

3D structure - 2D plate - 1D rod interaction problem

Matko Ljulj¹ and Josip Tambaca²

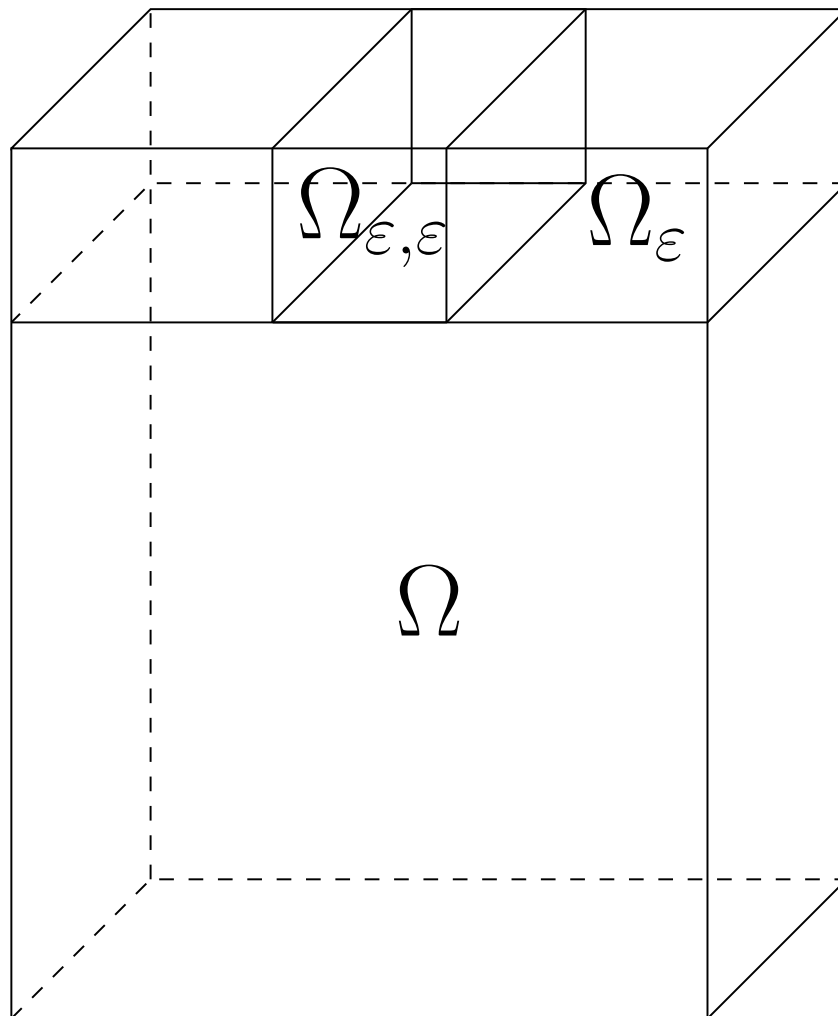
¹University of Zagreb Faculty of Science

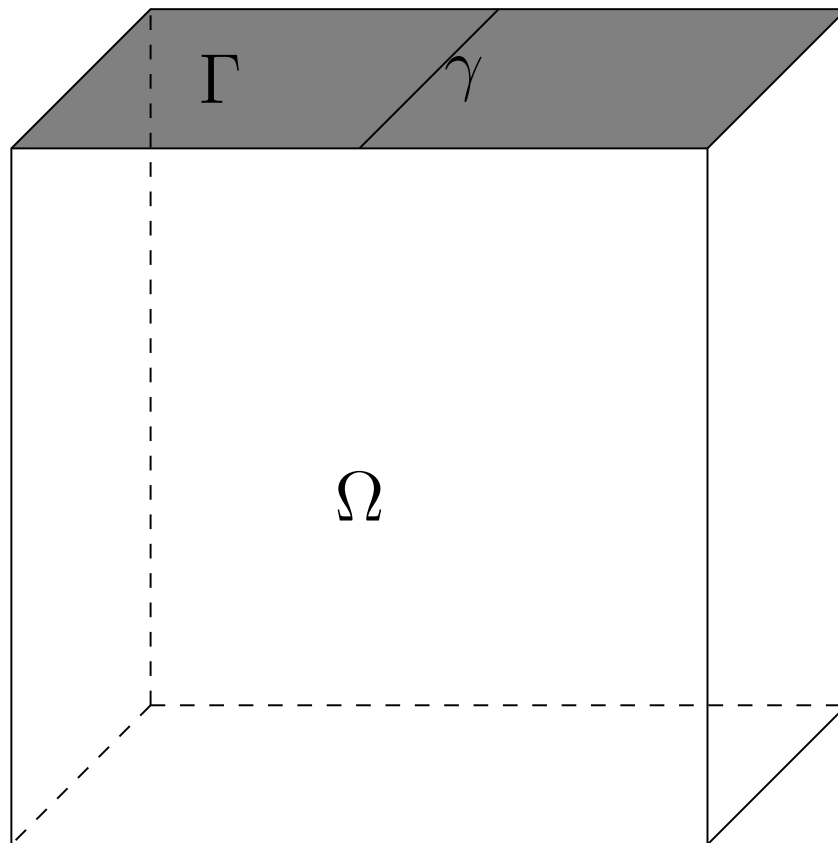
²University of Zagreb, Faculty of Science

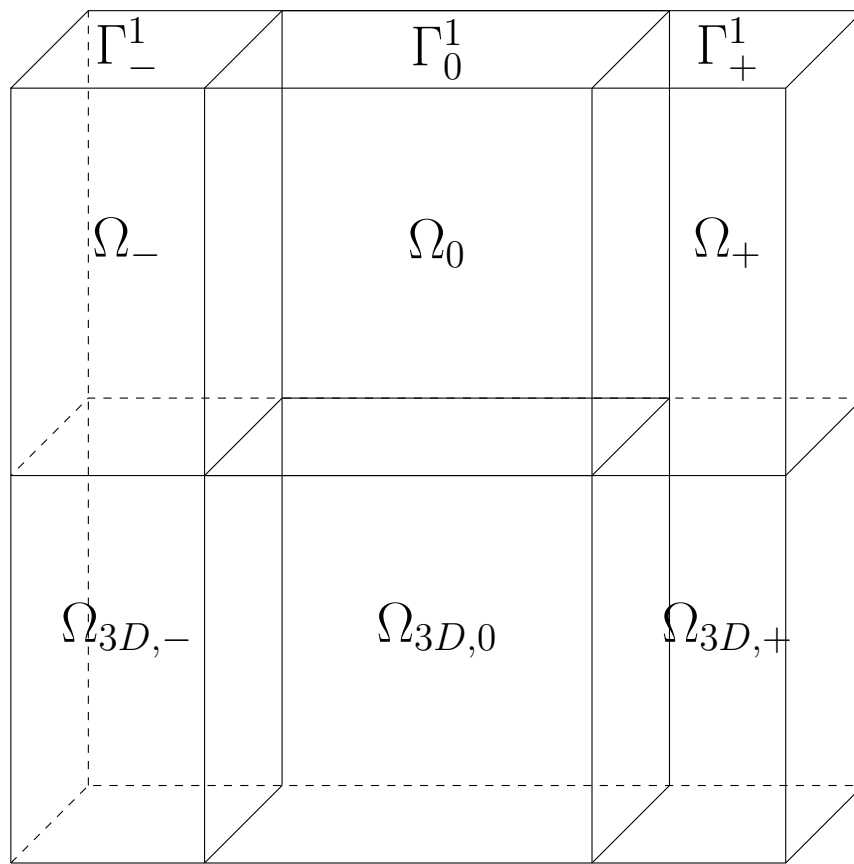
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Abstract

In this paper we consider the equilibrium problem of interaction of three elastic bodies of different elastic properties. The main body is the unit cube. On top of it is a thin layer/cuboid of thickness ϵ of material whose stiffness is of order $\frac{1}{\epsilon}$ that in the middle contains another cuboid which is of width and thickness ϵ that is made of material with elasticity coefficients of order $\frac{1}{\epsilon^q}$ for $q > 0$. We show that the family of solutions of linearized elasticity problems, when ϵ tends to zero, converges to a solution of a problem that is posed only on the unit cube with possibly additional elastic terms on the boundary related to the plate/rod energy of the thin elastic parts. It turns out that there are five different regimes related to different values of q ($q \in [0, 2), \{2\}, [2, 4), \{4\}, [4, \infty)$) with different limit problems. We further formulate a model posed on the unit cube that has the same asymptotics when ϵ tends to zero as the full 3d problem posed on the union of the unit cube and thin cuboids. This model then can be used as the approximating model in all regimes.







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August 25, 2022

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Abstract

In this paper we consider the equilibrium problem of interaction of three elastic bodies of different elastic properties. The main body is the unit cube. On top of it is a thin layer/cuboid of thickness ε of material whose stiffness is of order $\frac{1}{\varepsilon}$ that in the middle contains another cuboid which is of width and thickness ε that is made of material with elasticity coefficients of order $\frac{1}{\varepsilon^q}$ for $q > 0$. We show that the family of solutions of linearized elasticity problems, when ε tends to zero, converges to a solution of a problem that is posed only on the unit cube with possibly additional elastic terms on the boundary related to the plate/rod energy of the thin elastic parts. It turns out that there are five different regimes related to different values of q ($q \in \langle 0, 2 \rangle, \{2\}, \langle 2, 4 \rangle, \{4\}, \langle 4, \infty \rangle$) with different limit problems. We further formulate a model posed on the unit cube that has the same asymptotics when ε tends to zero as the full 3d problem posed on the union of the unit cube and thin cuboids. This model then can be used as the approximating model in all regimes.

Keywords: linearized elasticity, interaction, thin, plate, rod

*mljulj@math.hr, Department of Mathematics, Faculty of Science, University of Zagreb

†tambaca@math.hr, Department of Mathematics, Faculty of Science, University of Zagreb

1 Introduction

Interaction of two or more continua appears in many real life situations. Thus it is very important to have accurate model of interaction. In cases when one of continua is thin this turns out to be more delicate. On one hand numerical approximation for thin bodies leads to large meshes which is undesired property, on the other hand thin bodies allow efficient lower-dimensional models. However, coupling of these lower-dimensional models with three-dimensional is nontrivial and difficult if the lower-dimensional model is one-dimensional. Possible example is in hemodynamics. If you want to model interaction of the vessel wall and the stent inserted in it and you want to use a one-dimensional model for the stent struts (see [31]) and the vessel is thick enough so a two-dimensional model is not adequate (as in [5]). In this application the stiffness of the stent struts is much larger than the stiffness of the vessel wall, so it serves as motivation for the problem at hand. A similar and more simple problem of interaction of elastic three-dimensional cube and a plate-like three-dimensional body is already considered in [18], while its 2d–1d analogue is numerically investigated in [17].

In this paper we give rigorous derivation and justification of the following problem. Let $\varepsilon > 0$ be a small parameter that will describe the thickness of thin parts of the system. The elastic body consists of the unit cube $\Omega = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]$ with thin layer at the top $\Omega^\varepsilon = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [0, \varepsilon]$ which contains the thin strip $\Omega^{\varepsilon, \varepsilon} = [0, 1] \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \times [0, \varepsilon]$. The three parts of the domain Ω , $\Omega^\varepsilon \setminus \Omega^{\varepsilon, \varepsilon}$ and $\Omega^{\varepsilon, \varepsilon}$ are assumed to be made of different materials whose elasticity coefficients are related to the small parameter ε with orders $1, \frac{1}{\varepsilon}, \frac{1}{\varepsilon q}$, respectively. The parameter $q > 0$ serves to relate the stiffness of the elastic strip to the elastic properties of the cube and the thin plate. We fix the bottom of the cube ($x_3 = -1$) and apply the forcing at the top ($x_3 = \varepsilon$) of order ε^0 . We then perform the asymptotic analysis when ε tends to zero including the convergence proof of the associated linearized elasticity problem and obtain five different limit models depending on the value of $q > 0$. All limit models are given in the unit cube with additional terms at the top boundary. From [18] and the analysis of the 3d–2d model the stiffness order $\frac{1}{\varepsilon}$ for the plate-like body $\Omega^\varepsilon \setminus \Omega^{\varepsilon, \varepsilon}$ corresponds to the membrane behavior of the plate. Thus in all limit models the energy contains also the membrane energy at the top of the cube. For $q = 2$ the membrane energy of the rod is also included in the limit energy, while for $q = 4$ the flexural energy of the rod is included but only for the bending in direction tangential to the top of the cube. There is no flexural energy related to the bending in the normal direction since the plate gives no resistance to bending (displacement in the normal direction). This is done in Section 3 and presented in Theorem 4. Note here that in derivations of plate and rod theories they cannot sustain forcing of order ε^0 , and appropriate scaling of forces is necessary. However here the rod and the plate are supported by the three-dimensional cube and no such problem appears.

In real life situations it is unclear which of these models to use. Thus it is important that we are able to formulate a model, depending on ε and q , posed only on the unit cube (not including the thin cuboids) that has the same asymptotics when ε tends to zero as the original three-dimensional problem, see Theorem 10. The model contains the energy of the cube, the energy of the plate and the energy of the rod. The energy of the plate is of the Naghdi type from [32] with membrane, shear and flexural terms and given using both, displacement of the middle surface and infinitesimal rotation of the cross-section, as unknowns in the problem. The rod model that is used is built in the same manner also with membrane, shear and flexural terms and also with six unknowns, displacement of the middle curve and infinitesimal rotation of the cross-section, see (2.2). This rod model corresponds to the Naghdy/Timoshenko type rod model and can be found in [10]. See Theorem 1 for the main result of the paper.

As already mentioned the problem of interaction of different continua or continua of different dimensions is the area of great interest. For linearly elastic material there are several papers considering interaction of two continua, one of which is thin, see [7] and [6] for the flexural case, [4] and [18] for whole family of regimes. In the case of curved domain and membrane and flexural shell models the asymptotics is discussed in [3]. See also [1] for the variational approach to the thin inclusion problem and [2] in both linear and nonlinear elasticity. For hyperelastic materials and membrane

regime for the thin part the asymptotics of the 3d problem is discussed in [9] by Γ -convergence techniques. A similar analysis for micropolar elastic media is done in [28]. Interaction of the viscous fluid and the linearly elastic plate is considered in [22, 19]. An example from electromagnetism can be found in [24], from heat conduction in [16], for 1d elastic material in [20] and for modeling a thin elastic sheet on a liquid in [21]. This topic is also related to the problem of modeling of joints within both nonlinear and linearized elasticity, see [13, 11] or thin elastic interfaces, linear and nonlinear, isotropic and functionally graded, with or without constraints, see [14, 15, 8, 23, 30] and reference therein. For piezoelectric interfaces see [25, 27] and for thermoelasticity see [26, 29].

2 Definition of the problem and the main result

2.1 Formulation of the full 3D model with thin domains

Let us first formulate the full three-dimensional problem with thin parts of the domain. Let $\varepsilon > 0$ be the small parameter which will describe thickness of thin parts of the domain. The domain $\Omega_{3D+\varepsilon}$

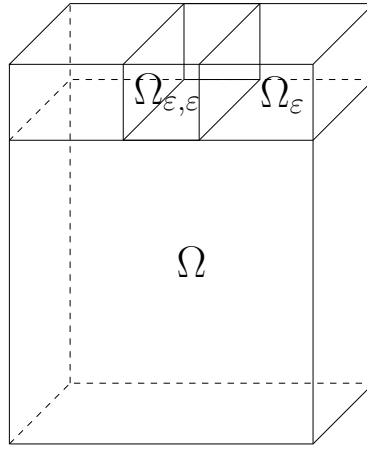


Figure 1: Domain $\Omega_{3D+\varepsilon}$ of the full 3D model.

consists of the cube $\Omega = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]$ with the thin layer at the top $\Omega_\varepsilon = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [0, \varepsilon]$ which contains a thin strip $\Omega_{\varepsilon, \varepsilon} = [0, 1] \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \times [0, \varepsilon]$, see Figure 1. As a rule $'$ is related to the variables x_1 and x_2 , for instance $x' = (x_1, x_2)$ and $\nabla' = [\partial_1 \ \partial_2]$. Forcing in the problem comes from the force density $\mathbf{f} : \Gamma_\varepsilon \rightarrow \mathbb{R}^3$ applied at the top surface $\Gamma_\varepsilon = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times \{\varepsilon\}$. The three parts of the domain Ω , $\Omega_\varepsilon \setminus \Omega_{\varepsilon, \varepsilon}$ and $\Omega_{\varepsilon, \varepsilon}$ are made of different material with elasticity tensors

$$\mathcal{C}_{3D}, \quad \frac{1}{\varepsilon} \mathcal{C}_{\text{plate}}, \quad \frac{1}{\varepsilon^q} \mathcal{C}_{\text{rod}},$$

with Lamé coefficients λ_{3D}, μ_{3D} , $\lambda_{\text{plate}}, \mu_{\text{plate}}$ and $\lambda_{\text{rod}}, \mu_{\text{rod}}$, respectively. The additional parameter $q > 0$ is related to the stiffness of the elastic rod $\Omega_{\varepsilon, \varepsilon}$. In addition we fix the bottom of the cube. Thus the three-dimensional problem is given by: find

$$\mathbf{u}^\varepsilon \in V(\Omega_{3D+\varepsilon}) = \{\mathbf{v} \in H^1(\Omega_{3D+\varepsilon}) : \mathbf{v}|_{x_3=-1} = 0\}$$

such that

$$\begin{aligned} \int_{\Omega_{3D}} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^\varepsilon) \cdot \mathbf{e}(\mathbf{v}) dx + \frac{1}{\varepsilon} \int_{\Omega_\varepsilon \setminus \Omega_{\varepsilon, \varepsilon}} \mathcal{C}_{\text{plate}} \mathbf{e}(\mathbf{u}^\varepsilon) \cdot \mathbf{e}(\mathbf{v}) dx + \frac{1}{\varepsilon^q} \int_{\Omega_{\varepsilon, \varepsilon}} \mathcal{C}_{\text{rod}} \mathbf{e}(\mathbf{u}^\varepsilon) \cdot \mathbf{e}(\mathbf{v}) dx \\ = \int_{\Gamma^\varepsilon} \mathbf{f} \cdot \mathbf{v} dx', \quad \mathbf{v} \in V(\Omega_{3D+\varepsilon}). \end{aligned} \tag{2.1}$$

2.2 Formulation of the 3D–2D–1D model

To formulate the associate 3D–2D–1D interface model we additionally define $\Gamma = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ which replaces the thin layer $\Omega_\varepsilon \setminus \Omega_{\varepsilon,\varepsilon}$ and $\gamma = [0, 1] \times \{0\} \times \{0\}$ which replaces the thin strip $\Omega_{\varepsilon,\varepsilon}$. See Figure 2.

