753	1.86479
$(-1, -1, \dots, -1)^T$	3	2.73408	1.33098	0.41064	1.39859

non-differentiable operators. Besides, by means of a dynamical study, we check that the iterative process considered has as good accessibility as Newton's method. We focus on the local and semilocal convergence study by using weaker conditions that in previous studies.

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