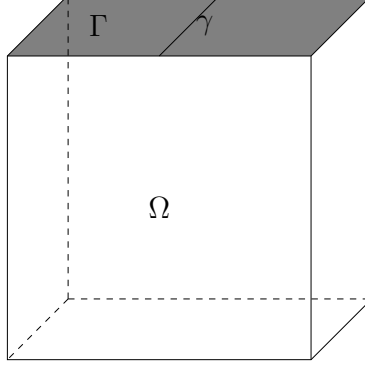


Figure 2: Domain of the 3D–2D–1D model.

Now the solution of the 3D–2D–1D model is the function from the product space

$$(\mathbf{u}^\varepsilon, \tilde{\boldsymbol{\omega}}^\varepsilon) \in V_{3\text{d}-2\text{d}-1\text{d}} = \{(\mathbf{v}, \tilde{\boldsymbol{\omega}}) \in H^1(\Omega; \mathbb{R}^3) \times H^1(\Gamma; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, \mathbf{v}|_\Gamma \in H^1(\Gamma; \mathbb{R}^3), (\mathbf{v}, \tilde{\boldsymbol{\omega}})|_\gamma \in H^1(\gamma; \mathbb{R}^6)\}$$

such that satisfies

$$\begin{aligned} & \int_\Omega \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^\varepsilon) \cdot \mathbf{e}(\mathbf{v}) dx + \int_\Gamma \mathcal{C}_m (\nabla' \mathbf{u}^\varepsilon + \mathbf{A} \tilde{\boldsymbol{\omega}}^\varepsilon) \cdot (\nabla' \mathbf{v} + \mathbf{A} \tilde{\boldsymbol{\omega}}) dx' + \frac{\varepsilon^2}{12} \int_\Gamma \mathcal{C}_f \nabla' \tilde{\boldsymbol{\omega}}^\varepsilon \cdot \nabla' \tilde{\boldsymbol{\omega}} dx' \\ & + \varepsilon^{2-q} \int_\gamma \mathbf{M} [\partial_1 \mathbf{u}^\varepsilon + \mathbf{e}_1 \times \tilde{\boldsymbol{\omega}}^\varepsilon] \cdot [\partial_1 \mathbf{v} + \mathbf{e}_1 \times \tilde{\boldsymbol{\omega}}] dx_1 + \varepsilon^{4-q} \int_\gamma \mathbf{H} \partial_1 \tilde{\boldsymbol{\omega}}^\varepsilon \cdot \partial_1 \tilde{\boldsymbol{\omega}} dx_1 \quad (2.2) \\ & = \int_\Gamma \mathbf{f} \cdot \mathbf{v} dx', \quad (\mathbf{v}, \tilde{\boldsymbol{\omega}}) \in V_{3\text{d}-2\text{d}-1\text{d}}. \end{aligned}$$

Here $\mathbf{A} \tilde{\boldsymbol{\omega}} = [\mathbf{e}_1 \times \tilde{\boldsymbol{\omega}} \quad \mathbf{e}_2 \times \tilde{\boldsymbol{\omega}}]$ and

$$\mathbf{M} = \begin{bmatrix} E_{\text{rod}} A & 0 & 0 \\ 0 & \frac{\mu_{\text{rod}} A}{\alpha_2} & 0 \\ 0 & 0 & \frac{\mu_{\text{rod}} A}{\alpha_3} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mu_{\text{rod}} K & 0 & 0 \\ 0 & E_{\text{rod}} I_2 & 0 \\ 0 & 0 & E_{\text{rod}} I_3 \end{bmatrix},$$

where A is the area, I_2 and I_3 are moments of inertia and K is the torsional rigidity of the cross-section of the rod; α_2 and α_3 depend on the properties of the cross-section, see [10] for details. Note that in our geometry setting the cross-section is the square of size 1 and then $A = 1$, $I_2 = I_3 = \frac{1}{12}$ and K torsional rigidity of the unit square (no closed formula). Note also that ε in (2.2) naturally corresponds to the coefficients in the equation related to the physical cross-section, namely the area of the cross-section A is equal ε^2 , while the moments of inertia and torsional rigidity are of order ε^4 . As already noted in the introduction in the limit for the rod only relevant will be extension and bending tangential to the plate. Thus from matrices \mathbf{M} and \mathbf{H} only relevant will be $m_{11} = E_{\text{rod}} A$ and $h_{22} = \frac{1}{12} E_{\text{rod}} I_2$.

Furthermore the elasticity tensors \mathcal{C}_m and \mathcal{C}_f , $\mathcal{C}_f : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}^{3 \times 2}$ are given by

$$\begin{aligned} \mathcal{C}_m \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} &= \mathcal{A} \mathbf{C} \cdot \mathbf{D} + \mu_{\text{plate}} \mathbf{c} \cdot \mathbf{d}, \\ \mathcal{C}_f \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} &= \mathcal{A} (\mathbf{J} \mathbf{C}) \cdot \mathbf{J} \mathbf{D} + \mathcal{B}_f \mathbf{c} \cdot \mathbf{d}, \end{aligned}$$

where

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{c}^T \end{bmatrix}, \quad \hat{\mathbf{D}} = \begin{bmatrix} \mathbf{D} \\ \mathbf{d}^T \end{bmatrix} \in \mathbb{R}^{3 \times 2}, \quad \mathbf{C}, \mathbf{D} \in \mathbb{R}^{2 \times 2}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^2, \quad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix $\mathcal{B}_f \in \mathbb{R}^{2 \times 2}$ is assumed to be positive definite and the elasticity tensor \mathcal{A} is given by

$$\mathcal{A}\mathbf{D} = \frac{2\lambda_{\text{plate}}\mu_{\text{plate}}}{\lambda_{\text{plate}} + 2\mu_{\text{plate}}}(\mathbf{I} \cdot \mathbf{D})\mathbf{I} + 2\mu_{\text{plate}}\mathbf{D}, \quad \mathbf{D} \in \mathbb{R}^{2 \times 2},$$

where λ_{plate} and μ_{plate} are the Lamé coefficients. We assume that $3\lambda_{\text{plate}} + 2\mu_{\text{plate}}, \mu_{\text{plate}} > 0$.

The main result of this paper is given in the following theorem. Namely we show that the asymptotics of the solution 3D model (2.1) and the 3D–2D–1D model (2.2) are the same for all $q > 0$. This implies that we can replace the full three-dimensional model with the problem on a more simple domain.

Theorem 1. *Let $q > 0$. Let $(\mathbf{u}^{\varepsilon, 3d})_{\varepsilon > 0} \in V(\Omega_{3D+\varepsilon})$ be a family of solutions of (2.1) and let $(\mathbf{u}^{\varepsilon, 3d-2d-1d}, \tilde{\boldsymbol{\omega}}^{\varepsilon, 3d-2d-1d})_{\varepsilon > 0} \in V_{3d-2d-1d}$ be a family of solutions of (2.2). Then for each $q > 0$ families $(\mathbf{u}^{\varepsilon, 3d}|_{\Omega})_{\varepsilon > 0}$ and $(\mathbf{u}^{\varepsilon, 3d-2d-1d})_{\varepsilon > 0}$ have the same limit. That limit differs depending on the five different regimes regarding the value of q : $q \in \langle 0, 2 \rangle, \{2\}, \langle 2, 4 \rangle, \{4\}, \langle 4, \infty \rangle$, given in both Theorem 4 and Theorem 10. Additionally, as $\varepsilon \rightarrow 0$, it holds*

$$\|\mathbf{u}^{\varepsilon, 3d} - \mathbf{u}^{\varepsilon, 3d-2d-1d}\|_{H^1(\Omega; \mathbb{R}^3)}^2 \rightarrow 0.$$

Proof. Follows from Theorem 4 and Theorem 10. □

3 Limits of the 3d equations

In this section we do the asymptotic analysis of the full 3D model (2.1). For that we need the Korn inequality, uniform with respect to ε , for the domain

$$\Omega_{3D+\varepsilon} = [0, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, \varepsilon].$$

Lemma 2 (Lemma 2 in [18]). *There is $C_K > 0$ such that for all $\varepsilon \in [0, 1]$ and for all $\mathbf{v} \in V(\Omega_{3D+\varepsilon})$ one has*

$$C_K \|\mathbf{v}\|_{H^1(\Omega_{3D+\varepsilon}; \mathbb{R}^3)}^2 \leq \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega_{3D+\varepsilon}; \mathbb{R}^{3 \times 3})}^2.$$

In order to get the *a priori* estimates, uniform with respect to ε , for the family of solutions of (2.1), as usual, we now rescale the domain $\Omega_{3D+\varepsilon}$ on the domain independent of ε , $\Omega_{\text{all}} = [0, 1] \times [-1, 1] \times [-1, 1]$, using the map $\mathbf{R}^\varepsilon : \Omega_{\text{all}} \rightarrow \Omega_{3D+\varepsilon}$, given by

$$\mathbf{R}^\varepsilon(x_1, x_2, x_3) = \begin{cases} (x_1, (1-\varepsilon)(x_2+1) - \frac{1}{2}, x_3), & (x_2, x_3) \in [-1, -\frac{1}{2}] \times [-1, 0], \\ (x_1, \varepsilon x_2, x_3), & (x_2, x_3) \in [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0], \\ (x_1, (1-\varepsilon)(x_2-1) + \frac{1}{2}, x_3), & (x_2, x_3) \in [\frac{1}{2}, 1] \times [-1, 0], \\ (x_1, (1-\varepsilon)(x_2+1) - \frac{1}{2}, \varepsilon x_3), & (x_2, x_3) \in [-1, -\frac{1}{2}] \times [0, 1], \\ (x_1, \varepsilon x_2, \varepsilon x_3), & (x_2, x_3) \in [-\frac{1}{2}, \frac{1}{2}] \times [0, 1], \\ (x_1, (1-\varepsilon)(x_2-1) + \frac{1}{2}, \varepsilon x_3), & (x_2, x_3) \in [\frac{1}{2}, 1] \times [0, 1] \end{cases}.$$

Respectively, we also need the following notation, see Figure 3,

$$\begin{aligned} \Omega_{3D,-} &= [0, 1] \times \left[-1, -\frac{1}{2}\right] \times [-1, 0], \quad \Omega_{3D,0} = [0, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 0], \quad \Omega_{3D,+} = [0, 1] \times \left[\frac{1}{2}, 1\right] \times [-1, 0], \\ \Omega_- &= [0, 1] \times \left[-1, -\frac{1}{2}\right] \times [0, 1], \quad \Omega_0 = [0, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1], \quad \Omega_+ = [0, 1] \times \left[\frac{1}{2}, 1\right] \times [0, 1], \\ \Gamma_-^1 &= [0, 1] \times \left[-1, -\frac{1}{2}\right] \times 1, \quad \Gamma_0^1 = [0, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times 1, \quad \Gamma_+^1 = [0, 1] \times \left[\frac{1}{2}, 1\right] \times 1. \end{aligned}$$

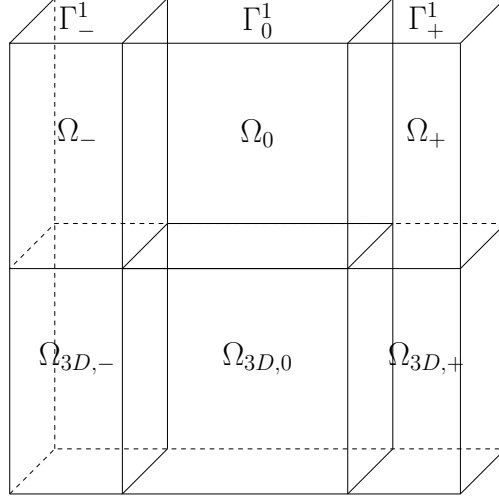


Figure 3: Domain of the rescaled full 3D problem.

Now the rescaled displacement $\mathbf{u}(\varepsilon) = \mathbf{u}^\varepsilon \circ \mathbf{R}^\varepsilon$ belongs to $V(\Omega_{\text{all}}) = \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0\}$ and satisfies (after the change of variables in (2.1))

$$B_{3D}^\varepsilon(\mathbf{u}(\varepsilon)) = L_{3D}^\varepsilon(\mathbf{v}), \quad \mathbf{v} \in V(\Omega_{\text{all}}), \quad (3.1)$$

where

$$\begin{aligned} L_{3D}^\varepsilon(\mathbf{v}) &= (1 - \varepsilon) \int_{\Gamma_\pm^1} \mathbf{f} \cdot \mathbf{v} dx' + \varepsilon \int_{\Gamma_0^1} \mathbf{f} \cdot \mathbf{v} dx', \\ B_{3D}^\varepsilon(\mathbf{u}, \mathbf{v}) &= (1 - \varepsilon) \int_{\Omega_{3D,\pm}} \mathcal{C}_{3D} \mathbf{e}_{3D,\pm}^\varepsilon(\mathbf{u}) \cdot \mathbf{e}_{3D,\pm}^\varepsilon(\mathbf{v}) dx + \varepsilon \int_{\Omega_{3D,0}} \mathcal{C}_{3D} \mathbf{e}_{3D,0}^\varepsilon(\mathbf{u}) \cdot \mathbf{e}_{3D,0}^\varepsilon(\mathbf{v}) dx \\ &\quad + (1 - \varepsilon) \int_{\Omega_\pm} \mathcal{C}_{\text{plate}} \mathbf{e}_\pm^\varepsilon(\mathbf{u}) \cdot \mathbf{e}_\pm^\varepsilon(\mathbf{v}) dx + \varepsilon^{2-q} \int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}_0^\varepsilon(\mathbf{u}) \cdot \mathbf{e}_0^\varepsilon(\mathbf{v}) dx, \end{aligned}$$

where

$$\begin{aligned} \mathbf{e}_{3D,\pm}^\varepsilon(\mathbf{v}) &= \mathbf{e}_1(\mathbf{v}) + \frac{1}{1-\varepsilon} \mathbf{e}_2(\mathbf{v}) + \mathbf{e}_3(\mathbf{v}), \\ \mathbf{e}_{3D,0}^\varepsilon(\mathbf{v}) &= \mathbf{e}_1(\mathbf{v}) + \frac{1}{\varepsilon} \mathbf{e}_2(\mathbf{v}) + \mathbf{e}_3(\mathbf{v}), \\ \mathbf{e}_\pm^\varepsilon(\mathbf{v}) &= \mathbf{e}_1(\mathbf{v}) + \frac{1}{1-\varepsilon} \mathbf{e}_2(\mathbf{v}) + \frac{1}{\varepsilon} \mathbf{e}_3(\mathbf{v}), \\ \mathbf{e}_0^\varepsilon(\mathbf{v}) &= \mathbf{e}_1(\mathbf{v}) + \frac{1}{\varepsilon} \mathbf{e}_2(\mathbf{v}) + \frac{1}{\varepsilon} \mathbf{e}_3(\mathbf{v}), \end{aligned}$$

and $\mathbf{e}_i(\mathbf{v})$ is the matrix function with derivatives, from symmetrized gradient, with respect to variable x_i only.

$$\mathbf{e}_i(\mathbf{v}) = \frac{1}{2} \sum_{j=1}^3 \partial_i v_j (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i).$$

Note that in (3.1) notation $\int_{\Omega_{3D,\pm}}$ means the the integrals over $\Omega_{3D,+}$ and $\Omega_{3D,-}$ have to be summed.

The problem (3.1) is just rescaled classical linearized elasticity problem. Thus the existence and uniqueness of its solution immediately follows.

Theorem 3. *The problem (3.1) has unique solution.*

Remark 1. In what follows we will (with or without mention) use several identifications of function spaces. Firstly, as usual in plate modelling,

$$\{v \in L^2(\Omega_-) : \partial_3 v = 0\} \cong L^2(\Gamma_-),$$

and analogously for $L^2(\Gamma_+)$, where

$$\Gamma_- := [0, 1] \times \left[-1, -\frac{1}{2}\right] \times 0, \quad \Gamma_+ := [0, 1] \times \left[\frac{1}{2}, 1\right] \times 0.$$

As usual in rod modelling,

$$\{v \in L^2(\Omega_0) : \partial_2 v = \partial_3 v = 0\} \cong L^2(\gamma).$$

Spaces $H^1(\Gamma_-)$, $H^1(\Gamma_+)$ and $H^1(\gamma)$ are analogously identified. Secondly, the limit functions \mathbf{u}^0 will be shown to satisfy $\mathbf{u}^0 \in H^1(\Omega_{3D,\pm})$ and $\partial_2 \mathbf{u}^0 = 0$ in $L^2(\Omega_{3D,0})$. By the trace theorem applied on $\Omega_{3D,\pm}$ this implies that $\mathbf{u}^0|_{x_2=\frac{1}{2}} = \mathbf{u}^0|_{x_2=-\frac{1}{2}}$. Thus we can neglect the middle part and identify the spaces

$$\{v \in H^1(\Omega_{3D,-}) \cap H^1(\Omega_{3D,+}) : v|_{x_2=-\frac{1}{2}} = v|_{x_2=\frac{1}{2}}\} \cong H^1(\Omega)$$

(each half of the cube Ω corresponds to one of the domains $\Omega_{3D,+}$ and $\Omega_{3D,-}$). Finally, in similar manner as above, we identify spaces

$$\{v \in H^1(\Omega_-) \cap H^1(\Omega_+) : \partial_3 v = 0 \text{ in } \Omega_- \cup \Omega_+, v|_{x_2=-\frac{1}{2}} = v|_{x_2=\frac{1}{2}}\} \cong H^1(\Gamma).$$

In the following theorem we formulate the main result of this section, the asymptotic behavior of the family of solutions of the problem (3.1). We obtain five different limit models corresponding to different values of the parameter q similarly as in [18]. Corresponding function spaces to these five models are as follows:

$$\begin{aligned} V^I &= \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}_{x_3=-1} = 0, v_\alpha \in H^1(\Gamma), \alpha = 1, 2\}, \\ V^{II} &= \{\mathbf{v} \in V^I : v_1 \in H^1(\gamma)\}, \\ V^{III} &= \{\mathbf{v} \in V^{II} : \partial_1 v_1 = 0\}, \\ V^{IV} &= \{\mathbf{v} \in V^{III} : v_2 \in H^2(\gamma)\}, \\ V^V &= \{\mathbf{v} \in V^{IV} : \partial_{11} v_2 = 0\}. \end{aligned}$$

Obviously

$$V^V \subset V^{IV} \subset V^{III} \subset V^{II} \subset V^I,$$

with V^I imposing regularity on the plate Γ , V^{II} imposing additional regularity at the rod γ , V^{III} imposing extensional rigidity of the rod γ , V^{IV} imposing further regularity for the flexural displacement of the rod but only in the \mathbf{e}_2 direction (!) and finally V^V imposing flexural stiffness of the rod.

Theorem 4. *Let $(\mathbf{u}^\varepsilon)_\varepsilon \in V(\Omega_{all})$ be a family of solutions of (3.1). Let $q > 0$. Then*

$$\|\mathbf{u}^\varepsilon - \mathbf{u}\|_{H^1(\Omega; \mathbb{R}^3)}^2 \rightarrow 0,$$

where \mathbf{u}^0 is the unique solution of one of the following problems

I) for $q \in \langle 0, 2 \rangle$, $\mathbf{u}^0 \in V^I$ is the unique solution of

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad \mathbf{v} \in V^I; \quad (3.2)$$

II) for $q = 2$, $\mathbf{u}^0 \in V^{II}$ is the unique solution of

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' + \int_{\gamma} E_{\text{rod}} \partial_1 u_1^0 \cdot \partial_1 v_1 dx_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad (3.3)$$

which holds for all $\mathbf{v} \in V^{II}$;

III) for $q \in \langle 2, 4 \rangle$, $\mathbf{u}^0 \in V^{III}$ is the unique solution of

$$\int_{\Omega} C_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad (3.4)$$

which holds for all $\mathbf{v} \in V^{III}$;

IV) for $q = 4$, $\mathbf{u}^0 \in V^{IV}$ is the unique solution of

$$\int_{\Omega} C_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' + \int_{\gamma} \frac{E_{\text{rod}}}{12} \partial_{11} u_2^0 \cdot \partial_{11} v_2 dx_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad (3.5)$$

which holds for all $\mathbf{v} \in V^{IV}$;

V) for $q \in \langle 4, \infty \rangle$, $\mathbf{u}^0 \in V^V$ is the unique solution of

$$\int_{\Omega} C_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad (3.6)$$

which holds for all $\mathbf{v} \in V^V$.

Furthermore we have the following convergences

$$\begin{aligned} \mathbf{u}^\varepsilon &\rightarrow \mathbf{u}^0 && \text{strongly in } L^2(\Omega_{3D,0}; \mathbb{R}^3), \\ \mathbf{u}^\varepsilon &\rightarrow \mathbf{u}^0 && \text{strongly in } L^2(\Omega_{\pm}; \mathbb{R}^3), \\ \sqrt{\varepsilon} \mathbf{e}_{3D,0}^\varepsilon(\mathbf{u}^\varepsilon) &\rightarrow \mathbf{e}^{3D,0} && \text{strongly in } L^2(\Omega_{3D,0}; \mathbb{R}^{3 \times 3}), \\ \mathbf{e}_{\pm}^\varepsilon(\mathbf{u}^\varepsilon) &\rightarrow \mathbf{e}^{\pm} && \text{strongly in } L^2(\Omega_{\pm}; \mathbb{R}^{3 \times 3}), \\ \varepsilon^{1-\frac{q}{2}} \mathbf{e}_0^\varepsilon(\mathbf{u}^\varepsilon) &\rightarrow \mathbf{e}^0 && \text{strongly in } L^2(\Omega_0; \mathbb{R}^{3 \times 3}), \end{aligned} \quad (3.7)$$

with $\mathbf{e}^{3D,0} = 0$,

$$\mathbf{e}^{\pm} = \begin{bmatrix} \partial_1 u_1^0 & \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) & 0 \\ \cdot & \partial_2 u_2^0 & 0 \\ \cdot & \cdot & -\frac{\lambda_{\text{plate}}}{\lambda_{\text{plate}} + 2\mu_{\text{plate}}}(\partial_1 u_1^0 + \partial_2 u_2^0) \end{bmatrix},$$

and the value \mathbf{e}^0 depends on the value q : $\mathbf{e}^0 = 0$ for $q \notin \{2, 4\}$,

$$\mathbf{e}^0 = \partial_1 u_1^0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} & 0 \\ 0 & 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} \end{bmatrix}$$

for $q = 2$, and

$$\mathbf{e}^0 = -x_2 \partial_{11} u_2^0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} & 0 \\ 0 & 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} \end{bmatrix}$$

for $q = 4$.

Remark 2. Note that for the limit function u_3^0 the only information we have is that it belongs to $H^1(\Omega)$. By the trace theorem, it also belongs to $L^2(\Gamma)$, but we cannot extract any more information in domain γ . This is related to the fact that in cases $q \geq 4$ condition in space and terms in limit models are related to flexural effects of the rod in direction \mathbf{e}_2 only.

Corollary 1. a) For all $\varepsilon \in \langle 0, 1 \rangle$, all $q > 0$ and all $\mathbf{v} \in V(\Omega_{all})$ one has

$$\begin{aligned} C_K \Big(& \|\mathbf{v}\|_{H^1(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \varepsilon \|\mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 + \varepsilon \|\nabla \mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3 \times \mathbb{R}^3})}^2 + \frac{1}{\varepsilon} \|\partial_2 \mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 \\ & + \varepsilon \|\mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3})}^2 + \varepsilon \|\nabla \mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3 \times \mathbb{R}^3})}^2 + \frac{1}{\varepsilon} \|\partial_3 \mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3})}^2 \\ & + \varepsilon^2 \|\mathbf{v}\|_{L^2(\Omega_0;\mathbb{R}^3)}^2 + \varepsilon^2 \|\partial_1 \mathbf{v}\|_{L^2(\Omega_0;\mathbb{R}^3)}^2 + \|\partial_2 \mathbf{v}\|_{L^2(\Omega_0;\mathbb{R}^3)}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega_0;\mathbb{R}^3)}^2 \Big) \leq B^\varepsilon(\mathbf{v}, \mathbf{v}). \end{aligned}$$

b) For all $\varepsilon \in \langle 0, 1 \rangle$, all $q > 0$ and all $\mathbf{v} \in V(\Omega_{all})$ one has

$$\begin{aligned} \|\mathbf{v}\|_{L^2(\Gamma_{\pm}^1;\mathbb{R}^3)}^2 & \leq \|\mathbf{v}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \varepsilon \|\mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3})}^2 + \frac{1}{\varepsilon} \|\partial_3 \mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3})}^2, \\ \varepsilon \|\mathbf{v}\|_{L^2(\Gamma_0^1;\mathbb{R}^3)}^2 & \leq \varepsilon \|\mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 + \varepsilon \|\partial_3 \mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 + \varepsilon^2 \|\mathbf{v}\|_{L^2(\Omega_0;\mathbb{R}^3)}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega_0;\mathbb{R}^3)}^2. \end{aligned}$$

c) For all $\varepsilon \in \langle 0, 1 \rangle$, all $q > 0$ and all $\mathbf{v} \in V(\Omega_{all})$ one has

$$\begin{aligned} \|\mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 & \leq C \left(\|\mathbf{v}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \|\nabla \mathbf{v}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3 \times \mathbb{R}^3})}^2 + \|\partial_2 \mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 \right), \\ \|\mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3})}^2 & \leq C \left(\|\mathbf{v}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega_{\pm;\mathbb{R}^3})}^2 \right). \end{aligned}$$

d) For all $\varepsilon \in \langle 0, 1 \rangle$, all $q > 0$ and all $\mathbf{v} \in V(\Omega_{all})$, $\alpha = 1, 2$ one has

$$\begin{aligned} \|v_\alpha\|_{L^2(\Omega_0)}^2 & \leq C \left(\|v_\alpha\|_{L^2(\Omega_{3D,\pm})}^2 + \|\partial_3 v_\alpha\|_{L^2(\Omega_{3D,\pm})}^2 + \|\partial_3 v_\alpha\|_{L^2(\Omega_{\pm})}^2 + \|\partial_2 v_\alpha\|_{L^2(\Omega_{\pm})}^2 + \|\partial_2 v_\alpha\|_{L^2(\Omega_0)}^2 \right). \end{aligned}$$

Proof. a) Here we use Lemma 2. Since $\varepsilon \leq 1$ we have that $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^q}$ are larger or equal to 1, so we can estimate the symmetrized gradient by the potential energy in (2.1). Then we rescale the obtained estimate to the canonical domain via \mathbf{R}^ε .

b) The first statement is proven in Corollary 3b) in [18], while the second one is analogous, multiplied by ε .

c) Using the Newton-Leibniz theorem as in Corollary 3b) in [18] we obtain

$$\|\mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 \leq C \left(\|\mathbf{v}|_{x_2=\pm\frac{1}{2}}\|_{L^2(\Omega_{3D,\pm;\mathbb{R}^3})}^2 + \|\partial_2 \mathbf{v}\|_{L^2(\Omega_{3D,0;\mathbb{R}^3})}^2 \right).$$

Now the trace theorem on $\Omega_{3D,\pm}$ implies the first estimate in c).

For the second estimate we take $x \in \Omega_{3D,\pm}$ and again apply the Newton-Leibniz theorem and the homogeneous boundary condition on $x_3 = -1$ to obtain

$$\mathbf{v}(x) = \mathbf{v}(x_1, x_2, 0) + \int_0^{x_3} \partial_3 \mathbf{v} dx_3 = \int_{-1}^0 \partial_3 \mathbf{v} dx_3 + \int_0^{x_3} \partial_3 \mathbf{v} dx_3.$$

Then we integrate and estimate to obtain the second estimate in c).

d) Again we use the Newton-Leibniz formula for $x \in \Omega_0$

$$\mathbf{v}(x) = \int_{-1}^0 \partial_3 \mathbf{v} dx_3 + \int_0^{x_3} \partial_3 \mathbf{v} dx_3 + \int_{x_2}^{-1/2} \partial_2 \mathbf{v} dx_2 + \int_{-1/2}^{x_2} \partial_2 \mathbf{v} dx_2$$

and after integration and estimates we obtain d). □

For the test function $\mathbf{v} = \mathbf{u}(\varepsilon)$ in (3.1) we have

$$B_{3D}^\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon)) = L_{3D}^\varepsilon(\mathbf{u}(\varepsilon)) \leq \|\mathbf{f}\|_{L^2(\Gamma_{\pm}^1)} \|\mathbf{u}(\varepsilon)\|_{L^2(\Gamma_{\pm}^1)} + \varepsilon \|\mathbf{f}\|_{L^2(\Gamma_0^1)} \|\mathbf{u}(\varepsilon)\|_{L^2(\Gamma_0^1)}.$$

Now the application of the estimate b) and then a) from Corollary 1, after noting that all terms in the right hand side of b) appear in the left hand side of the estimate a), implies

$$B_{3D}^\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon)) = L_{3D}^\varepsilon(\mathbf{u}(\varepsilon)) \leq \|\mathbf{f}\|_{L^2(\Gamma_\pm^1)} \|\mathbf{u}(\varepsilon)\|_{L^2(\Gamma_\pm^1)} + \varepsilon \|\mathbf{f}\|_{L^2(\Gamma_0^1)} \|\mathbf{u}(\varepsilon)\|_{L^2(\Gamma_0^1)} \leq C \sqrt{B_{3D}^\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon))}.$$

This implies that all terms in $B_{3D}^\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon))$ and all terms in the left hand side of estimates a) and c) in Corollary 1 are bounded. Weak compactness then implies that on a subsequences we have some weak convergences listed in the following corollary.

In the sequel we will use compactness argument for different bounded ε -families and extract subsequences. For each extraction we will keep the same notation k or ε_k . At the end, due to the uniqueness of the obtained limits, we will get that actually the whole families converge.

Corollary 2. *For all $q > 0$ there are sequences $(\varepsilon_k)_k \subset [0, 1]$ and $(\mathbf{u}^k)_k \subset V(\Omega_{all})$ and limits $\mathbf{u}^0 \in L^2(\Omega_{all} \setminus \Omega_0; \mathbb{R}^3)$, $\mathbf{e}^{3D,0} \in L^2(\Omega_{3D,0}; \mathbb{R}^{3 \times 3})$, $\mathbf{e}^\pm \in L^2(\Omega_\pm; \mathbb{R}^{3 \times 3})$, $\mathbf{w}^\alpha \in L^2(\Omega_0; \mathbb{R}^3)$, $\alpha = 1, 2$ and $\mathbf{e}^0 \in L^2(\Omega_0; \mathbb{R}^{3 \times 3})$ such that $\varepsilon_k \rightarrow 0$ and*

$$\begin{aligned} (i) \quad & \mathbf{u}^k \rightharpoonup \mathbf{u}^0 && \text{weakly in } H^1(\Omega_{3D,\pm}; \mathbb{R}^3), \\ (ii) \quad & \mathbf{u}^k \rightharpoonup \mathbf{u}^0 && \text{weakly in } L^2(\Omega_{3D,0}; \mathbb{R}^3), \\ (iii) \quad & \partial_2 \mathbf{u}^k \rightarrow 0 && \text{strongly in } L^2(\Omega_{3D,0}; \mathbb{R}^3), \\ (iv) \quad & \mathbf{u}^k \rightharpoonup \mathbf{u}^0 && \text{weakly in } L^2(\Omega_\pm; \mathbb{R}^3), \\ (v) \quad & \partial_3 \mathbf{u}^k \rightarrow 0 && \text{strongly in } L^2(\Omega_\pm; \mathbb{R}^3), \\ (vi) \quad & \sqrt{\varepsilon_k} \mathbf{e}_{3D,0}^{\varepsilon_k}(\mathbf{u}^k) \rightharpoonup \mathbf{e}^{3D,0} && \text{weakly in } L^2(\Omega_{3D,0}; \mathbb{R}^{3 \times 3}), \\ (vii) \quad & \mathbf{e}_\pm^{\varepsilon_k}(\mathbf{u}^k) \rightharpoonup \mathbf{e}^\pm && \text{weakly in } L^2(\Omega_\pm; \mathbb{R}^{3 \times 3}), \\ (viii) \quad & \partial_\beta \mathbf{u}^k \rightharpoonup \mathbf{w}^\beta && \text{weakly in } L^2(\Omega_0; \mathbb{R}^3), \quad \beta \in \{2, 3\}, \\ (ix) \quad & \varepsilon^{1-\frac{q}{2}} \mathbf{e}_0^{\varepsilon_k}(\mathbf{u}^k) \rightharpoonup \mathbf{e}^0 && \text{weakly in } L^2(\Omega_0; \mathbb{R}^{3 \times 3}). \end{aligned} \tag{3.8}$$

3.1 Preliminary analysis

From the convergences (ii) and (iii) in (3.8) we see that $\partial_2 \mathbf{u}^0 = 0$ in $\Omega_{3D,0}$, i.e., \mathbf{u}^0 is independent of x_2 . Thus the traces of \mathbf{u}^0 on $x_2 = \pm \frac{1}{2}$ coincide. Therefore the limit \mathbf{u}^0 restricted to $\Omega_{3D,+} \cup \Omega_{3D,0} \cup \Omega_{3D,-}$ belongs to a space isomorphic to $H^1(\Omega; \mathbb{R}^3)$, see Remark 1. Similarly, from convergences (iv) and (v), \mathbf{u}^0 restricted to Ω_- and Ω_+ belongs to spaces isomorphic to $L^2(\Gamma_-; \mathbb{R}^3)$ and $L^2(\Gamma_+; \mathbb{R}^3)$, respectively.

From (iv) and (vii) in (3.8) we obtain that

$$e_{11}^\pm = \partial_1 u_1^0, \quad e_{22}^\pm = \partial_2 u_2^0, \quad e_{12}^\pm = e_{21}^\pm = \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0)$$

(all equations are on Ω_\pm), so by the 2D Korn inequality

$$\sum_{\alpha_1, \alpha_2=1}^2 \|\partial_{\alpha_1} v_{\alpha_2}\|_{H^1(\Omega_\pm; \mathbb{R}^3)} \leq C(\|v_1\|_{L^2(\Omega_\pm)}^2 + \|v_2\|_{L^2(\Omega_\pm)}^2) + \frac{1}{2} \sum_{\alpha_1, \alpha_2=1}^2 \|\partial_{\alpha_1} v_{\alpha_2} + \partial_{\alpha_2} v_{\alpha_1}\|_{L^2(\Omega_\pm)}^2$$

we have that $u_1^0, u_2^0 \in H^1(\Gamma_-, \mathbb{R}^3)$ and $u_1^0, u_2^0 \in H^1(\Gamma_+, \mathbb{R}^3)$. Using the 2D Korn inequality we also conclude

$$\begin{aligned} (u_1^k, u_2^k) &\rightharpoonup (u_1^0, u_2^0) && \text{weakly in } H^1(\Omega_+; \mathbb{R}^2), \\ (u_1^k, u_2^k) &\rightharpoonup (u_1^0, u_2^0) && \text{weakly in } H^1(\Omega_-; \mathbb{R}^2). \end{aligned}$$

Now we apply Corollary 1.d) to $(u_\alpha^k)_k$ to conclude that for $\alpha = 1, 2$, $(\|u_\alpha^k\|_{L^2(\Omega_0; \mathbb{R}^3)}^2)_k$ is bounded as well, thus $(u_\alpha^k)_k$ converges to u_α^0 weakly in $L^2(\Omega_0)$ (up to a subsequence). By the uniqueness of the limit, we can partly identify the limits \mathbf{w}^β from (3.8) by $w_\alpha^\beta = \partial_\beta u_\alpha^0$, $\alpha = 1, 2, \beta = 2, 3$.

Furthermore $\partial_1 u_2^k \rightharpoonup \partial_1 u_2^0$ in $H^{-1}(\Omega_0)$, which implies $\varepsilon_k \partial_1 u_2^k \rightarrow 0$ in $H^{-1}(\Omega_0)$. This, together with the last convergence in (3.8) multiplied by $\varepsilon_k^{\frac{q}{2}}$ for $q > 0$, implies $\partial_2 u_1^k \rightarrow 0$ in $H^{-1}(\Omega_0)$. Since we already know that $\partial_2 u_1^k$ is bounded in $L^2(\Omega_0)$, we finally conclude

$$\partial_2 u_1^k \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega_0).$$

Directly from the last convergence in (3.8) on position $(2, 2)$ multiplied by $\varepsilon_k^{\frac{q}{2}}$ we obtain $\partial_2 u_2^k \rightarrow 0$ in $L^2(\Omega_0)$. This is enough to conclude that $\partial_2 u_1^0 = \partial_2 u_2^0 = 0$ on Ω_0 . Thus we have proved the following result.

Lemma 5. *We have that $u_\alpha^k \rightharpoonup u_\alpha^0$ weakly in $L^2(\Omega_0)$ and*

$$\partial_2 u_1^0 = \partial_2 u_2^0 = 0 \quad \text{in } \Omega_0, \quad \partial_3 u_1^0 = \partial_3 u_2^0 = 0 \quad \text{in } \Omega_\pm.$$

This lemma then implies that traces of limits u_1^0 and u_2^0 on Ω_+ and Ω_- on $\{x_2 = \pm \frac{1}{2}\}$ coincide and that on $\Omega_\pm \cup \Omega_0$ limit functions u_1^0 and u_2^0 belong to the space isomorphic to $H^1(\Gamma)$ (in the view of Remark 1). Finally, the limit \mathbf{u}^0 belongs to a space isomorphic to,

$$V^I := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, v_\alpha|_\Gamma \in H^1(\Gamma; \mathbb{R}^3), \alpha = 1, 2\}.$$

Let us insert $\mathbf{v} \in V(\Omega_{\text{all}})$ such that $\mathbf{v} = 0$ for $x_2 \geq -\frac{1}{2}$ in (3.1) (then only integrals over Ω_- and $\Omega_{3D,-}$ are nonzero). Then we multiply the equation (3.1) by ε_k , and let $\varepsilon_k \rightarrow 0$. We obtain

$$\int_{\Omega_-} \mathcal{C}_{\text{plate}} \mathbf{e}^- \cdot \mathbf{e}_3(\mathbf{v}) dx = 0.$$

This is the same situation as in the plate equation derivation, see [18, Section 3.2]. Then $(\mathcal{C}_{\text{plate}} \mathbf{e}^-)_{i3} = 0, i = 1, 2, 3$. The same can be obtained for Ω_+ . We conclude that

$$\mathbf{e}^\pm = \begin{bmatrix} \partial_1 u_1^0 & \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) & 0 \\ \cdot & \partial_2 u_2^0 & 0 \\ \cdot & \cdot & -\frac{\lambda_{\text{plate}}}{\lambda_{\text{plate}} + 2\mu_{\text{plate}}}(\partial_1 u_1^0 + \partial_2 u_2^0) \end{bmatrix}. \quad (3.9)$$

Also, using notation $\mathbf{e}'(\mathbf{v}) = \mathbf{e}_1(\mathbf{v}) + \mathbf{e}_2(\mathbf{v})$ we get

$$\mathcal{C}_{\text{plate}} \mathbf{e}^\pm \cdot \mathbf{e}'(\mathbf{v}) = \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}), \quad \mathbf{v} \in H^1(\Gamma_\pm; \mathbb{R}^3) \quad (3.10)$$

and

$$\mathcal{C}_{\text{plate}} \mathbf{e}^\pm \cdot \mathbf{e}_3(\mathbf{v}) = 0, \quad \mathbf{v} \in H^1(\Omega_\pm; \mathbb{R}^3). \quad (3.11)$$

In order to prove the strong convergence we additionally define

$$\begin{aligned} \Lambda(k) &:= (1 - \varepsilon_k) \int_{\Omega_{3D,\pm}} \mathcal{C}_{3D} \left(\mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k) - \mathbf{e}(\mathbf{u}^0) \right) \cdot \left(\mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k) - \mathbf{e}(\mathbf{u}^0) \right) dx \\ &\quad + \int_{\Omega_{3D,0}} \mathcal{C}_{3D} \left(\sqrt{\varepsilon_k} \mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k) - \mathbf{e}^{3D,0} \right) \cdot \left(\sqrt{\varepsilon_k} \mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k) - \mathbf{e}^{3D,0} \right) dx \\ &\quad + (1 - \varepsilon_k) \int_{\Omega_\pm} \mathcal{C}_{\text{plate}} \left(\mathbf{e}_\pm^{\varepsilon_k}(\mathbf{u}^k) - \mathbf{e}^\pm \right) \cdot \left(\mathbf{e}_\pm^{\varepsilon_k}(\mathbf{u}^k) - \mathbf{e}^\pm \right) dx \\ &\quad + \int_{\Omega_0} \mathcal{C}_{\text{rod}} \left(\varepsilon_k^{1-\frac{q}{2}} \mathbf{e}_0(\mathbf{u}^k) - \mathbf{e}^0 \right) \cdot \left(\varepsilon_k^{1-\frac{q}{2}} \mathbf{e}_0(\mathbf{u}^k) - \mathbf{e}^0 \right) dx. \end{aligned}$$

Then we use the equation (3.1) to replace the quadratic terms and obtain

$$\begin{aligned} \Lambda(k) &= (1 - \varepsilon_k) \int_{\Omega_{3D,\pm}} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \left(\mathbf{e}(\mathbf{u}^0) - 2\mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k) \right) dx \\ &\quad + \int_{\Omega_{3D,0}} \mathcal{C}_{3D} \mathbf{e}^{3D,0} \cdot \left(\mathbf{e}^{3D,0} - 2\sqrt{\varepsilon_k} \mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k) \right) dx \\ &\quad + (1 - \varepsilon_k) \int_{\Omega_\pm} \mathcal{C}_{\text{plate}} \mathbf{e}^\pm \cdot \left(\mathbf{e}^\pm - 2\mathbf{e}_\pm^{\varepsilon_k}(\mathbf{u}^k) \right) dx + \int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \left(\mathbf{e}^0 - 2\varepsilon_k^{1-\frac{q}{2}} \mathbf{e}_0(\mathbf{u}^k) \right) dx \\ &\quad + (1 - \varepsilon_k) \int_{\Gamma_\pm} \mathbf{f} \cdot \mathbf{v} dx' + \varepsilon_k \int_{\Gamma_0} \mathbf{f} \cdot \mathbf{v} dx'. \end{aligned}$$

Now we let k to infinity and use (3.8) and (3.9) to obtain that $\Lambda(k) \rightarrow \Lambda$, where

$$\begin{aligned} \Lambda := & - \int_{\Omega_{3D,\pm}} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) dx - \int_{\Omega_{3D,0}} \mathcal{C}_{3D} \mathbf{e}^{3D,0} \cdot \mathbf{e}^{3D,0} dx \\ & - \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{u}^0) dx' - \int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \mathbf{e}^0 dx + \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx'. \end{aligned} \quad (3.12)$$

It is clear that $\Lambda \geq 0$ as the limit of a nonnegative sequence. In all cases that follow the obtained limit model implies that the limit Λ is equal to zero. The form of Λ will then imply that both or one of the terms \mathbf{e}^0 and $\mathbf{e}^{3D,0}$ are equal to zero, and that $\mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k)$, $\sqrt{\varepsilon_k} \mathbf{e}_{3D,\pm}^{\varepsilon_k}(\mathbf{u}^k)$, $\mathbf{e}_{\pm}^{\varepsilon_k}(\mathbf{u}^k)$ and $\varepsilon_k^{1-\frac{q}{2}} \mathbf{e}_0(\mathbf{u}^k)$ converge strongly in L^2 on corresponding domains. Then together with uniqueness of the solution, we obtain strong convergences for all ε -families from (3.7).

3.2 The case $0 < q < 2$

As noted in the previous subsection, the limit function belongs to V^I . Let us take a test function $\mathbf{v} \in V(\Omega_{\text{all}})$ such that $\partial_3 \mathbf{v} = 0$ in Ω_{\pm} , $\partial_2 \mathbf{v} = 0$ in $\Omega_{3D,0}$, $\partial_2 \mathbf{v} = \partial_3 \mathbf{v} = 0$ in Ω_0 and let ε_k to zero. In the limit, using (3.10), we obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad (3.13)$$

which by density holds for all $\mathbf{v} \in V^I$. This is the same model as obtained in [18, Section 3.2.], for which the well-posedness is proved by coercivity inequality suited for natural norm in the space V^I :

$$\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}'(\mathbf{v})\|_{L^2(\Gamma; \mathbb{R}^{3 \times 3})}^2 \geq c \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 \right). \quad (3.14)$$

This shows uniqueness of the limit \mathbf{u}^0 . From (3.13) and (3.12) we obtain that $\Lambda = 0$, and that both terms \mathbf{e}^0 and $\mathbf{e}^{3D,0}$ are equal to zero, so all desired strong convergences hold and the all ε -families converge.

3.3 The case $q = 2$

From (1,1) coordinate of the last convergence in (3.8) we obtain $\partial_1 u_1^k \rightarrow \partial_1 u_1^0$ in $L^2(\Omega_0)$. Thus, $e_{11}^0 = \partial_1 u_1^0$ and $u_1^0 \in H^1(\gamma)$ ($\partial_2 u_1^0 = \partial_3 u_1^0 = 0$), so the limit \mathbf{u}^0 belongs to

$$V^{II} := \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, v_{\alpha}|_{\Gamma} \in H^1(\Gamma; \mathbb{R}), v_1 \in H^1(\gamma) \} = \{ \mathbf{v} \in V^I : v_1 \in H^1(\gamma) \}.$$

Let us multiply (3.1) by ε_k and let $\varepsilon_k \rightarrow 0$. We obtain

$$\int_{\Omega_{\pm}} \mathcal{C}_{\text{plate}} \mathbf{e}^{\pm} \cdot \mathbf{e}_3(\mathbf{v}) dx + \int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot (\mathbf{e}_2(\mathbf{v}) + \mathbf{e}_3(\mathbf{v})) dx = 0. \quad (3.15)$$

Due to (3.11), the first integral is equal to zero. Thus

$$\int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot (\mathbf{e}_2(\mathbf{v}) + \mathbf{e}_3(\mathbf{v})) dx = 0. \quad (3.16)$$

By inserting $\mathbf{v} = \psi(x_1) x_2 \mathbf{e}_1$ for arbitrary $\psi \in H^1([0,1])$, we obtain

$$\int_{\Omega_0} e_{12}^0 \psi(x_1) dx = 0. \quad (3.17)$$

Similarly we obtain

$$\int_{\Omega_0} e_{13}^0 \psi(x_1) dx = 0. \quad (3.18)$$

Next we insert $\mathbf{v} = \psi(x_1)x_2\mathbf{e}_2 + \psi(x_1)x_3\mathbf{e}_3$ in (3.16) for arbitrary $\psi \in H^1([0, 1])$. After algebraic simplifications, we obtain

$$\int_{\Omega_0} (e_{22}^0 + e_{33}^0)\psi(x_1)dx = - \int_{\Omega_0} \frac{\lambda_{\text{rod}}}{\lambda_{\text{rod}} + \mu_{\text{rod}}} e_{11}^0 \psi(x_1)dx = - \int_{\Omega_0} \frac{\lambda_{\text{rod}}}{\lambda_{\text{rod}} + \mu_{\text{rod}}} \partial_1 u_1^0 \psi(x_1)dx. \quad (3.19)$$

From (3.17)–(3.19), for $\mathbf{v} \in H^1(\gamma, \mathbb{R}^3)$ (i.e., $\mathbf{v} \in H^1(\Omega^0; \mathbb{R}^3)$) such that $\partial_2 \mathbf{v} = \partial_3 \mathbf{v} = 0$ we obtain

$$\begin{aligned} \int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \mathbf{e}_1(\mathbf{v})dx &= \int_{\Omega_0} (\lambda_{\text{rod}} \text{tr } \mathbf{e}^0 + 2\mu_{\text{rod}} e_{11}^0) \partial_1 v_1 + 2\mu_{\text{rod}} e_{12}^0 \partial_1 v_2 + 2\mu_{\text{rod}} e_{13}^0 \partial_1 v_3 dx \\ &= \int_{\Omega_0} E_{\text{rod}} e_{11}^0 \partial_1 v_1 dx = \int_{\Omega_0} E_{\text{rod}} \partial_1 u_1^0 \partial_1 v_1 dx = \int_{\gamma} E_{\text{rod}} \partial_1 u_1^0 \partial_1 v_1 dx_1. \end{aligned} \quad (3.20)$$

Let us now take a test function $\mathbf{v} \in V(\Omega_{\text{all}})$ in (3.1) such that $\partial_3 \mathbf{v} = 0$ in Ω_{\pm} , $\partial_2 \mathbf{v} = 0$ in $\Omega_{3D,0}$, $\partial_2 \mathbf{v} = \partial_3 \mathbf{v} = 0$ in Ω_0 and let ε to zero. By using (3.20) we obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v})dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v})dx' + \int_{\gamma} E_{\text{rod}} \partial_1 u_1^0 \partial_1 v_1 dx_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad (3.21)$$

which by density holds for all $\mathbf{v} \in V^{II}$.

From (3.14) and the trace theorem for v_1 on Γ , we obtain the bound

$$\begin{aligned} \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}'(\mathbf{v})\|_{L^2(\Gamma; \mathbb{R}^{3 \times 3})}^2 + \|\partial_1 v_1\|_{L^2(\gamma)}^2 \\ \geq c \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 \right) + \|\partial_1 v_1\|_{L^2(\gamma)}^2 \\ \geq c' \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 + \|v_1\|_{H^1(\gamma)}^2 \right). \end{aligned} \quad (3.22)$$

This shows coercivity inequality for natural norm in the space V^{II} , so we conclude uniqueness of the limit \mathbf{u}^0 , and that the whole ε -family converges to the same limit.

For the strong convergence, let us firstly define matrix

$$\mathbf{E}_{\text{rod}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} & 0 \\ 0 & 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} \end{bmatrix}. \quad (3.23)$$

For such matrix it is easy to see that it holds

$$\mathcal{C}_{\text{rod}} \mathbf{E}_{\text{rod}} = \begin{bmatrix} E_{\text{rod}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and consequently

$$\mathcal{C}_{\text{rod}} \mathbf{E}_{\text{rod}} \cdot \mathbf{E}_{\text{rod}} = E_{\text{rod}} \quad \text{and} \quad \mathcal{C}_{\text{rod}} \mathbf{E}_{\text{rod}} \cdot \mathbf{F} = 0,$$

for each $\mathbf{F} \in \mathbb{M}^{3 \times 3}$ with element on the position (1, 1) equal to zero.

Let us define matrices $\tilde{\mathbf{e}}^0 := \partial_1 u_1^0 \mathbf{E}_{\text{rod}}$ and $\tilde{\tilde{\mathbf{e}}}^0 = \mathbf{e}^0 - \tilde{\mathbf{e}}^0$. Since $e_{11}^0 = \partial_1 u_1^0$, the element on the position (1, 1) in the matrix $\tilde{\tilde{\mathbf{e}}}^0$ is equal to zero, so we have

$$\mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \mathbf{e}^0 = \mathcal{C}_{\text{rod}} (\tilde{\mathbf{e}}^0 + \tilde{\tilde{\mathbf{e}}}^0) \cdot (\tilde{\mathbf{e}}^0 + \tilde{\tilde{\mathbf{e}}}^0) = \mathcal{C}_{\text{rod}} \tilde{\mathbf{e}}^0 \cdot \tilde{\mathbf{e}}^0 + \mathcal{C}_{\text{rod}} \tilde{\tilde{\mathbf{e}}}^0 \cdot \tilde{\tilde{\mathbf{e}}}^0 = E_{\text{rod}} (\partial_1 u_1^0)^2 + \mathcal{C}_{\text{rod}} \tilde{\tilde{\mathbf{e}}}^0 \cdot \tilde{\tilde{\mathbf{e}}}^0.$$

By inserting this in (3.12), by using (3.21), we obtain that

$$\Lambda = - \int_{\Omega_{3D,0}} \mathcal{C}_{3D} \mathbf{e}^{3D,0} \cdot \mathbf{e}^{3D,0} dx - \int_{\Omega_0} \mathcal{C}_{\text{rod}} \tilde{\tilde{\mathbf{e}}}^0 \cdot \tilde{\tilde{\mathbf{e}}}^0 dx$$

is equal to zero. Thus $\mathbf{e}^{3D,0} = \tilde{\tilde{\mathbf{e}}}^0 = 0$ and

$$\mathbf{e}^0 = \tilde{\mathbf{e}}^0 = \partial_1 u_1^0 \mathbf{E}_{\text{rod}} = \begin{bmatrix} \partial_1 u_1^0 & 0 & 0 \\ 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} \partial_1 u_1^0 & 0 \\ 0 & 0 & \frac{-\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} \partial_1 u_1^0 \end{bmatrix}.$$

As in the previous case, we also conclude desired strong convergences.

3.4 The case $2 < q < 4$

From the last convergence in (3.8) we obtain that $\partial_1 u_1^0 = 0$ on Ω_0 , thus the limit belongs to

$$V^{III} := \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, v_\alpha|_\Gamma \in H^1(\Gamma; \mathbb{R}^3), \partial_1 v_1 = 0 \text{ on } \gamma \} = \{ \mathbf{v} \in V^{II} : \partial_1 v_1 = 0 \text{ on } \gamma \}.$$

Let us choose test functions $\mathbf{v}^0 + \varepsilon_k \mathbf{v}^1$ with $\mathbf{v}^0, \mathbf{v}^1 \in V(\Omega_{\text{all}})$ for (3.1) such that $\partial_3 \mathbf{v}^0 = 0$ in Ω_\pm , $\partial_2 \mathbf{v}^0 = 0$ in $\Omega_{3D,0}$, $\mathbf{v}^0 \in H^2(\Omega_0; \mathbb{R}^3)$ dependent only on x_1 , $\partial_1 v_1^0 = 0$ on Ω_0 , and $\mathbf{v}^1 = (-x_2 \partial_1 v_2^0 - (x_3 - \frac{1}{2}) \partial_1 v_3^0) \mathbf{e}_1$ on Ω_0 . Then

$$\mathbf{e}_0^{\varepsilon_k}(\mathbf{v}^0 + \varepsilon_k \mathbf{v}^1) = \mathbf{e}_1(\mathbf{v}^0) + \mathbf{e}_2(\mathbf{v}^1) + \mathbf{e}_3(\mathbf{v}^1) + \varepsilon_k \mathbf{e}_1(\mathbf{v}^1) = \varepsilon_k \mathbf{e}_1(\mathbf{v}^1).$$

Thus $\varepsilon_k^{1-\frac{q}{2}} \mathbf{e}_0^{\varepsilon_k}(\mathbf{v}^0 + \varepsilon_k \mathbf{v}^1) \rightarrow 0$ strongly in L^2 . Therefore in the limit of (3.1) we obtain

$$\int_\Omega C_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}^0) dx + \int_\Gamma \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}^0) dx' = \int_\Gamma \mathbf{f} \cdot \mathbf{v}^0 dx', \quad (3.24)$$

which by density holds for all $\mathbf{v} \in V^{III}$.

From (3.22) for $\mathbf{v} \in V^{III}$ we obtain the bound

$$\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}'(\mathbf{v})\|_{L^2(\Gamma; \mathbb{R}^{3 \times 3})}^2 \geq c \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 + \|v_1\|_{H^1(\gamma)}^2 \right). \quad (3.25)$$

so we again conclude well-posedness, uniqueness of the limit \mathbf{u}^0 , and that the whole ε -family converges to the same limit. From (3.24) and (3.12) we obtain that $\Lambda = 0$, and that both terms \mathbf{e}^0 and $\mathbf{e}^{3D,0}$ are equal to zero, so all desired strong convergences hold.

3.5 The case $q = 4$

Lemma 6. *Let the family $(\mathbf{v}^\varepsilon)_{\varepsilon>0} \subseteq H^1(\Omega_0; \mathbb{R}^3)$ satisfies*

$$v_\alpha^\varepsilon \rightharpoonup v_\alpha \quad \text{weakly in } H^1(\Omega_0) \quad \alpha = 1, 2, \quad (3.26)$$

$$\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon} \rightharpoonup \phi_1 \quad \text{weakly in } L^2(\Omega_0), \quad (3.27)$$

$$\partial_2 v_3^\varepsilon \rightharpoonup \phi_2 \quad \text{weakly in } L^2(\Omega_0), \quad (3.28)$$

$$\partial_3 v_3^\varepsilon \rightharpoonup \phi_3 \quad \text{weakly in } L^2(\Omega_0), \quad (3.29)$$

$$\frac{1}{\varepsilon} \mathbf{e}_0^\varepsilon(\mathbf{v}^\varepsilon) \rightharpoonup \mathbf{e}^0 \quad \text{weakly in } L^2(\Omega_0; \mathbb{R}^{3 \times 3}), \quad (3.30)$$

where $\overline{\mathbf{w}} := \frac{1}{|\Omega_0|} \int_{\Omega_0} \mathbf{w} dx$. Additionally, let $v_\alpha|_{x_2=-1/2}$ be independent of x_3 . Then:

a) it holds

$$\nabla \mathbf{v}^\varepsilon - \overline{\partial_1 v_3^\varepsilon} \mathbf{e}_3 \mathbf{e}_1^T = \begin{bmatrix} \partial_1 v_1^\varepsilon & \partial_2 v_1^\varepsilon & \partial_3 v_1^\varepsilon \\ \partial_1 v_2^\varepsilon & \partial_2 v_2^\varepsilon & \partial_3 v_2^\varepsilon \\ \partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon} & \partial_2 v_3^\varepsilon & \partial_3 v_3^\varepsilon \end{bmatrix} \rightharpoonup \begin{bmatrix} 0 & 0 & 0 \\ \partial_1 v_2 & 0 & 0 \\ \phi_1 & 0 & 0 \end{bmatrix} \quad \text{weakly in } L^2(\Omega_0; \mathbb{R}^{9 \times 9}),$$

furthermore, the convergence of components converging to zero is strong;

b) the limit function v_1 is a constant, the limit function v_2 is independent of variables x_2, x_3 and belongs to $H^2(0, 1)$ and the limit function ϕ_1 is independent of variables x_2, x_3 and belongs to $H^1(0, 1)$;

c) the limit functions satisfy following conditions:

$$\partial_2 e_{11}^0 = -\partial_{11} v_2, \quad \partial_3 e_{11}^0 = -\partial_1 \phi_1;$$

d) if the convergence (3.30) is strong, then all convergences in a) are also strong.

Proof. From assumptions, we have

$$\varepsilon^{-1} \partial_1 v_1^\varepsilon \rightharpoonup e_{11}^0 \quad \text{weakly in } L^2(\Omega_0), \quad (3.31)$$

$$\varepsilon^{-2} \partial_2 v_2^\varepsilon \rightharpoonup e_{22}^0 \quad \text{weakly in } L^2(\Omega_0), \quad (3.32)$$

$$\varepsilon^{-2} \partial_3 v_3^\varepsilon \rightharpoonup e_{33}^0 \quad \text{weakly in } L^2(\Omega_0), \quad (3.33)$$

$$\varepsilon^{-1} \partial_1 v_2^\varepsilon + \varepsilon^{-2} \partial_2 v_1^\varepsilon \rightharpoonup e_{12}^0 \quad \text{weakly in } L^2(\Omega_0), \quad (3.34)$$

$$\varepsilon^{-1} \partial_1 v_3^\varepsilon + \varepsilon^{-2} \partial_3 v_1^\varepsilon \rightharpoonup e_{13}^0 \quad \text{weakly in } L^2(\Omega_0), \quad (3.35)$$

$$\varepsilon^{-2} \partial_2 v_3^\varepsilon + \varepsilon^{-2} \partial_3 v_2^\varepsilon \rightharpoonup e_{23}^0 \quad \text{weakly in } L^2(\Omega_0). \quad (3.36)$$

From (3.26) specially

$$v_\alpha^\varepsilon \rightarrow v_\alpha \quad \text{strongly in } L^2(\Omega_0) \quad \alpha = 1, 2. \quad (3.37)$$

From (3.31), (3.32) and (3.33) multiplied by $\varepsilon, \varepsilon^2, \varepsilon^2$ respectively we obtain

$$\partial_1 v_1^\varepsilon, \partial_2 v_2^\varepsilon, \partial_3 v_3^\varepsilon \rightarrow 0 \quad \text{strongly in } L^2(\Omega_0). \quad (3.38)$$

Thus $\partial_1 v_1 = \partial_2 v_2 = \phi_3 = 0$. From (3.26) we have $\varepsilon \partial_1 v_2^\varepsilon \rightarrow 0$ strongly in $L^2(\Omega_0)$. Then from (3.34) multiplied by ε^2 we obtain

$$\partial_2 v_1^\varepsilon \rightarrow 0 = \partial_2 v_1 \quad \text{strongly in } L^2(\Omega_0).$$

From (3.27) after differentiating and multiplying by ε we have

$$\varepsilon \nabla \partial_1 v_3^\varepsilon = \varepsilon \nabla (\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}) \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0; \mathbb{R}^3).$$

Then from (3.35) multiplied by ε^2 , after differentiation we obtain

$$\nabla \partial_3 v_1^\varepsilon \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0; \mathbb{R}^3).$$

Since from (3.26) we have $\partial_3 v_1^\varepsilon \rightharpoonup \partial_3 v_1$ weakly in $L^2(\Omega_0)$, by using the Lions lemma we conclude that $\partial_3 v_1^\varepsilon \rightarrow \partial_3 v_1$ strongly in $L^2(\Omega_0)$, and that $\nabla \partial_3 v_1 = 0$. Together with $\partial_1 v_1 = \partial_2 v_1 = 0$ this implies that v_1 is affine in x_3 . Since by assumption $v_1|_{x_2=-1/2}$ is independent of x_3 this implies that $\partial_3 v_1 = 0$ and thus v_1 is a constant.

By (3.34) and (3.35) we have

$$\varepsilon \partial_3 (\varepsilon^{-1} \partial_1 v_2^\varepsilon + \varepsilon^{-2} \partial_2 v_1^\varepsilon) - \varepsilon \partial_2 (\varepsilon^{-1} \partial_1 v_3^\varepsilon + \varepsilon^{-2} \partial_3 v_1^\varepsilon) = \partial_1 (\partial_3 v_2^\varepsilon - \partial_2 v_3^\varepsilon) \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0).$$

Comparing with (3.36) differentiated and multiplied by ε^2 , we obtain that

$$\partial_{13} v_2^\varepsilon, \partial_{12} v_3^\varepsilon \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0). \quad (3.39)$$

Let us now prove that $\partial_3 v_2^\varepsilon \rightarrow 0$ strongly in $L^2(\Omega_0)$. From (3.26) we have that $\partial_3 v_2^\varepsilon$ converges strongly in $H^{-1}(\Omega_0)$. From (3.39) and (3.38), respectively we have that $\partial_{13} v_2^\varepsilon \rightarrow 0$ and $\partial_{23} v_2^\varepsilon \rightarrow 0$, both strongly in $H^{-1}(\Omega_0)$. Finally, from (3.36) and (3.33) we have

$$\partial_{33} v_2^\varepsilon = \varepsilon^2 \partial_3 (\varepsilon^{-2} \partial_2 v_3^\varepsilon + \varepsilon^{-2} \partial_3 v_2^\varepsilon) - \varepsilon^2 \partial_2 (\varepsilon^{-2} \partial_3 v_3^\varepsilon) \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0).$$

From Lions lemma, $\partial_3 v_2^\varepsilon$ converges strongly in $L^2(\Omega_0)$ to $\partial_3 v_2$ which is a constant since $\nabla \partial_3 v_2 = 0$. Thus v_2 is an affine function in x_3 with coefficients in x_1 , a.e. Since, by assumption, $v_2|_{x_1=-1/2}$ is independent of x_3 , v_2 is a function of x_1 only. Thus

$$\partial_3 v_2^\varepsilon \rightarrow 0 = \partial_3 v_2 \quad \text{strongly in } L^2(\Omega_0). \quad (3.40)$$

By multiplying (3.36) by ε^2 , we obtain that $\partial_2 v_3^\varepsilon \rightarrow 0 = \phi_2$ strongly in $L^2(\Omega_0)$.

From (3.39) and (3.38), respectively, we obtain that

$$\partial_2 (\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}) = \partial_{12} v_3, \quad \partial_3 (\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}) = \partial_{13} v_3 \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0).$$

Thus $\partial_2\phi_1 = \partial_3\phi_1 = 0$ and ϕ_1 is independent of x_2, x_3 . Since

$$\partial_1 (\partial_1 v_3^\varepsilon + \varepsilon^{-1} \partial_3 v_1^\varepsilon) = \partial_1 (\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}) + \partial_3 (\varepsilon^{-1} \partial_1 v_1^\varepsilon), \quad (3.41)$$

convergences (3.31) and (3.27) applied on the term on the right hand side and convergence (3.35) multiplied by ε applied on the left hand side imply that in the limit we have the equality

$$0 = \partial_1 \phi_1 + \partial_3 e_{11}^0.$$

Let ξ and θ be such that $\xi = \xi(x_1) \in C_c^\infty(\langle 0, 1 \rangle)$, $\theta = \theta(x_2, x_3) \in C_c^\infty(\langle 0, 1 \rangle^2)$, $\int_{[0,1]^2} \theta d\mathbf{x}' = 1$. In (3.41) we now have

$$\int_0^1 \phi_1 \partial_1 \xi dx_1 = \int_{\Omega_0} \phi_1 \theta \partial_1 \xi d\mathbf{x} = - \int_{\Omega_0} e_{11}^0 \partial_3 \theta \xi d\mathbf{x} = \int_0^1 \left(\int_{[0,1]^2} -e_{11}^0 \partial_3 \theta d\mathbf{x}' \right) \xi dx_1.$$

Since $x_1 \mapsto \int_{[0,1]^2} -e_{11}^0 \partial_3 \theta d\mathbf{x}'$ belongs to $L^2(0, 1)$ due to arbitrariness of ξ we conclude that $\phi_1 \in H^1(0, 1)$.

Taking derivative with respect to x_1 of (3.34) multiplied by ε we obtain

$$\partial_{11} v_2^\varepsilon + \varepsilon^{-1} \partial_{21} v_1^\varepsilon \rightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_0) \quad (3.42)$$

Using convergences (3.26) and (3.31) in the limit we obtain $\partial_{11} v_2 + \partial_2 e_{11}^0$. For the same ξ and θ as above we obtain

$$\int_0^1 \partial_1 v_2 \partial_1 \xi dx_1 = \int_{\Omega_0} \partial_1 v_2 \theta \partial_1 \xi d\mathbf{x} = - \int_{\Omega_0} e_{11}^0 \partial_2 \theta \xi d\mathbf{x} = \int_0^1 \left(\int_{[0,1]^2} -e_{11}^0 \partial_2 \theta d\mathbf{x}' \right) \xi dx_1.$$

As before this implies $\partial_1 v_2 \in H^1(0, 1)$.

Let us now assume that the convergence in (3.30) is strong, i.e. all convergences (3.31)–(3.36) are strong.

From (3.26) $\partial_1 v_2^\varepsilon$ converges strongly in $H^{-1}(\Omega_0)$. Taking derivative with respect to x_1 of (3.32) multiplied by ε^2 we obtain that $\partial_{21} v_2^\varepsilon$ converges strongly in $H^{-1}(\Omega_0)$. From (3.40) we know that $\partial_3 v_2^\varepsilon$ converges strongly to zero in $L^2(\Omega_0)$, which implies that $\partial_{31} v_2^\varepsilon$ converges strongly in $H^{-1}(\Omega_0)$. Now (3.42) with strong convergence in (3.31) implies that $\partial_{11} v_2^\varepsilon$ converges strongly in $H^{-1}(\Omega_0)$. Lions lemma now implies that $\partial_1 v_2^\varepsilon$ converges strongly in $L^2(\Omega_0)$.

From (3.27) $\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}$ converges strongly in $H^{-1}(\Omega_0)$. Since $\nabla(\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}) = \nabla \partial_1 v_3^\varepsilon$ we can apply the same arguments as for $\partial_1 v_2^\varepsilon$ above to conclude that convergence in (3.27) is strong. \square

Lemma 7. *Let $v_\alpha|_{x_2=-1/2}$ be independent of x_3 . Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ and all $\mathbf{v} \in H^1(\Omega_0; \mathbb{R}^3)$ one has*

$$\begin{aligned} & C_K \left(\|v_1\|_{H^1(\Omega_0)}^2 + \|v_2\|_{H^1(\Omega_0)}^2 + \|\partial_1 v_3 - \overline{\partial_1 v_3}\|_{L^2(\Omega_0; \mathbb{R}^3)}^2 + \|\partial_2 v_3\|_{L^2(\Omega_0; \mathbb{R}^3)}^2 + \|\partial_3 v_3\|_{L^2(\Omega_0; \mathbb{R}^3)}^2 \right) \\ & \leq \frac{1}{\varepsilon^2} \|e_0^\varepsilon(\mathbf{v})\|_{L^2(\Omega_0; \mathbb{R}^{3 \times 3})}^2 + \|v_1\|_{L^2(\Omega_0)}^2 + \|v_2\|_{L^2(\Omega_0)}^2, \end{aligned}$$

where $\overline{\mathbf{w}} := \frac{1}{|\Omega_0|} \int_{\Omega_0} \mathbf{w} d\mathbf{x}$.

Proof. Let us suppose the opposite. Then there exists a sequence still labelled by $(\mathbf{v}^\varepsilon)_{\varepsilon>0}$ such that

$$\|v_1^\varepsilon\|_{H^1(\Omega_0)}^2 + \|v_2^\varepsilon\|_{H^1(\Omega_0)}^2 + \|\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}\|_{L^2(\Omega_0; \mathbb{R}^3)}^2 + \|\partial_2 v_3^\varepsilon\|_{L^2(\Omega_0; \mathbb{R}^3)}^2 + \|\partial_3 v_3^\varepsilon\|_{L^2(\Omega_0; \mathbb{R}^3)}^2 = 1, \quad (3.43)$$

such that the terms on the right hand side of the inequality tend to zero. More precisely, we have

$$\begin{aligned} v_\alpha^\varepsilon & \rightharpoonup v_\alpha & \text{weakly in } H^1(\Omega_0) & \alpha = 1, 2, \\ \partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon} & \rightharpoonup \phi_1 & \text{weakly in } L^2(\Omega_0), \\ \partial_2 v_3^\varepsilon & \rightharpoonup \phi_2 & \text{weakly in } L^2(\Omega_0), \\ \partial_3 v_3^\varepsilon & \rightharpoonup \phi_3 & \text{weakly in } L^2(\Omega_0), \\ \frac{1}{\varepsilon} e_0^\varepsilon(\mathbf{v}^\varepsilon) & \rightarrow 0 & \text{strongly in } L^2(\Omega_0; \mathbb{R}^{3 \times 3}), \\ v_\alpha^\varepsilon & \rightarrow 0 & \text{strongly in } L^2(\Omega_0) & \alpha = 1, 2. \end{aligned} \quad (3.44)$$

From the last convergence we have $v_1 = v_2 = 0$. Further, all assumptions of the Lemma 6 are satisfied. Thus all convergences in (3.44) are strong and $\phi_2 = \phi_3 = 0$.

From $\frac{1}{\varepsilon} \mathbf{e}_0^\varepsilon(\mathbf{v}^\varepsilon) \rightarrow 0$ specially $e_{11}^0 = 0$. From part c) of Lemma 6, we have that $\partial_1 \phi_1 = 0$. Together with part b) of the same lemma, ϕ_1 is a constant, so we have

$$\phi_1 = \frac{1}{|\Omega_0|} \int_{\Omega_0} \phi_1 dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega_0|} \int_{\Omega_0} (\partial_1 v_3^\varepsilon - \overline{\partial_1 v_3^\varepsilon}) dx = 0.$$

Thus the left hand side of (3.43) converges to 0, so we obtained a contradiction. \square

We apply Lemmas 6 and 7 on $\mathbf{u}^{\varepsilon_k}$ since for $q \geq 4$ the assumptions are fulfilled. We conclude that u_1^0 is a constant and that $u_2^0 \in H^2(\Omega_0; \mathbb{R}^3)$ depends only on x_1 . Thus \mathbf{u}^0 belongs to

$$\begin{aligned} V^{IV} &:= \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, v_\alpha|_\Gamma \in H^1(\Gamma; \mathbb{R}^3), \partial_1 v_1 = 0 \text{ on } \gamma, v_2 \in H^2(\gamma) \} \\ &= \{ \mathbf{v} \in V^{III} : v_2 \in H^2(\gamma) \}. \end{aligned}$$

We also conclude that $\partial_1 u_3^{\varepsilon_k} - \overline{\partial_1 u_3^{\varepsilon_k}}$ weakly converges to $\psi_1 \in H^1(\Omega_0)$ on Ω_0 , dependent only on x_1 . From part c) of Lemma 6 we conclude that

$$\frac{1}{\varepsilon_k} (e_0^{\varepsilon_k})_{11} \rightharpoonup e_{11}^0 = e_{11}^H - x_2 \partial_{11} u_2^0 - \left(x_3 - \frac{1}{2} \right) \partial_1 \psi_1, \quad (3.45)$$

for a function $e_{11}^H \in L^2(\Omega_0; \mathbb{R}^3)$ dependent only on x_1 .

Let us choose arbitrary test function $\mathbf{v} \in V(\Omega_{\text{all}})$ for (3.1), multiply the equation by ε^2 , and let $\varepsilon \rightarrow 0$. We obtain

$$\int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot (\mathbf{e}_2(\mathbf{v}) + \mathbf{e}_3(\mathbf{v})) dx = 0. \quad (3.46)$$

Firstly, in (3.46) we choose a test function $\mathbf{v} = \frac{x_2^2}{2} \phi_2(x_1) \mathbf{e}_2$, for arbitrary $\phi_2 \in H^1([0, 1])$, to obtain

$$\int_{\Omega_0} (\lambda_{\text{rod}} e_{11}^0 + (\lambda_{\text{rod}} + 2\mu_{\text{rod}}) e_{22}^0 + \lambda_{\text{rod}} e_{33}^0) x_2 \phi_2(x_1) dx = 0. \quad (3.47)$$

For a test function $\mathbf{v} = x_2 \left(x_3 - \frac{1}{2} \right) \phi_3(x_1) \mathbf{e}_2 - \frac{x_2^2}{2} \phi_3(x_1) \mathbf{e}_3$, for arbitrary $\phi_3 \in H^1([0, 1])$, from (3.46) we obtain

$$\int_{\Omega_0} (\lambda_{\text{rod}} e_{11}^0 + (\lambda_{\text{rod}} + 2\mu_{\text{rod}}) e_{22}^0 + \lambda_{\text{rod}} e_{33}^0) \left(x_3 - \frac{1}{2} \right) \phi_3(x_1) dx = 0. \quad (3.48)$$

We obtain results analogous to (3.47) and (3.48) for term $(\lambda_{\text{rod}} e_{11}^0 + \lambda_{\text{rod}} e_{22}^0 + (\lambda_{\text{rod}} + 2\mu_{\text{rod}}) e_{33}^0)$, and sum everything to obtain

$$\begin{aligned} & \int_{\Omega_0} (e_{22}^0 + e_{33}^0) \left(x_2 \phi_2(x_1) + \left(x_3 - \frac{1}{2} \right) \phi_3(x_1) \right) dx \\ &= - \int_{\Omega_0} \frac{\lambda_{\text{rod}}}{\lambda_{\text{rod}} + \mu_{\text{rod}}} e_{11}^0 \left(x_2 \phi_2(x_1) + \left(x_3 - \frac{1}{2} \right) \phi_3(x_1) \right) dx. \end{aligned} \quad (3.49)$$

Let us choose test functions $\mathbf{v}^0 + \varepsilon_k \mathbf{v}^1$ with $\mathbf{v}^0, \mathbf{v}^1 \in V(\Omega_{\text{all}})$ for (3.1) such that $\partial_3 \mathbf{v}^0 = 0$ in Ω_\pm , $\partial_2 \mathbf{v}^0 = 0$ in $\Omega_{3D,0}$, $\mathbf{v}^0 \in H^2(\Omega_0; \mathbb{R}^3)$ dependent only on x_1 , $\partial_1 v_1^0 = 0$ on Ω_0 , and $\mathbf{v}^1 = (-x_2 \partial_1 v_2^0 - (x_3 - \frac{1}{2}) \partial_1 v_3^0) \mathbf{e}_1$. Then on Ω_0 we have

$$\mathbf{e}_0^{\varepsilon_k}(\mathbf{v}^0 + \varepsilon_k \mathbf{v}^1) = \mathbf{e}_1(\mathbf{v}^0) + \mathbf{e}_2(\mathbf{v}^1) + \mathbf{e}_3(\mathbf{v}^1) + \varepsilon_k \mathbf{e}_1(\mathbf{v}^1) = \varepsilon_k \mathbf{e}_1(\mathbf{v}^1).$$

After letting $\varepsilon \rightarrow 0$ we obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}^0) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}^0) dx' + \int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \mathbf{e}_1(\mathbf{v}^1) dx = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v}^0 dx'. \quad (3.50)$$

By (3.49) and (3.45), the last term on the left hand side can be rewritten as

$$\begin{aligned}
\int_{\Omega_0} \mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \mathbf{e}_1(\mathbf{v}^1) dx &= \int_{\Omega_0} ((\lambda_{\text{rod}} + 2\mu_{\text{rod}})e_{11}^0 + \lambda_{\text{rod}}(e_{22}^0 + e_{33}^0)) \left(-x_2 \partial_{11} v_2^0 - \left(x_3 - \frac{1}{2} \right) \partial_{11} v_3^0 \right) dx \\
&= \int_{\Omega_0} E_{\text{rod}} e_{11}^0 \left(-x_2 \partial_{11} v_2^0 - \left(x_3 - \frac{1}{2} \right) \partial_{11} v_3^0 \right) dx \\
&= \int_{\Omega_0} E_{\text{rod}} \left(e_{11}^H - x_2 \partial_{11} u_2^0 - \left(x_3 - \frac{1}{2} \right) \partial_1 \psi_1 \right) \left(-x_2 \partial_{11} v_2^0 - \left(x_3 - \frac{1}{2} \right) \partial_{11} v_3^0 \right) dx \\
&= \int_{\gamma} \frac{E_{\text{rod}}}{12} \partial_{11} u_2^0 \cdot \partial_{11} v_2^0 dx_1 + \int_{\gamma} \frac{E_{\text{rod}}}{12} \partial_1 \psi_1 \cdot \partial_{11} v_3^0 dx_1.
\end{aligned}$$

Let us observe (3.50) for the test function function \mathbf{v}^0 with $v_1^0 = v_2^0 = 0$:

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(v_3^0 \mathbf{e}_3) dx + \int_{\Omega_0} \frac{E_{\text{rod}}}{12} \partial_1 \psi_1 \cdot \partial_{11} v_3^0 dx = \int_{\Gamma} f_3 v_3^0 dx'.$$

Let us choose arbitrary $\nu \in H^1(\Omega_{\text{all}})$ with $\nu|_{x_3=-1} = 0$, $\partial_2 \nu = 0$ on $\Omega_{3D,0}$, $\partial_3 \nu = 0$ on Ω_{\pm} , $\partial_2 \nu = \partial_3 \nu = 0$ on Ω_0 . Let us take $\bar{\nu} \mathbf{e}_3 \in V^I$ as a function isomorphic to $\nu \mathbf{e}_3$, see Remark 1. Since γ in Ω is of capacity zero, there exists a sequence $(\bar{\nu}_n)_n \subset V^I$ with $\bar{\nu}_n = 0$ on γ such that strongly converges to $\bar{\nu}$ in $H^1(\Omega)$ (see Theorem 2.44, [12]). When in the last equation we plug in functions $\nu_n \subset H^1(\Omega_{\text{all}})$ (isomorphic to elements of sequence $(\bar{\nu}_n)_n$) and let $n \rightarrow \infty$, we obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\nu \mathbf{e}_3) dx = \int_{\Gamma} f_3 \nu dx',$$

so

$$\int_{\gamma} \frac{E_{\text{rod}}}{12} \partial_1 \psi_1 \cdot \partial_{11} \nu dx_1 = 0$$

for all $\nu \in H^2([0, 1])$. Consequently, $\partial_1 \psi_1 = 0$. Thus the model for $q = 4$ reads: find $\mathbf{u}^0 \in V^{IV}$ such that for all $\mathbf{v} \in V^{IV}$ one has

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' + \int_{\gamma} \frac{E_{\text{rod}}}{12} \partial_{11} u_2^0 \cdot \partial_{11} v_2 dx_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx'. \quad (3.51)$$

From (3.25) and the trace theorem for v_2 on Γ , we obtain the bound

$$\begin{aligned}
&\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}'(\mathbf{v})\|_{L^2(\Gamma; \mathbb{R}^{3 \times 3})}^2 + \|\partial_{11} v_2\|_{L^2(\gamma)}^2 \\
&\geq c \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 + \|v_1\|_{H^1(\gamma)}^2 \right) + \|\partial_{11} v_2\|_{L^2(\gamma)}^2 \\
&\geq c' \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 + \|v_1\|_{H^1(\gamma)}^2 + \|v_2\|_{H^2(\gamma)}^2 \right).
\end{aligned} \quad (3.52)$$

This shows coercivity inequality for natural norm in the space V^{IV} , so we conclude uniqueness of the limit \mathbf{u}^0 , and that the whole ε -family converges to the same limit.

For the strong convergence, we use similar idea as in the case $q = 2$. We define matrices $\tilde{\mathbf{e}}^0 := e_{11}^0 \mathbf{E}_{\text{rod}}$ and $\tilde{\tilde{\mathbf{e}}}^0 = \mathbf{e}^0 - \tilde{\mathbf{e}}^0$, where matrix \mathbf{E}_{rod} is defined in (3.23). We analogously conclude that

$$\mathcal{C}_{\text{rod}} \mathbf{e}^0 \cdot \mathbf{e}^0 = E_{\text{rod}} (e_{11}^0)^2 + \mathcal{C}_{\text{rod}} \tilde{\tilde{\mathbf{e}}}^0 \cdot \tilde{\tilde{\mathbf{e}}}^0.$$

Since e_{11}^H and u_2^0 are both only x_1 dependent, it holds

$$\int_{\Omega_0} E_{\text{rod}} (e_{11}^0)^2 dx = \int_{\Omega_0} E_{\text{rod}} (e_{11}^H - x_2 \partial_{11} u_2^0)^2 dx = \int_{\Omega_0} E_{\text{rod}} (e_{11}^H)^2 dx + \int_{\Omega_0} \frac{E_{\text{rod}}}{12} (\partial_{11} u_2^0)^2 dx.$$

By inserting the last two equations in (3.12), by using (3.51), we obtain that

$$\Lambda = - \int_{\Omega_{3D,0}} \mathcal{C}_{3D} \mathbf{e}^{3D,0} \cdot \mathbf{e}^{3D,0} dx - \int_{\Omega_0} E_{\text{rod}} (e_{11}^H)^2 dx - \int_{\Omega_0} \mathcal{C}_{\text{rod}} \tilde{\tilde{\mathbf{e}}}^0 \cdot \tilde{\tilde{\mathbf{e}}}^0 dx.$$

is equal to zero. Thus $\mathbf{e}^{3D,0} = e_{11}^H = \tilde{\mathbf{e}}^0 = 0$, $e_{11}^0 = -x_2 \partial_{11} u_2^0$, and

$$\mathbf{e}^0 = \tilde{\mathbf{e}}^0 = \begin{bmatrix} -x_2 \partial_{11} u_2^0 & 0 & 0 \\ 0 & \frac{\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} x_2 \partial_{11} u_2^0 & 0 \\ 0 & 0 & \frac{\lambda_{\text{rod}}}{2(\lambda_{\text{rod}} + \mu_{\text{rod}})} x_2 \partial_{11} u_2^0 \end{bmatrix}.$$

As in the previous cases, we also conclude desired strong convergences. Additionally, the statement d) in Lemma 6 then implies the strong convergence of u_α^ε in $H^1(\Omega_0)$.

3.6 The case $q > 4$

From the last convergence in (3.8) we conclude $\frac{1}{\varepsilon_k} \mathbf{e}_0^{\varepsilon_k}(\mathbf{u}^\varepsilon) \rightarrow 0$ strongly in $L^2(\Omega_0; \mathbb{R}^{3 \times 3})$. We apply Lemma 6 (parts a), c) and d)) to conclude

$$\begin{aligned} \partial_2 \mathbf{u}^{\varepsilon_k} &\rightarrow 0 && \text{strongly in } L^2(\Omega_0; \mathbb{R}^3), \\ \partial_3 \mathbf{u}^{\varepsilon_k} &\rightarrow 0 && \text{strongly in } L^2(\Omega_0; \mathbb{R}^3), \\ \partial_1 u_1^{\varepsilon_k} &\rightarrow 0 && \text{strongly in } L^2(\Omega_0; \mathbb{R}^3), \\ \partial_{11} u_2^{\varepsilon_k}, \partial_{11} u_1^{\varepsilon_k} &\rightarrow 0 && \text{strongly in } H^{-1}(\Omega_0; \mathbb{R}^3). \end{aligned}$$

Furthermore, we conclude that the limit belongs to

$$\begin{aligned} V^V &:= \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, v_\alpha|_\Gamma \in H^1(\Gamma; \mathbb{R}^3), \partial_1 v_1 = 0 \text{ on } \gamma, \partial_{11} v_2 = 0 \text{ on } \gamma \} \\ &= \{ \mathbf{v} \in V^{IV} : \partial_{11} v_2 = 0 \text{ on } \gamma \}. \end{aligned}$$

Let us choose arbitrary $\nu \in H^1(\Omega_{\text{all}})$ with $\partial_2 \nu = 0$ on $\Omega_{3D,0}$, $\partial_3 \nu = 0$ on Ω_\pm , $\partial_2 \nu = \partial_3 \nu = 0$ on Ω_0 . Let us take $\bar{\nu} \mathbf{e}_3 \in V^I$ as a function isomorphic to $\nu \mathbf{e}_3$. Again by [12, Theorem 2.44], there exists a sequence $(\bar{\nu}_n)_n \subset V^I$ with $\bar{\nu}_n = 0$ on γ such that strongly converges to $\bar{\nu}$ in $H^1(\Omega)$. Then for the test functions $\nu_n \mathbf{e}_3$ in (3.1) (where functions ν_n are isomorphic to elements of sequence $(\bar{\nu}_n)_n$) we obtain

$$\int_\Omega \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\nu \mathbf{e}_3) dx = \int_\Gamma f_3 \nu dx'.$$

We now choose arbitrary $v_1, v_2 \in H^2(\Omega_{\text{all}})$ with $\partial_2 v_\alpha = 0$ on $\Omega_{3D,0}$, $\partial_3 v_\alpha = 0$ on Ω_\pm , $\partial_2 v_\alpha = \partial_3 v_\alpha = 0$ on Ω_0 , and with $\partial_1 v_1 = \partial_{11} v_2 = 0$ on Ω_0 . For test functions of the form $v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 - \varepsilon x_2 \partial_1 v_2 \mathbf{e}_1$ we obtain

$$\int_\Omega \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) dx + \int_\Gamma \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) dx' = \int_\Gamma f_1 v_1 + f_2 v_2 dx'.$$

Summing the last two equations, by density we obtain the model: find $\mathbf{u}^0 \in V^V$ such that for all $\mathbf{v} \in V^V$ one has

$$\int_\Omega \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_\Gamma \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_\Gamma \mathbf{f} \cdot \mathbf{v} dx'. \quad (3.53)$$

From (3.52) for $\mathbf{v} \in V^V$ we obtain the bound

$$\begin{aligned} &\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}'(\mathbf{v})\|_{L^2(\Gamma; \mathbb{R}^{3 \times 3})}^2 + \|\partial_{11} v_2\|_{L^2(\gamma)}^2 \\ &\geq c \left(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|v_1\|_{H^1(\Gamma)}^2 + \|v_2\|_{H^1(\Gamma)}^2 + \|v_3\|_{L^2(\Gamma)}^2 + \|v_1\|_{H^1(\gamma)}^2 + \|v_2\|_{H^2(\gamma)}^2 \right). \end{aligned} \quad (3.54)$$

so we again conclude well-posedness, uniqueness of the limit \mathbf{u}^0 , and that the whole ε -family converges to the same limit. From (3.53) and (3.12) we obtain that $\Lambda = 0$, and that both terms \mathbf{e}^0 and $\mathbf{e}^{3D,0}$ are equal to zero, so all desired strong convergences hold.

4 Limits of the 3d-plate-rod model

In this model we analyze, case by case, a simplified model (2.2) in which the thin parts are replaced by the plate and rod equations.

Theorem 8. *The problem (2.2) has unique solution.*

The result follows by Lax-Milgram lemma using the $V_{3d-2d-1d}$ -ellipticity of the form which follows from the following estimate.

Lemma 9. *There is $C > 0$ such that for all $(\mathbf{v}, \tilde{\mathbf{w}}) \in V_{3d-2d-1d}$ one has*

$$\begin{aligned} & \|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|\mathbf{v}\|_{H^1(\Gamma; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{w}}\|_{H^1(\Gamma; \mathbb{R}^3)}^2 + \|\mathbf{v}\|_{H^1(\gamma; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{w}}\|_{H^1(\gamma; \mathbb{R}^3)}^2 \\ & \leq C \left(\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\nabla' \mathbf{v} + \mathbf{A} \tilde{\mathbf{w}}\|_{L^2(\Gamma; \mathbb{R}^{3 \times 2})}^2 + \|\nabla' \tilde{\mathbf{w}}\|_{L^2(\Gamma; \mathbb{R}^{3 \times 2})}^2 \right. \\ & \quad \left. + \|\partial_1 \mathbf{v} + \mathbf{e}_1 \times \tilde{\mathbf{w}}\|_{L^2(\gamma; \mathbb{R}^3)}^2 + \|\partial_1 \tilde{\mathbf{w}}\|_{L^2(\gamma; \mathbb{R}^3)}^2 \right). \end{aligned}$$

Proof. First we use the estimate [18, Lemma 9] for the estimate

$$\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|\mathbf{v}\|_{H^1(\Gamma; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{w}}\|_{H^1(\Gamma; \mathbb{R}^3)}^2 \leq C \left(\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 + \|\nabla' \mathbf{v} + \mathbf{A} \tilde{\mathbf{w}}\|_{L^2(\Gamma; \mathbb{R}^{3 \times 2})}^2 + \|\nabla' \tilde{\mathbf{w}}\|_{L^2(\Gamma; \mathbb{R}^{3 \times 2})}^2 \right).$$

Then using the trace inequality terms $\|\mathbf{v}\|_{L^2(\gamma; \mathbb{R}^3)}$ and $\|\tilde{\mathbf{w}}\|_{L^2(\gamma; \mathbb{R}^3)}$ are also estimated. Then the additional terms $\|\partial_1 \mathbf{v} + \mathbf{e}_1 \times \tilde{\mathbf{w}}\|_{L^2(\gamma; \mathbb{R}^3)}^2 + \|\partial_1 \tilde{\mathbf{w}}\|_{L^2(\gamma; \mathbb{R}^3)}^2$ are used to estimate $\|\mathbf{v}\|_{H^1(\gamma; \mathbb{R}^3)}$ and $\|\tilde{\mathbf{w}}\|_{H^1(\gamma; \mathbb{R}^3)}$. \square

Also in an usual way, since $\|\mathbf{v}\|_{L^2(\Gamma; \mathbb{R}^3)} \leq C \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}$ by the trace theorem and the Korn inequality, we obtain the *a priori* estimates for the terms in the elastic energy of the system. Then the following convergences hold.

Corollary 3. *There is a sequence $(\varepsilon_k)_k$ such that $\varepsilon_k \rightarrow 0$ and $\mathbf{u}^0 \in V(\Omega)$, $\mathbf{e}_P^m, \mathbf{e}_P^f \in L^2(\Gamma; \mathbb{R}^{3 \times 2})$ and $\mathbf{e}_R^m, \mathbf{e}_R^f \in L^2(\gamma; \mathbb{R}^3)$ such that*

$$\begin{aligned} & \mathbf{u}^{\varepsilon_k} \rightharpoonup \mathbf{u}^0 \quad \text{weakly in } H^1(\Omega; \mathbb{R}^3), \\ & (\nabla' \mathbf{u}^{\varepsilon_k} + \mathbf{A} \tilde{\mathbf{w}}^{\varepsilon_k}) \rightharpoonup \mathbf{e}_P^m \quad \text{weakly in } L^2(\Gamma; \mathbb{R}^{3 \times 2}), \\ & \varepsilon_k \nabla \tilde{\mathbf{w}}^{\varepsilon_k} \rightharpoonup \mathbf{e}_P^f \quad \text{weakly in } L^2(\Gamma; \mathbb{R}^{3 \times 2}), \\ & \varepsilon_k^{1-q/2} (\partial_1 \mathbf{u}^{\varepsilon_k} + \mathbf{e}_1 \times \tilde{\mathbf{w}}^{\varepsilon_k}) \rightharpoonup \mathbf{e}_R^m \quad \text{weakly in } L^2(\gamma; \mathbb{R}^3), \\ & \varepsilon_k^{2-q/2} \partial_1 \tilde{\mathbf{w}}^{\varepsilon_k} \rightharpoonup \mathbf{e}_R^f \quad \text{weakly in } L^2(\gamma; \mathbb{R}^3). \end{aligned} \tag{4.1}$$

Theorem 10. *Let $(\mathbf{u}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)_\varepsilon \subset V_{3d-2d-1d}$ be the family of solutions of (2.2). Let $q > 0$. Then*

$$\begin{aligned} & \mathbf{u}^{\varepsilon_k} \rightarrow \mathbf{u}^0 \quad \text{strongly in } H^1(\Omega; \mathbb{R}^3), \\ & (\nabla' \mathbf{u}^{\varepsilon_k} + \mathbf{A} \tilde{\mathbf{w}}^{\varepsilon_k}) \rightarrow \mathbf{e}_P^m \quad \text{strongly in } L^2(\Gamma; \mathbb{R}^{3 \times 2}), \\ & \varepsilon_k \nabla \tilde{\mathbf{w}}^{\varepsilon_k} \rightarrow \mathbf{e}_P^f \quad \text{strongly in } L^2(\Gamma; \mathbb{R}^{3 \times 2}), \\ & \varepsilon_k^{(2-q)/2} (\partial_1 \mathbf{u}^{\varepsilon_k} + \mathbf{e}_1 \times \tilde{\mathbf{w}}^{\varepsilon_k}) \rightarrow \mathbf{e}_R^m \quad \text{strongly in } L^2(\gamma; \mathbb{R}^3), \\ & \varepsilon_k^{(4-q)/2} \partial_1 \tilde{\mathbf{w}}^{\varepsilon_k} \rightarrow \mathbf{e}_R^f \quad \text{strongly in } L^2(\gamma; \mathbb{R}^3), \\ & u_\alpha^{\varepsilon_k} \rightarrow u_\alpha^0 \quad \text{strongly in } H^1(\Gamma), \quad \alpha = 1, 2, \\ & u_\alpha^{\varepsilon_k} \rightarrow u_\alpha^0 \quad \text{strongly in } L^2(\gamma), \quad \alpha = 1, 2. \end{aligned} \tag{4.2}$$

The limit function \mathbf{u}^0 belongs to V^I and is unique as well as $\mathbf{e}_R^m, \mathbf{e}_R^f$ (both depending on q), and

$$\mathbf{e}_P^m = \begin{bmatrix} \partial_1 u_1^0 & \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) \\ \partial_1 u_2^0 + \partial_2 u_1^0 & \partial_2 u_2^0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e}_P^f = 0.$$

I) For $q \in \langle 0, 2 \rangle$, $\mathbf{u}^0 \in V^I$ is the unique solution of (3.2) and $\mathbf{e}_R^m = \mathbf{e}_R^f = 0$.

II) For $q = 2$, $\mathbf{u}^0 \in V^{II}$ is the unique solution of (3.3), $\mathbf{e}_R^f = 0$ and

$$\mathbf{e}_R^m = \begin{bmatrix} \partial_1 u_1^0 \\ 0 \\ 0 \end{bmatrix}.$$

III) For $q \in \langle 2, 4 \rangle$, $\mathbf{u}^0 \in V^{III}$ is the unique solution of (3.4) and $\mathbf{e}_R^m = \mathbf{e}_R^f = 0$.

IV) For $q = 4$, $\mathbf{u}^0 \in V^{IV}$ is the unique solution of (3.5), $\mathbf{e}_R^m = 0$ and

$$\mathbf{e}_R^f = \begin{bmatrix} 0 \\ 0 \\ \partial_{11} u_2^0 \end{bmatrix}.$$

V) For $q \in \langle 4, \infty \rangle$, $\mathbf{u}^0 \in V^V$ is the unique solution of (3.6) and $\mathbf{e}_R^m = \mathbf{e}_R^f = 0$.

Additionally,

- for $q \geq 2$, $u_1^\varepsilon \rightarrow u_1^0$ strongly in $H^1(\gamma)$;
- for $q > 2$, $\tilde{\omega}_3^\varepsilon \rightarrow \partial_1 u_2^0$ strongly in $H^{-1}(\gamma)$;
- for $q \geq 4$, $\tilde{\omega}_3^\varepsilon \rightarrow \partial_1 u_2^0$ strongly in $H^1(\gamma)$.

From the first convergence in (4.1) and the trace theorem we obtain $\mathbf{u}^{\varepsilon_k} \rightharpoonup \mathbf{u}^0$ in $L^2(\Gamma; \mathbb{R}^3)$. From the second convergence in (4.1), by observing all components, we obtain

$$\begin{aligned} \partial_1 u_1^{\varepsilon_k} &\rightharpoonup (\mathbf{e}_P^m)_{11}, & \partial_1 u_2^{\varepsilon_k} - \tilde{\omega}_3^{\varepsilon_k} &\rightharpoonup (\mathbf{e}_P^m)_{21}, & \partial_1 u_3^{\varepsilon_k} + \tilde{\omega}_2^{\varepsilon_k} &\rightharpoonup (\mathbf{e}_P^m)_{31}, \\ \partial_2 u_1^{\varepsilon_k} + \tilde{\omega}_3^{\varepsilon_k} &\rightharpoonup (\mathbf{e}_P^m)_{12}, & \partial_2 u_2^{\varepsilon_k} &\rightharpoonup (\mathbf{e}_P^m)_{22}, & \partial_2 u_3^{\varepsilon_k} - \tilde{\omega}_1^{\varepsilon_k} &\rightharpoonup (\mathbf{e}_P^m)_{32}, \end{aligned}$$

all weakly in $L^2(\Gamma)$. By the 2d Korn type inequality

$$\|\mathbf{v}\|_{H^1(\Gamma; \mathbb{R}^2)} \leq C(\|\mathbf{e}(\mathbf{v})\|_{L^2(\Gamma; \mathbb{R}^{2 \times 2})} + \|\mathbf{v}\|_{L^2(\Gamma; \mathbb{R}^2)})$$

applied on the coordinates $(\mathbf{e}_P^m)_{11}$, $(\mathbf{e}_P^m)_{21} + (\mathbf{e}_P^m)_{12}$, and $(\mathbf{e}_P^m)_{22}$, we obtain that, on a subsequence not relabeled, $u_\alpha^{\varepsilon_k} \rightharpoonup u_\alpha^0$ in $H^1(\Gamma; \mathbb{R}^3)$. By the trace theorem $u_\alpha^{\varepsilon_k} \rightharpoonup u_\alpha^0$ in $L^2(\gamma; \mathbb{R})$. Thus the limit \mathbf{u}^0 belongs to V^I . Furthermore we now get that $\tilde{\omega}_3^{\varepsilon_k} \rightharpoonup \tilde{\omega}_3$ weakly in $L^2(\Gamma)$ and

$$(\mathbf{e}_P^m)_{11} = \partial_1 u_1^0, \quad (\mathbf{e}_P^m)_{22} = \partial_2 u_2^0, \quad (\mathbf{e}_P^m)_{12} + (\mathbf{e}_P^m)_{21} = \partial_1 u_2^0 + \partial_2 u_1^0. \quad (4.3)$$

Let us take $\mathbf{v} = 0$ and for arbitrary $\tilde{\mathbf{w}} \in H^1(\Gamma; \mathbb{R}^3)$ with $\tilde{\mathbf{w}} = 0$ on γ , let $(\tilde{\mathbf{w}}_\varepsilon)_\varepsilon$ be a sequence of smooth functions converging strongly to $\tilde{\mathbf{w}}$ in $H^1(\Gamma; \mathbb{R}^3)$ such that $\tilde{\mathbf{w}}_\varepsilon = 0$ on $[0, 1] \times [-\varepsilon, \varepsilon]$. After inserting such \mathbf{u} and $\tilde{\mathbf{w}}_\varepsilon$ into (2.2) and letting $\varepsilon \rightarrow 0$, we obtain

$$\int_\Gamma \mathcal{C}_m \mathbf{e}_P^m \cdot (\mathbf{A} \tilde{\mathbf{w}}) dx' = 0 \quad (4.4)$$

for all $\tilde{\mathbf{w}} \in H^1(\Gamma; \mathbb{R}^3)$ with $\tilde{\mathbf{w}} = 0$ on γ . By varying $\tilde{\mathbf{w}}$, we obtain $(\mathbf{e}_P^m)_{31} = (\mathbf{e}_P^m)_{32} = 0$, and $(\mathbf{e}_P^m)_{21} - (\mathbf{e}_P^m)_{12} = 0$. From (4.3) then we have $(\mathbf{e}_P^m)_{21} = (\mathbf{e}_P^m)_{12} = \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0)$. Also (4.4) holds for arbitrary $\tilde{\mathbf{w}} \in L^2(\Gamma; \mathbb{R}^3)$ as well. Then for arbitrary $\mathbf{v} \in H^1(\Gamma; \mathbb{R}^3)$ we have

$$\mathcal{C}_m \mathbf{e}_P^m \cdot \nabla' \mathbf{v} = \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}).$$

Thus we have just proved the following lemma.

Lemma 11. *The limit function \mathbf{u}^0 belongs to V^I ,*

$$\mathbf{e}_P^m = \begin{bmatrix} \partial_1 u_1^0 & \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) \\ \partial_1 u_2^0 + \partial_2 u_1^0 & \partial_2 u_2^0 \\ 0 & 0 \end{bmatrix},$$

$\mathcal{C}_m \mathbf{e}_P^m \cdot \nabla' \mathbf{v} = \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v})$ for any $\mathbf{v} \in H^1(\Gamma; \mathbb{R}^3)$ and the following convergences hold

$$\begin{aligned} u_\alpha^{\varepsilon_k} &\rightharpoonup u_\alpha^0 && \text{weakly in } H^1(\Gamma), \quad \alpha = 1, 2, \\ u_\alpha^{\varepsilon_k} &\rightharpoonup u_\alpha^0 && \text{weakly in } L^2(\gamma), \quad \alpha = 1, 2, \\ \tilde{\omega}_3^{\varepsilon_k} &\rightharpoonup \tilde{\omega}_3^0 && \text{weakly in } L^2(\Gamma). \end{aligned}$$

In order to prove the strong convergences we additionally define

$$\begin{aligned} \Lambda(k) &:= \int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^{\varepsilon_k} - \mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^{\varepsilon_k} - \mathbf{u}^0) dx + \int_{\Gamma} \mathcal{C}_m (\nabla' \mathbf{u}^{\varepsilon} + \mathbf{A} \tilde{\omega}^{\varepsilon} - \mathbf{e}_P^m) \cdot (\nabla' \mathbf{u}^{\varepsilon} + \mathbf{A} \tilde{\omega}^{\varepsilon} - \mathbf{e}_P^m) dx' \\ &+ \frac{1}{12} \int_{\Gamma} \mathcal{C}_f \nabla' (\varepsilon \tilde{\omega}^{\varepsilon} - \mathbf{e}_P^f) \cdot (\varepsilon \tilde{\omega}^{\varepsilon} - \mathbf{e}_P^f) dx' \\ &+ \int_{\gamma} \mathbf{M} (\varepsilon^{1-q/2} [\partial_1 \mathbf{u}^{\varepsilon} + \mathbf{e}_1 \times \tilde{\omega}^{\varepsilon}] - \mathbf{e}_R^m) \cdot (\varepsilon^{1-q/2} [\partial_1 \mathbf{u}^{\varepsilon} + \mathbf{e}_1 \times \tilde{\omega}^{\varepsilon}] - \mathbf{e}_R^m) dx_1 \\ &+ \int_{\gamma} \mathbf{H} (\varepsilon^{2-q/2} \partial_1 \tilde{\omega}^{\varepsilon} - \mathbf{e}_R^f) \cdot (\varepsilon^{2-q/2} \partial_1 \tilde{\omega}^{\varepsilon} - \mathbf{e}_R^f) dx_1. \end{aligned}$$

After eliminating quadratic terms using the equation (2.2), due to Corollary 3 and Lemma 11, we obtain that $\Lambda(k)$ converges to the limit

$$\begin{aligned} \Lambda &:= \int_{\Gamma} \mathbf{f} \cdot \mathbf{u}^0 dx' - \int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) dx - \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{u}^0) dx' \\ &- \frac{1}{12} \int_{\Gamma} \mathcal{C}_f \mathbf{e}_P^f \cdot \mathbf{e}_P^f dx' - \int_{\gamma} \mathbf{M} \mathbf{e}_R^m \cdot \mathbf{e}_R^m dx_1 - \int_{\gamma} \mathbf{H} \mathbf{e}_R^f \cdot \mathbf{e}_R^f dx_1. \end{aligned} \quad (4.5)$$

It is clear that $\Lambda \geq 0$ as the limit of a nonnegative sequence. In all cases that follow the obtained limit model implies that the limit Λ is equal to zero. It will imply that some or all $\mathbf{e}_P^f, \mathbf{e}_R^m, \mathbf{e}_R^f$ are zero and that all convergences in (4.1) are strong. Also, since the limit problems are the same as obtained in previous section, for which we proved uniqueness of solutions, the whole ε -families will converge to the same limit.

4.1 The case $q < 2$

We take arbitrary $(\mathbf{v}, \tilde{\omega}) \in V_{3d-2d-1d}$ with $\tilde{\omega} = 0$ and let ε tends to zero. From Lemma 11 we obtain the limit model: find $\mathbf{u}^0 \in V^I$ such that

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx' \quad (4.6)$$

that holds for all $\mathbf{v} \in V^I$ due to density.

From (4.5) and (4.6) $\Lambda = 0$, and thus all $\mathbf{e}_P^f, \mathbf{e}_R^m, \mathbf{e}_R^f$ are equal to zero. Furthermore the strong convergences in (4.1) hold.

4.2 The case $q = 2$

Componentwise the fourth convergence in (4.1) is given by

$$\partial_1 u_1^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^m)_1, \quad \partial_1 u_2^{\varepsilon_k} - \tilde{\omega}_3^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^m)_2, \quad \partial_1 u_3^{\varepsilon_k} + \tilde{\omega}_2^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^m)_3,$$

all weakly in $L^2(\gamma)$. Since from Lemma 11 we know that $u_1^{\varepsilon_k} \rightharpoonup u_1^0$ weakly in $L^2(\gamma)$ we can identify the limit $(\mathbf{e}_R^m)_1 = \partial_1 u_1^0$ and thus $\partial_1 u_1^{\varepsilon_k} \rightarrow \partial_1 u_1^0$ in $L^2(\gamma)$. Therefore, the limit \mathbf{u}^0 belongs to V^{II} .

Now let us take an arbitrary $\tilde{\mathbf{w}} \in H^1(\Gamma; \mathbb{R}^3) \cap H^1(\gamma; \mathbb{R}^3)$ and $\mathbf{v} = 0$ in (2.2) and let ε tends to zero. We obtain

$$\int_{\Gamma} \mathcal{C}_m \mathbf{e}_P^m \cdot (\mathbf{A} \tilde{\mathbf{w}}) dx' + \int_{\gamma} \mathbf{M} \mathbf{e}_R^m \cdot [\mathbf{e}_1 \times \tilde{\mathbf{w}}] dx_1 = 0,$$

which due to (4.4) in fact implies

$$\int_{\gamma} \mathbf{M} \mathbf{e}_R^m \cdot [\mathbf{e}_1 \times \tilde{\mathbf{w}}] dx_1 = 0$$

for arbitrary $\tilde{\mathbf{w}} \in L^2(\gamma; \mathbb{R}^3)$ (by density). This implies $(\mathbf{e}_R^m)_2 = (\mathbf{e}_R^m)_3 = 0$. Now we take arbitrary $(\mathbf{v}, \tilde{\mathbf{w}}) \in V_{3d-2d-1d}$ with $\tilde{\mathbf{w}} = 0$ and let $\varepsilon \rightarrow 0$, to obtain the model: find $\mathbf{u}^0 \in V^{II}$ such that

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' + \int_{\gamma} E_{\text{rod}} \partial_1 u_1^0 \cdot \partial_1 v_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx' \quad (4.7)$$

for all $\mathbf{v} \in V^{II}$ (again by density).

From above we have

$$\mathbf{e}_R^m = \begin{bmatrix} \partial_1 u_1^0 \\ 0 \\ 0 \end{bmatrix},$$

then from (4.5) and (4.7) $\Lambda = 0$, and thus $\mathbf{e}_P^f, \mathbf{e}_R^f$ are equal to zero and the strong convergences in (4.1) hold.

4.3 The case $2 < q < 4$

From the fourth convergence in (4.1) we additionally conclude $\partial_1 u_1^{\varepsilon_k} \rightarrow 0$ strongly in $L^2(\gamma)$. Thus $\partial_1 u_1^0 = 0$, so the limit \mathbf{u}^0 belongs to V^{III} .

By taking arbitrary test function $(\mathbf{v}, \tilde{\mathbf{w}}) \in V_{3d-2d-1d}$ with $\tilde{\mathbf{w}} = 0$ and $\partial_1 v_1 = 0$ on γ , we obtain the model: find $\mathbf{u}^0 \in V^{III}$ such that

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx', \quad \mathbf{v} \in V^{III}. \quad (4.8)$$

From (4.5) and (4.8) $\Lambda = 0$, and thus all $\mathbf{e}_P^f, \mathbf{e}_R^m, \mathbf{e}_R^f$ are equal to zero and the strong convergences in (4.1) hold.

4.4 The case $q = 4$

The components of the last convergence in (4.1) are

$$\partial_1 \tilde{\omega}_1^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^f)_1, \quad \partial_1 \tilde{\omega}_2^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^f)_2, \quad \partial_1 \tilde{\omega}_3^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^f)_3,$$

all weakly in $L^2(\gamma)$. From Lemma 11 we know that $u_2^{\varepsilon_k} \rightharpoonup u_2^0$ in $L^2(\gamma)$, so $\partial_1 u_2^{\varepsilon_k} \rightharpoonup \partial_1 u_2^0$ in $H^{-1}(\gamma)$. Then the fourth convergence in (4.1), $\partial_1 u_2^{\varepsilon_k} - \tilde{\omega}_3^{\varepsilon_k} \rightarrow 0$ strongly in $L^2(\gamma)$, implies that $\tilde{\omega}_3^{\varepsilon_k} \rightharpoonup \tilde{\omega}_3^0 = \partial_1 u_2^0$ weakly in $H^{-1}(\gamma; \mathbb{R}^3)$. Then we can identify $(\mathbf{e}_R^f)_3 = \partial_1 \tilde{\omega}_3^0 = \partial_{11} u_2^0$ from $\partial_1 \tilde{\omega}_3^{\varepsilon_k} \rightharpoonup (\mathbf{e}_R^f)_3$ weakly in $L^2(\gamma)$ and, by two applications of the Lions lemma, conclude that $\tilde{\omega}_3^0 \in H^1(\gamma)$ and that $\tilde{\omega}_3^{\varepsilon_k} \rightharpoonup \tilde{\omega}_3^0$ in $H^1(\gamma)$. Now since $\partial_1 u_2^0 = \tilde{\omega}_3^0 \in H^1(\gamma)$ we also obtain $u_2^0 \in H^2(\gamma)$, $u_2^{\varepsilon_k} \rightarrow u_2^0$ in $H^1(\gamma)$ and $\partial_1 \tilde{\omega}_3^{\varepsilon_k} \rightharpoonup \partial_{11} u_2^0$ in $L^2(\gamma)$. Thus, the limit \mathbf{u}^0 belongs to V^{IV} .

Let $(\mathbf{v}, \tilde{\mathbf{w}}) \in V_{3d-2d-1d}$ be an arbitrary test function such that $\partial_1 \mathbf{v} + \mathbf{e}_1 \times \tilde{\mathbf{w}} = 0$ on γ . More precisely let:

$$\begin{aligned} \mathbf{v} &\in H^1(\Omega; \mathbb{R}^3) \cap H^1(\Gamma; \mathbb{R}^3) \cap H^2(\gamma; \mathbb{R}^3), \quad \tilde{\mathbf{w}} \in H^1(\Gamma; \mathbb{R}^3) \cap H^1(\gamma; \mathbb{R}^3), \\ \partial_1 v_1 &= 0, \quad \partial_1 v_2 - \tilde{w}_3 = 0, \quad \partial_1 v_3 + \tilde{w}_2 = 0 \quad \text{on } \gamma. \end{aligned}$$

By letting $\varepsilon \rightarrow 0$ we obtain

$$\begin{aligned} \int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{C}_m \mathbf{e}_P^m \cdot (\nabla \mathbf{v} + \mathbf{A} \tilde{\mathbf{w}}) dx' \\ + \int_{\gamma} h_{11}(\mathbf{e}_R^f)_1 \cdot \partial_1 \tilde{w}_1 - h_{22}(\mathbf{e}_R^f)_2 \cdot \partial_{11} v_3 + h_{33} \partial_{11} u_2^0 \cdot \partial_{11} v_2 dx_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx'. \end{aligned} \quad (4.9)$$

By choosing $\mathbf{v} = 0$, $\tilde{w}_2 = \tilde{w}_3 = 0$, and by (4.4), we obtain

$$\int_{\gamma} h_{11}(\mathbf{e}_R^f)_1 \cdot \partial_1 \tilde{w}_1 dx_1 = 0.$$

Since \tilde{w}_1 can be arbitrary function from $H^1(\gamma; \mathbb{R}^3)$, we obtain $(\mathbf{e}_R^f)_1 = 0$. Let us now choose test functions such that $v_1 = v_2 = 0$, $\tilde{w}_1 = \tilde{w}_3 = 0$. We obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(v_3 \mathbf{e}_3) dx - \int_{\gamma} h_{22}(\mathbf{e}_R^f)_2 \cdot \partial_{11} v_3 dx_1 = \int_{\Gamma} f_3 v_3 dx'. \quad (4.10)$$

Let $\nu \in H^1(\Omega)$ be arbitrary. Since γ is of capacity zero in Ω , by [12, Theorem 2.44] there exists a sequence $(\nu_n)_n \subset H^1(\Omega)$ with $\nu_n = 0$ on γ such that strongly converges to ν in $H^1(\Omega)$. In (4.10) we insert $v_3 = \nu_n$ and let n tends to infinity. In the limit we obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\nu \mathbf{e}_3) dx = \int_{\Gamma} f_3 \nu dx',$$

for all $\nu \in H^1(\Omega)$. Consequently, from (4.10) we obtain

$$\int_{\gamma} h_{22}(\mathbf{e}_R^f)_2 \cdot \partial_{11} v_3 dx_1 = 0$$

for all $v_3 \in H^2(\gamma)$. Thus $(\mathbf{e}_R^f)_2 = 0$. From (4.9) we conclude that the limit $\mathbf{u}^0 \in V^{IV}$ satisfies

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' + \int_{\gamma} h_{33} \partial_{11} u_2^0 \cdot \partial_{11} v_2 dx_1 = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx' \quad (4.11)$$

for all $\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) \cap H^1(\Gamma; \mathbb{R}^3) \cap H^2(\gamma; \mathbb{R}^3)$ with $\partial_1 v_1 = 0$ on γ . By density argument, the equation is satisfied for all $\mathbf{v} \in V^{IV}$ as well.

From above we have

$$\mathbf{e}_R^f = \begin{bmatrix} 0 \\ 0 \\ \partial_{11} u_2^0 \end{bmatrix},$$

then from (4.5) and (4.11) $\Lambda = 0$, and thus $\mathbf{e}_P^f, \mathbf{e}_R^m$ are equal to zero and the strong convergences in (4.1) hold.

4.5 The case $q > 4$

From the last convergence in (4.1) we additionally conclude

$$\partial_1 \tilde{\omega}_1^{\varepsilon_k} \rightarrow 0, \quad \partial_1 \tilde{\omega}_2^{\varepsilon_k} \rightarrow 0, \quad \partial_1 \tilde{\omega}_3^{\varepsilon_k} \rightarrow 0,$$

all strongly in $L^2(\gamma)$.

By differentiating conclusions from the case $q = 4$, we obtain $\partial_{11} u_2^0 = 0$ on γ , thus the limit belongs to V^V .

For arbitrary $\nu \in H^1(\Omega)$, by [12, Theorem 2.44] there again exists a sequence $(\nu_n)_n \subset H^1(\Omega)$ with $\nu_n = 0$ on γ such that strongly converges to ν in $H^1(\Omega)$. For test functions $(\mathbf{v}, \tilde{\mathbf{w}}) = (\nu_n \mathbf{e}_3, 0)$ in (2.2) we let n to infinity and obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\nu \mathbf{e}_3) dx = \int_{\Gamma} f_3 \nu dx'.$$

Now we choose test functions $\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) \cap H^1(\Gamma; \mathbb{R}^3) \cap H^2(\gamma; \mathbb{R}^3)$, such that $v_3 = 0$ on Ω , $\partial_1 v_1 = 0$, $\partial_{11} v_2 = 0$ and $\tilde{\mathbf{w}} = \partial_1 v_2 \mathbf{e}_3$ on γ . Then $\partial_1 v_2$ is a constant and $\tilde{\mathbf{w}} = (0, 0, \partial_1 v_2) \in H^1(\Gamma; \mathbb{R}^3)$. We obtain

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) dx' = \int_{\Gamma} f_1 v_1 + f_2 v_2 dx'.$$

By summing up last two equations and by density, we obtain the model: find $\mathbf{u}^0 \in V^V$ such that for all $\mathbf{v} \in V^V$

$$\int_{\Omega} \mathcal{C}_{3D} \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}) dx' = \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} dx'. \quad (4.12)$$

From (4.5) and (4.12) $\Lambda = 0$, and thus all $\mathbf{e}_P^f, \mathbf{e}_R^m, \mathbf{e}_R^f$ are equal to zero and the strong convergences in (4.1) hold.

4.6 Additional claims

Let us prove last three claims from the Theorem 10.

For $q \geq 2$, the fourth claim from (4.2) implies that $\partial_1 u_1^\varepsilon$ converges strongly in $L^2(\gamma)$. Together with the seventh claim from (4.2) we obtain that $u_1^\varepsilon \rightarrow u_1^0$ strongly in $H^1(\gamma)$.

For $q > 2$, the fourth claim from (4.2) implies that $\partial_1 \tilde{u}_2^\varepsilon - \tilde{\omega}_3^\varepsilon \rightarrow 0$ strongly in $L^2(\gamma)$. After differentiating seventh claim from (4.2) we have $\partial_1 u_2^\varepsilon \rightarrow \partial_1 u_2^0$ strongly in $H^{-1}(\gamma)$. By combining these two results we obtain $\tilde{\omega}_3^\varepsilon \rightarrow \partial_1 u_2^0$ strongly in $H^{-1}(\gamma)$.

For $q \geq 4$, the fifth claim from (4.2) implies that $\partial_1 \tilde{\omega}_3^\varepsilon$ converges strongly in $L^2(\gamma)$. Together with the result from above for $q > 2$, after applying Lions lemma several times we obtain $\tilde{\omega}_3^\varepsilon \rightarrow \partial_1 u_2^0$ strongly in $H^1(\gamma)$.

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References

- [1] E. Acerbi, G. Buttazzo and D. Percivale, *Thin inclusions in linear elasticity: a variational approach*, J Reine Angew Math **386** (1988), 99–115.
- [2] A.-L. Bessoud, F. Krasucki and G. Michaille, *Multi-materials with strong interface: variational modelings*, Asymptot. Anal. **61** (2009), 1–19.
- [3] A.-L. Bessoud, F. Krasucki and M. Serpilli, *Asymptotic analysis of shell-like inclusions with high rigidity*, J. Elasticity **103** (2011), 153–172.
- [4] Caillerie D., The effect of a thin inclusion of high rigidity in an elastic body, *Mathematical Methods in the Applied Sciences* **2** (1980), 3, 251–270.
- [5] S. Čanić, M. Galić, M. Ljulj, B. Muha, J. Tambača and Y. Wang, *Analysis of a linear 3D fluid-mesh-shell interaction problem*, Z. Angew. Math. Phys. **70** (2019), 2, no. 44, 38 pp.
- [6] P.G. Ciarlet, *Mathematical elasticity. Vol. II. Theory of plates*. North-Holland Publishing Co., Amsterdam, 1997.
- [7] P.G. Ciarlet, H. Le Dret and R. Nzingwa, *Junctions between three-dimensional and two-dimensional linearly elastic structures*, J. Math. Pures Appl. **68** (1989), 261–295.
- [8] S. Dumont, F. Lebon and R. Rizzoni, *Imperfect interfaces with graded materials and unilateral conditions: theoretical and numerical study*, Math. Mech. Solids **23** (2018), 445–460.

- [9] E. El Bachari, *Modelling of a 3d-2d junction in nonlinear elasticity by G-convergence* C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), 1241–1246.
- [10] P. Germain, *Mécanique des milieux continus*, Mason, Pariz, 1962.
- [11] G. Geymonat, F. Krasucki and S. Lenci, *Mathematical analysis of a bonded joint with a soft thin adhesive*, Math Mech Solids **4** (1999), 201–225.
- [12] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications, Inc., Mineola, 2006.
- [13] F. Krasucki, A. Münch and Y. Ousset, *Mathematical analysis of nonlinear bonded joint models*, Math Models Meth Appl Sci **14** (2004), 1–22.
- [14] F. Lebon and R. Rizzoni, *Asymptotic analysis of a thin interface: the case involving similar rigidity*, Int J Eng Sci **48** (2010), 473–486.
- [15] F. Lebon and R. Rizzoni, *Asymptotic behavior of a hard thin linear interphase: an energy approach*, Int J Solids Struct **48** (2011), 441–449.
- [16] M. Ljulj, E. Marušić-Paloka, I. Pažanin and J. Tambača, *Mathematical model of heat transfer through a conductive pipe*, M2AN **55** (2021), 627–658.
- [17] M. Ljulj and J. Tambača, *Numerical investigation of the 2d-1d structure interaction model*, Mathematics and Mechanics of Solids **26** (2021), 1876–1895.
- [18] M. Ljulj and J. Tambača, *3D structure – 2D plate interaction problem*, Mathematics and Mechanics of Solids **24** (2019), 3354–3377.
- [19] I. Malakhova-Ziablova, G. Panasenko, R. Stavre, Appl. Anal. **95** (2016), no. 7, 1467–1506.
- [20] B. Méjri, *Carleman estimate for a 1D linear elastic problem involving interfaces: application to an inverse problem*, Math. Methods Appl. Sci. **44** (2021), 13, 10686–10702.
- [21] D. Padilla-Garza, *An asymptotic variational problem modeling a thin elastic sheet on a liquid, lifted at one end*, Math. Methods Appl. Sci. **43** (2020) 8, 4956–4973.
- [22] G.P. Panasenko and R. Stavre, *Asymptotic analysis of a viscous fluid–thin plate interaction: periodic flow*, Math. Models Methods Appl. Sci. **24** (2014), 1781–1822.
- [23] M. L. Raffa, F. Lebon and R. Rizzoni, *Derivation of a model of imperfect interface with finite strains and damage by asymptotic techniques: an application to masonry structures*, Meccanica **53** (2018), 1645–1660.
- [24] K. Schmidt and S. Tordeux, *Asymptotic modelling of conductive thin sheets*, Z. Angew. Math. Phys. **61** (2010), 4, 603–626.
- [25] M. Serpilli, *Asymptotic analysis of a multimaterial with a thin piezoelectric interphase*, Meccanica **49** (2014), 1641–1652.
- [26] M. Serpilli, *Asymptotic interface models in magneto-electro-thermo-elastic composites*, Meccanica **52** (2017), 1407–1424.
- [27] M. Serpilli, *Mathematical modeling of weak and strong piezoelectric interfaces*, J. Elasticity **121** (2015), 235–254.
- [28] M. Serpilli, *On modeling interfaces in linear micropolar composites*, Math. Mech. Solids **23** (2018), 667–685.

- [29] M. Serpilli, S. Dumont, R. Rizzoni and F. Lebon, *Interface models in coupled thermoelasticity*, Technologies **9** (2021), 1, no. 17, pp. 16.
- [30] M. Serpilli, R. Rizzoni, F. Lebon and S. Dumont, *An asymptotic derivation of a general imperfect interface law for linear multiphysics composites*, International Journal of Solids and Structures **180-181** (2019), 97–107.
- [31] J. Tambača, M. Kosor, S. Čanić and D. Paniagua, *Mathematical modeling of vascular stents*, SIAM J. Appl. Math. **70** (2010), 6, 1922–1952.
- [32] J. Tambača and Z. Tutek, *A new linear Naghdi type shell model for shells with little regularity* Appl. Math. Model. **40** (2016), 10549–10562.